World Applied Sciences Journal 34 (4): 415-422, 2016
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2016.34.4.129

Different Methods of Analytical Advection Diffusion Equation

Khaled S.M. Essa and Sawsan E.M. Elsaid

Mathematics and Theoretical Physics, NRC, Atomic Energy Authority, Cairo- Egypt

Abstract: In this paper, the advection diffusion equation is solved using two methods to evaluate crosswind integrated of pollutant concentration per emission rate in three dimensions with constant wind speed and eddy diffusivity under steady state. The used data set was observed from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark. The tracer sulfur hexafluoride (SF_6) was released from a tower at a height of 115m without buoyancy. Comparison between the observed and predicated there are some predicated data which are agreement with observed data (one to one) and others lie inside the factor of two and factor of four.

Key words: Advection diffusion equation · Observed and predicated concentration · Sulfur hexafluoride

INTRODUCTION

Advection diffusion equations are used to stimulate a variety of a different phenomenon and industrial applications. advection diffusion equation describes the transport occurring in fluid through the combination of advection and diffusion. Its analytical / numerical solution along with an initial condition and two boundary conditions help to understand the contaminant or pollutant concentration distribution behavior through an open medium like air, rivers, lakes and porous medium like aquifer, on the basis of which remedial processes to reduce or eliminate the damages may be enforced. In initial works while obtaining the analytical solution of dispersion problems in the ideal condition, the basic approach was to reduce the advection –diffusion equation into a diffusion equation by eliminating the advection term. Advection diffusion equation with constant and variable has a wide range of practical and industrial application (Amruta Daga and Pradhan V.H. [1]). The contaminants in aquifer systems migrate with ground water flow, may factors that may affect groundwater flow are also likely to influence the migration of contaminants in equifers.because contaminants are chemicals or bacteria or virus which are mostly physically, chemically and biologically active, the transport of contaminants are subject to physical, chemical and biological activities, such as contaminant density, adsorption and desorption, retardation, degradation and chemical- biological reactions. Analytical solution, numerical simulations and experiment and filed observation are used to address groundwater flowed contaminant transport problems in aquifers. Contaminant (solute) transport through a medium is described by a partial differential equation of parabolic type and it is usually know as advection – dispersion equation. Advection – dispersion equation is applicable in many disciplines like groundwatehydrology, chemical engineering bio sciences, environmental sciences and petroleum engineering to describe the behavior of solute concentration. In earlier, this equation along with a set of initial and boundary conditions has been solved for uniform dispersion and velocity. Analytical solutions are obtained for a one –dimensional advection dispersion equation with variable coefficient in a longitudinal domain. (Dilip Kumar Jaiswal and Atul Kumar [2]). One-dimensional advection–diffusion equation with variable coefficients is solved for three dispersion problems: (i) solute dispersion along steady flow through an inhomogeneous medium, (ii) temporally dependent solute dispersion along uniform flow through homogeneous medium and (iii) solute dispersion along temporally dependent flow through inhomogeneous medium. Continuous point sources of uniform and increasing nature are considered in an initially solute free semi-infinite medium. Analytical solutions are obtained using Laplace transformation technique. The in homogeneity of the medium is expressed by spatially dependent flow. Its velocity is defined by a function

Corresponding Author: Khaled S.M. Essa, Mathematics and Theoretical Physics, NRC, Atomic Energy Authority, Cairo- Egypt.

interpolated linearly in a finite domain in which concentration values are to be evaluated. The dispersion is considered proportional to square of the spatially dependent velocity. The solutions of the third problem may help understand the concentration dispersion pattern along a sinusoidal varying unsteady flow through an inhomogeneous medium. New independent variables are introduced through separate transformations, in terms of which the advection–diffusion equation in each problem is reduced into the one with the constant coefficients. The effects of spatial and temporal dependence on the concentration dispersion are studied with the help of respective parameters and are shown graphically (Atul Kumar and *et al.* [3]). In order to evaluate such scenarios one need efficient procedures, which yield immediate results, for instance evaluating the ground level concentration of pollutants and especially the maximum concentration and its position [4].

In this paper, the advection diffusion equation is solved using Laplace transform and separation of variables methods to evaluate crosswind integrated of pollutant concentration per emission rate in three dimensions with constant wind speed and eddy diffusivity under steady state. The used data set was observed from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark. The tracer sulfur hexafluoride (SF_6) was released from a tower at a height of 115m without buoyancy. That there are some predicated data which are agreement with observed data (one to one) and others lie inside the factor of two and factor of four.

Mathematical Solution: The basic gradient transport model can be written [5]:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=S+\frac{\partial}{\partial x}K_x\frac{\partial C}{\partial x}+\frac{\partial}{\partial y}K_y\frac{\partial C}{\partial y}+\frac{\partial}{\partial z}K_z\frac{\partial C}{\partial z} \quad (1)$$

where:
C is the average concentration of diffusing point (x, y and z) (kg/m^3).
U is mean wind velocity along the x-axis (m/s).
K_x, k_y and k_z are the eddy diffusivities coefficients along x, y and z axes respectively (m^2/s).
x is along –winds coordinate measured in wind direction from the source (m).
y is cross-wind coordinate direction (m).
z is vertical coordinate measured from the ground (m).
S is source/ sinks term (kg/m^3-s).
$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}$ is time rate of change and advection of the cloud by the mean wind.
$\frac{\partial}{\partial y}K_y\frac{\partial C}{\partial y}$ and $\frac{\partial}{\partial z}K_z\frac{\partial C}{\partial z}$ represent turbulent diffusion of material relative to the center of the pollutant cloud. (The cloud will expand over time due to these terms).

S source term which represents net production (or destruction) of pollutant due to sources (or removal).

The mean wind components (u, v and w) and mean concentration (C) represent average over a time scale (T_a) and apace scale (x_a).

Equation (1) is impossible to solve analytically for completely general functional forms for the diffusivity K and wind speeds u, v and w.

One Dimensional Equation(x), under Steady State, Constant K and Wind Speed:

$$u\frac{\partial C}{\partial x}=K\frac{\partial^2 C}{\partial x^2}\Rightarrow\frac{\partial^2 C}{\partial x^2}-\frac{u}{K}\frac{\partial C}{\partial x}=0$$

where K =0.04*u*x (2)

The equation (2) subject to the boundary condition:

$$C(x)=0 \text{ at } x\to\infty \quad (i)$$

C(x) =δ(x) at x=0 (ii)

Taking Laplace transform on the equation (2) we get that:

$$L\left(\frac{\partial^2 C}{\partial x^2}-\frac{u}{K}\frac{\partial C}{\partial x}\right)=L(0)\Rightarrow L\left(\frac{\partial^2 C}{\partial x^2}\right)-L\left(\frac{u}{K}\frac{\partial C}{\partial x}\right)=0 \quad (3)$$

$$S^2\tilde{C}(S)-S\delta(x)-\delta'(x)-\frac{uS}{K}\tilde{C}(S)+\frac{u}{K}C(0)=0$$

$$\therefore \tilde{C}(S)=\frac{\delta(x)}{S}+\frac{\delta'(x)}{S\left(S-\frac{u}{K}\right)} \quad (4)$$

Taking Laplace on the boundary condition, we get that

L(δ(x))=1 , L(δ'(x))= S

Substitute the equation (ii) on equation (4), we obtain that:

$$\tilde{C}(S)=\frac{1}{S}+\frac{S}{S\left(S-\frac{u}{K}\right)}=\frac{1}{S}+\frac{1}{\left(S-\frac{u}{K}\right)} \quad (5)$$

Taking Laplace inverse on equation (5) we get that :

$$C(x)=1+e^{\left(\frac{u}{K}\right)x} \quad (\mathrm{I})$$

where

$$L^{-1}\left(\frac{1}{s}\right)=1 and \qquad L^{-1}\left(\frac{1}{S-\frac{u}{K}}\right)=e^{\frac{u}{K}x}$$

Two-Dimensional Equation(x, Y), under Steady State, Constant K and Wind Speed U. Instantaneous Area Source:

$$u\frac{\partial C}{\partial x}=K\frac{\partial^2 C}{\partial y^2}\Rightarrow\frac{\partial^2 C}{\partial y^2}-\frac{u}{K}\frac{\partial C}{\partial x}=0 \quad (6)$$

The equation (6) subject to boundary condition:

C(x, y) =0 at x→∞ and y→± ∞ (c)

C(x, y) = δ(y) at x=0

$\frac{\partial C}{\partial y}=0$ at y=0

Taking Laplace transform respect to x we get that:

$$L\left(\frac{\partial^2 C}{\partial y^2}\right)-L\left(\frac{u}{K}\frac{\partial C}{\partial x}\right)=0\Rightarrow\frac{\partial^2\tilde{C}(s,y)}{\partial y^2}-\frac{su}{K}\tilde{C}(s,y)+\frac{u}{K}c(0,y)=0 \tag{7}$$

Substitute from (d) on equation (7) we get that:

$$\frac{\partial^2\tilde{C}(s,y)}{\partial y^2}-\frac{su}{K}\tilde{C}(s,y)+\frac{u}{K}\delta(y)=0 \tag{8}$$

Taking Laplace transform respect to y we get that:

$$\frac{\partial^2\tilde{\tilde{C}}(s,y)}{\partial y^2}-\frac{su}{K}\tilde{\tilde{C}}(s,y)+\frac{u}{K}\delta(y)=0\Rightarrow$$

$$p^2\tilde{\tilde{C}}(s,p)-pc(s,0)-c'(s,0)-\frac{su}{K}\tilde{\tilde{C}}(s,p)+\frac{u}{K}=0$$

$$\therefore\tilde{\tilde{C}}(s,p)=\frac{pc(s,0)+c'(s,0)-\frac{u}{K}}{\left(p^2-\frac{su}{K}\right)}=\frac{pc(s,0)}{\left(p^2-\frac{su}{K}\right)}+\frac{c'(s,0)-\frac{u}{K}}{\left(p^2-\frac{su}{K}\right)}$$

Substituting from boundary conditions we get that:

$$\frac{\partial^2\tilde{\tilde{C}}(s,y)}{\partial y^2}-\frac{su}{K}\tilde{\tilde{C}}(s,y)+\frac{u}{K}\delta(y)=0\Rightarrow$$

$$p^2\tilde{\tilde{C}}(s,p)-pc(s,0)-c'(s,0)-\frac{su}{K}\tilde{\tilde{C}}(s,p)+\frac{u}{K}=0$$

$$\therefore\tilde{\tilde{C}}(s,p)=\frac{\frac{u}{K}-\left(pc(s,0)+c'(s,0)\right)}{\left(p^2-\frac{su}{K}\right)}=\frac{\frac{u}{K}}{\left(p^2-\frac{su}{K}\right)}-\frac{pc(s,0)}{\left(p^2-\frac{su}{K}\right)}$$

Taking Laplace inverse transform on equation (9) respect to y, we get that:

$$\tilde{C}(s,y)=\frac{u}{K}\frac{\sinh\sqrt{\frac{su}{K}}\,y}{\sqrt{\frac{su}{K}}}-c(s,0)\cosh\sqrt{\frac{su}{K}}\,y \tag{10}$$

where $\cosh\sqrt{\frac{su}{K}}y=\frac{e^{\sqrt{\frac{su}{K}}y}+e^{-\sqrt{\frac{su}{K}}y}}{2},\sinh\sqrt{\frac{su}{K}}y=\frac{e^{\sqrt{\frac{su}{K}}y}-e^{-\sqrt{\frac{su}{K}}y}}{2}$ Substitute from (iii) on equation (8), we get that

$$\tilde{C}(s,y)=\sqrt{\frac{u}{K}}\left(\frac{e^{\sqrt{\frac{su}{K}}y}-e^{-\sqrt{\frac{su}{K}}y}}{2\sqrt{S}}\right)-c(s,0)\left(\frac{e^{\sqrt{\frac{su}{K}}y}+e^{-\sqrt{\frac{su}{K}}y}}{2}\right) \tag{11}$$

Using (i) in equation (11) we get that:

$$c(S,0)=\sqrt{\frac{u}{SK}} \tag{a}$$

Substitute from (a) on equation (11), we get that

$$\tilde{C}(s,y)=\sqrt{\frac{u}{K}}\left(\frac{e^{-\sqrt{\frac{su}{K}}y}}{\sqrt{s}}\right)$$

Taking Lablace inverse on this equation

$$C(x,y)=\frac{1}{\sqrt{\pi x}}e^{-\frac{uy^2}{4Kx}} \tag{II}$$

where

$$C(x,y)=L^{-1}\left(\tilde{C}(s,y)\right), and\ L^{-1}\left(\frac{e^{-\sqrt{\frac{su}{K}}y}}{\sqrt{s}}\right)=\frac{e^{-\frac{uy^2}{4xK}}}{\sqrt{\pi x}}$$

Two Dimensional Equation(x,z), under Steady State, Constant K and Wind Speed, Instantaneous Area Source:

$$u\frac{\partial C}{\partial x}=K\frac{\partial^2 C}{\partial z^2}\Rightarrow\frac{\partial^2 C}{\partial z^2}-\frac{u}{K}\frac{\partial C}{\partial x}=0 \tag{12}$$

The equation (2) subject to the boundary condition:

C(x, z) =0 at x→∞, z→∞ (i)

C(x, z) = δ (z-h) at x =0 (ii)

$K\frac{\partial C}{\partial z}$ =0 at z=0, h (iii)

Using separation of variable, assuming that:

C(x, z) = F(x) G (z)

Differnation respect to x, z

$$\frac{\partial C}{\partial x}=F'(x)G(z)$$
$$\frac{\partial C}{\partial z}=F(x)G'(z)$$
$$\frac{\partial^2 C}{\partial z^2}=F(x)G''(z) \tag{13}$$

Substituting from equation (13) in (12), we get that

$$F(x)G''(z)-\frac{u}{K}F'(x)G(z)=0$$

Divided on F(x), G (z) we get that:

$$\frac{G''(z)}{G(z)}-\frac{u}{K}\frac{F'(x)}{F(x)}=0\Rightarrow\frac{G''(z)}{G(z)}=\frac{u}{K}\frac{F'(x)}{F(x)}=-\lambda^2$$

where ë*2 is called the separation constant and is arbitrary.

$$F'(x)=\frac{-K\lambda^2}{u}F(x)$$
$$\therefore F(x)=c_1 e^{\frac{-K\lambda^2 x}{u}} \tag{14}$$

Substituting from equation (ii) in (14), we get that

$$c_1=\delta(z-h)$$

$$F'(x)=\frac{-K\lambda^2}{u}F(x)$$

$$\therefore C(x,z)=\delta(z-h)e^{\frac{-K\sqrt{2\pi x}}{u\sqrt{h}}}c_2\cos\frac{2\pi}{h}z$$

$$\frac{G''(z)}{G(z)}=-\lambda^2$$
$$\Rightarrow G''(z)+\lambda^2 G(z)=0\Rightarrow$$
$$G(z)=c_2\cos\lambda z+c_3\sin\lambda z \tag{15}$$

Substituting from equation (iii) at z=0 in (15), we get that

$$G'(z)=c_3\lambda\cos 0-c_2\lambda\sin 0=0$$
$$\therefore c_3\lambda=0\Rightarrow c_3=0 \tag{16}$$

Substituting from equation (16) in (15), we get that

$$G(z)=c_2\cos\lambda z \tag{17}$$

Substituting from equation (iii) at z=h in (17), we get that

$$G'(z)=-c_2\lambda\sin\lambda h=0$$
$$\therefore\lambda h=2\pi\Rightarrow\lambda=\frac{n\pi}{h}$$

$$G(z)=c_2\cos\frac{n\pi}{h}z \tag{18}$$

$$C(x,z) = \delta(z-h) e^{\frac{-K\sqrt{2\pi}x}{u\sqrt{h}}} c_2 \cos\frac{n\pi}{h} z \tag{19}$$

Substituting from equation (ii) at z=h in (19), we get that

$$c_2 = 1$$

n=0, 1, 2, 3,4, ….

$$C(x,z) = \left(e^{\frac{-Kx}{u}\sqrt{\frac{2\pi}{h}}} \right) \cos n\pi \tag{III}$$

Multiplayer (I), (II), (III) in Q/u, we obtained of the pollutant concentration per emission rate:

$$C(x,y,z) = \frac{Q\cos n\pi}{u\sqrt{\pi x}} \left(1 + e^{\frac{u}{K}x} \right) \left(e^{-\left(\frac{uy^2}{4Kx} + \frac{Kx}{u}\sqrt{\frac{2\pi}{h}} \right)} \right) \tag{20}$$

To obtained crosswind integrated of pollutant concentration per emission rate with constant wind speed and eddy diffusivity under steady state. We integrate the equation (20) respect to y from zero to y. we get that:

$$\bar{C}_y(x,y,z) = 2QK\cos n\pi \sqrt{\frac{\pi kx}{u^3}} \left(1 + e^{\frac{u}{K}x} \right) e^{-\left(\frac{K\sqrt{2\pi}x}{u\sqrt{h}} \right)} erf.\left(\frac{y}{2}\sqrt{\frac{u}{kx}} \right) \tag{21}$$

where

$$erf.fuction = \int_0^y e^{-\frac{uy^2}{4Kx}} dy$$

RESULTS AND DISCUSSION

The used data set was observed from the atmospheric diffusion experiments conducted at the northern part of Copenhagen, Denmark, under unstable conditions [6, 7]. The tracer sulfur hexafluoride (SF_6) was released from a tower at a height of 115m without buoyancy. The values of different parameters such as stability, wind speed at 10m (U_{10}), wind speed at 115m (U_{115}) and downwind distance during the experiment are represented in (Table 1). Comparison between the predicated and observed crosswind- integrated concentration normalized with the emission source rate at different downwind distance, wind speed and distance for the different runs.

Table 1: Comparison between the predicated and observed crosswind- integrated concentration normalized with the emission source rate at different downwind distance, wind speed and distance for the different runs

					C/Q (10^{-4}s/m^2)		
Run no.	h (m)	U (m/s)	X(m)	K= 0.04*u*x	observed	Predicated	Ref.predicated (5)
1	1980	3.34	1900	0.25	6.48	1.76	5.50
1	1980	3.34	3700	0.49	2.31	8.57	3.10
2	1920	3.82	2100	0.32	5.38	6.21	3.60
2	1920	3.82	4200	0.64	2.95	2.24	1.20
3	1120	3.82	1900	0.29	8.2	5.34	6.20
3	1120	4.93	3700	0.73	6..22	4.95	5.40
3	1120	4.93	5400	1.06	4..3	5.82	3.30
5	820	4.93	2100	0.41	6.72	4.81	5.80
5	820	6.52	4200	1.10	5..84	4.42	3.60
5	820	6.52	6100	1.59	4.97	5.20	2.30
6	1300	6.52	2000	0.52	3.96	3.38	2.80
6	1300	6.68	4200	1.12	2.22	1.12	1.20
6	1300	6.68	5900	1.58	1.83	6.46	1.40
7	1850	6.68	2000	0.53	6.7	3.30	6.40
7	1850	7.79	4100	1.28	3.25	8.98	5.20
7	1850	7.79	5300	1.65	2..23	4.41	2.10
8	810	8.11	1900	0.62	4.16	2.52	3.20
8	810	8.11	3600	1.17	2..02	2.18	2.01
8	810	8.11	5300	1.72	1.52	2.67	1.40
9	2090	11.45	2100	0.96	4..58	2.07	2.20
9	2090	11.45	4200	1.92	3.11	7.66	3.00
9	2090	11.45	6000	2.75	2..59	5.22	1.62

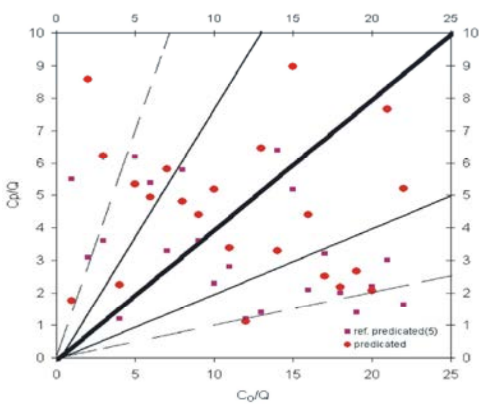

Fig. 1: Comparison between the predicated and observed crosswind- integrated concentration normalized with the emission source rate

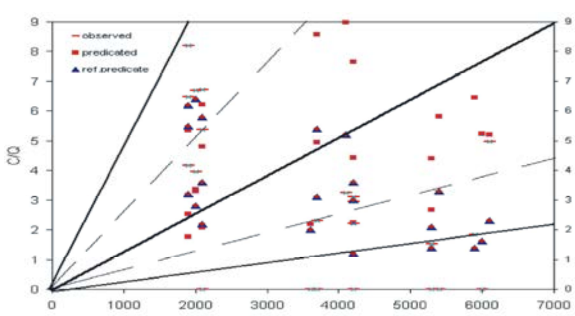

Fig. 2: Comparison between downwind distance and crosswind-integrated concentration normalized with the emission source rate

CONCLUSION

The advection diffusion equation is solved using Laplace transform and separation of variables methods to evaluate crosswind integrated of pollutant concentration per emission rate in three dimensions with constant wind speed and eddy diffusivity under steady state. The used data set was observed from the atmospheric diffusion experiments conducted at the Northern part of Copenhagen, Denmark. The tracer sulfur hexafluoride (SF_6) was released from a tower at a height of 115m without buoyancy. Comparison between the observed and predicated shows that there are some predicated data which are agreement with observed data (one to one) and others lie inside the factor of two and others factor of four.

REFERENCES

1. Amruta Daga, Pradhan V.H., 2013. Analytical solution of advection diffusion equation in homogeneous medium. International journal of science, spirituality, business and technology (IJSSBT), 2(1): ISSN 2261-7277.
2. Dilip Kumar Jaiswal and Atul Kumar, 2011. Analytical solutions of advection –dispersion equation for varying pulse type input point source in one –dimension. International Journal of Engineering Science and Technology, 3(1): 22-29.

3. Atul Kumar, Dilip Kumar Jaiswal and Naveen Kumar, 2009. Analytical Solutions to One-Dimensional Advection–Diffusion with Variable Coefficients in Semi-Infinite Media. Journal of Hydrology 01/2010; 380(3-4): 330-337. DOI: 10. 1016 / j. j hydrol. 2009.11.008.
4. Tiziano Tirabassi, Alessandro Tiesi, Marco T. Vilhena, Bardo E.J. Bodmann and Daniela Buske, 2011. An Analytical Simple Formula for the Ground Level Concentration from A point Source, Atmosphere2011,2,21-35;doi:103390/atoms2020021.
5- Hanna S. R. Briggs G. A. and Hosker R. P. Jr. (1982). Handbook on Atmospheric Diffusion, Published by Technical information center U. S. Departmentof Energy, DOE/TIC/-1122(DE82002045).
6. Gryning, S.E. and E. Lyck, 1984. Atmospheric dispersion from elevated sources in an urban area: Comparison between tracer experiments and model calculations, J. Climate Appl. Meteor., 23: 651-660.
7. Gryning, S.E., A.A.M. Holtslag, J.S. Irwin and B. Sivertsen, 1987. Applied dispersion modeling based on meteorological scaling parameters, Atmos. Environ., 21(1): 79-89.