

## New Exact Solutions for the Nonlinear Potential Boussinesq and ZKBBM Equations by Using $(1/G')$ -Expansion Method

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**Abstract:** In this paper, the  $(1/G')$ -expansion method with the aid of Maple are used to obtain new exact traveling wave solutions of the Nonlinear potential Boussinesq and ZKBBM Equations. The travelling wave solutions are expressed by the hyperbolic functions, the trigonometric functions and the rational functions. It is shown that the proposed method provides a powerful mathematical tool for solving nonlinear wave equations in mathematical physics and engineering.

**Key word:**

### INTRODUCTION

Finding exact solutions of nonlinear evolution equations (NEEs) is a very important part of nonlinear physical phenomena. It is a fact that exact solutions provide much physical information and help one to understand the mechanism that governs some physical models, such as plasma physics, optical fibers, biology, solid state physics, chemical physics and so on.

In recent years, different methods for finding exact solutions of nonlinear evolution equations have been proposed, developed and extended.

These are the Jacobi elliptic function method [1], the Hirota bilinear transformation [2], the Weierstrass function method [3], the Darboux and Backlund transform [4], the wronskian technique [5], homotopy perturbation method [6], the theta function method [7], symmetry method [8, 9], the homogeneous balance method [10, 11], sine-cosine method [12-14], F-expansion method [15], exp- function method [16-19], the Painleve expansion method [20], the transformed rational function method [21], the inverse scattering method [22-24],  $(G'/G)$ -expansion method [25-29]. The key idea of the original  $(G'/G)$ -expansion method is that the exact solutions of nonlinear partial differential equations (PDEs) can be expressed by a polynomial in one variable  $(G'/G)$  in which  $G = G(\xi)$  satisfies the second ordinary differential equation  $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$  where  $\lambda$  and  $\mu$  are constants.

In the present paper, we will use the  $(1/G')$ -expansion method, the main idea of  $(1/G')$ -expansion method is that our solutions can be expressed by a polynomial  $(1/G')$  and  $G = G(\xi)$  satisfies a second-order linear ODE

$$G''(\xi) + \lambda G'(\xi) + \mu = 0$$

where  $\lambda$  and  $\mu$  are constants, the degree of the polynomials can be determined by considering the homogeneous balance between the highest-order derivatives and nonlinear terms in the given nonlinear PDEs.  $(1/G')$ -expansion method has first been introduced by Yokus [30]. The present paper investigates the applicability and effectiveness of the  $(1/G')$ -expansion method on nonlinear evolution equations and systems of NEEs.

**Description of  $(1/G')$ -Expansion Method:** Suppose we have the following NLPDEs in the form:

$$F(u, u_t, u_x, u_{xx}, u_{tt}, \dots) = 0 \quad (1)$$

**Step 1:** The traveling wave variable

$$u(x, t) = u(\xi), \quad \xi = x + Vt \quad (2)$$

reduces (1) to an ODE in the form

$$P(u, u', u'', \dots) = 0, \quad (3)$$

where  $V$  is a constant and  $P$  is a polynomial in  $u$  and its total derivatives, while  $' = d/d\xi$ .

**Step 2:** Suppose that the solutions of (3) can be expressed by a polynomial () as follows:

$$u(\xi) = \sum_{i=0}^N a_i \left(\frac{1}{G'}\right)^i. \quad (4)$$

where  $a_i (i = 0, 1, \dots, N)$ ,  $\lambda$ ,  $\mu$  are constants determine Later where  $G = G(\xi)$  satisfies the second order LODE

$$G''(\xi) + \lambda G'(\xi) + \mu = 0 \quad (5)$$

**Step 3:** Determine the positive integer  $N$  in (4) by using the homogeneous balance between the highest -order derivatives and the nonlinear term in (3).

**Step 4:** The solution of the differential Eq. (5) is

$$G(\xi) = -\frac{\mu\xi}{\lambda} + C_1 e^{-\lambda\xi} + C_2 \quad (6)$$

Then

$$\frac{1}{G'(\xi)} = \frac{\lambda}{-\mu - \lambda^2 C_1 [\cosh(\xi\lambda) - \sinh(\xi\lambda)]} \quad (7)$$

**Step 5:** By substituting (4) into (3) and using second order LODE (5), the left-hand side of (3) can be converted into a polynomial in terms of  $(1/G')$ . Equating each coefficient of the polynomial to zero yields a system of algebraic equations and solving the algebraic equations by Maple we obtain  $a_i$ ,  $c$ ,  $\lambda$  and  $\mu$  constants.

### An Application

**The potential Boussinesq Equation:** In this section, we apply the method to find the exact traveling wave solutions of the nonlinear The potential Boussinesq Equation

$$u_{tt} + u_x u_{xx} + u_{xxx} = 0. \quad (8)$$

To this end, we see that the traveling wave variable (2) permits us To convert (8) into the following ODE

$$V^2 u'' + u' u'' + u''' = 0 \quad (9)$$

By balancing  $u''$  with  $u' u''$  in (9), we get  $N = 1$  Consequently, we get

$$u(\xi) = a_0 + a_1 \left(\frac{1}{G'}\right)^1 \quad (10)$$

where  $a_0$  and  $a_1$  are constants.

Substituting (10) into (9) and using (5), the left hand side of (9) becomes a polynomial in  $(1/G')$  Setting the coefficients of this polynomial to zero yields a system of algebraic equations in  $a_0, a_1$  and  $V$  as follows:

$$\left(\frac{1}{G'}\right)^1: V^2 a_1 \lambda^2 + a_1 \lambda^4 = 0, \quad (11)$$

$$\left(\frac{1}{G'}\right)^2: 3V^2 a_1 \lambda \mu + a_1^2 \lambda^3 + 15a_1 \lambda^3 \mu = 0, \quad (12)$$

$$\left(\frac{1}{G'}\right)^3: 2V^2 a_1 \mu^2 + 4a_1^2 \lambda^2 \mu + 50a_1 \lambda^2 \mu^2 = 0, \quad (13)$$

$$\left(\frac{1}{G'}\right)^4: 5a_1^2 \mu^2 \lambda + 60a_1 \lambda \mu^3 = 0, \quad (14)$$

$$\left(\frac{1}{G'}\right)^5: 2a_1^2 \mu^3 + 24a_1 \mu^4 = 0. \quad (15)$$

Solving the algebraic equations by the Maple or Mathematica, we get the following results.

$$\{ V = \sqrt{-1}, a_0 = a_0, a_1 = -12\mu \} \quad (16)$$

By substituting (16) into (10) using (6) we obtain

$$u(\xi) = a_0 - \frac{12\mu\lambda}{-\mu - c_1 \lambda^2 [\cosh(\xi\lambda) - \sinh(\xi\lambda)]} \quad (17)$$

where  $\xi = x + \sqrt{-1} t$ .

**The ZKBBM Equation:** We first consider the ZKBBM equation.

$$u_t + u_x - 2auu_x - bu_{xxt} = 0 \quad (18)$$

To this end, we see that the traveling wave variable (2) permits us To convert (18) into the following ODE

$$(1 + V)u' - 2auu' - bVu''' = 0 \quad (19)$$

By balancing  $u'''$  with  $uu'$  in (19), we get  $N = 2$ . Consequently, we get

$$u(\xi) = a_0 + a_1 \left(\frac{1}{G'}\right)^1 + a_2 \left(\frac{1}{G'}\right)^2 \quad (20)$$

where  $a_0$ ,  $a_1$  and  $a_2$  are constants substituting (20) into (19) and using (5), the left hand side of (19) becomes a polynomial in  $(1/G')$  Setting the coefficients of this polynomial to zero yields a system of algebraic equations in  $a_0$ ,  $a_1$ ,  $a_2$ , and  $V$  as follows:

$$\left(\frac{1}{G'}\right)^1: Va_1\lambda + a_1\lambda - 2aa_0a_1\lambda - bVa_1\lambda^3 \quad (21)$$

$$\left(\frac{1}{G'}\right)^2: -7bVa_1\lambda^2\mu - 2aa_1^2\lambda + Va_2\lambda + 2a_2\lambda + a_1\mu - 2aa_0a_1\mu - 4aa_0a_2\lambda - 8bVa_2\lambda^3 - 2aa_0a_1\mu Va_1\mu \quad (22)$$

$$\left(\frac{1}{G'}\right)^3: 2a_2\mu + Va_2\mu - 12bVa_1\lambda\mu^2 - 6aa_0a_2\lambda - 2aa_1^2\mu - 4aa_0a_2\mu - 38bVa_2\lambda^2\mu \quad (23)$$

$$\left(\frac{1}{G'}\right)^4: -6aa_1a_2\mu - 4aa_2^2 - 6bVa_1\mu^3 - 54bVa_2\lambda\mu^2 \quad (24)$$

$$\left(\frac{1}{G'}\right)^5: -4aa_2^2\mu - 24bVa_2\mu^3 \quad (25)$$

Solving the algebraic equations by the Maple or Mathematica , we get the following results.

$$\{ V = V, a_0 = \frac{V+1-bV\lambda^2}{2a}, a_1 = -\frac{bV\lambda\mu}{a}, a_2 = -\frac{bV\lambda\mu^2}{a} \}. \quad (26)$$

By substituting (26) into (20) using (6) we obtain

$$u(\xi) = \frac{V+1-bV\lambda^2}{2a} - \frac{\frac{bV\lambda^2\mu}{a}}{-\mu-c_1\lambda^2[\cosh(\xi\lambda)-\sinh(\xi\lambda)]} - \frac{\frac{bV\lambda\mu^2}{a} \left[ \frac{\lambda}{-\mu-c_1\lambda^2[\cosh(\xi\lambda)-\sinh(\xi\lambda)]^2} \right]^2} \quad (27)$$

where  $\xi = x + Vt$ .

#### Classical Drinfel'd-Sokolov-Wilson System:

$$u_t + pvv_x = 0, \quad (28)$$

$$v_t + qv_{xxx} + ruv_x + su_xv = 0, \quad (29)$$

where  $p$ ,  $q$ ,  $r$  and  $s$  are arbitrary constants, to this end , we see that the traveling wave variable

$u(x, t) = u(\xi)$ ,  $\xi = x - ct$  permits us To convert (28) and (29) into the following ODE

$$-cu' + pvv' = 0, \quad (30)$$

$$-cv' + qv''' + ruv' + su'v = 0, \quad (31)$$

According to step3, we get  $M=2$  for  $u$  and  $N=1$  for  $v$ , we assume that equations (30) and (31) have the following formal solutions

$$u(\xi) = a_0 + a_1 \left(\frac{1}{G'}\right)^1 + a_2 \left(\frac{1}{G'}\right)^2 \quad (32)$$

$$v(\xi) = b_0 + b_1 \left(\frac{1}{G'}\right)^1 \quad (33)$$

where  $a_0, a_1, a_2, b_0$  and  $b_1$  are constants.

Substituting (32) and (33) into (30) and (31) using (5), the left hand side of (30) and (31) becomes a polynomial in  $(1/G')$ . Setting the coefficients of this polynomial to zero yields a system of algebraic equations in  $a_0, a_1, a_2, b_0, b_1$  and  $C$  which can be solved by Maple to find following results:

$$\left. \begin{aligned} a_0 &= \frac{-1}{12} \left[ \frac{12q^2\mu^2\lambda^2(r-s) + b_1^2(r+2s)^2}{(r+2s)q\mu^2r} \right], & a_1 &= -\frac{6\lambda\mu q}{(r+2s)}, & a_2 &= \\ & -\frac{6\mu^2 q}{(r+2s)}, & b_0 &= \frac{\lambda b_1}{2\mu}, & b_1 &= b_1, & c &= -\frac{(r+2s)pb_1^2}{12q\mu^2} \end{aligned} \right\} \quad (34)$$

By substituting (34) into (32) and (33) using (6) we obtain

$$u(\xi) = \frac{-1}{12} \left[ \frac{12q^2\mu^2\lambda^2(r-s) + b_1^2(r+2s)^2}{(r+2s)q\mu^2r} \right] - \frac{\frac{6\lambda^2\mu q}{(r+2s)}}{-\mu - c_1\lambda^2[\cosh(\xi\lambda) - \sinh(\xi\lambda)]} - \frac{6\lambda^2\mu^2 q}{(r+2s)} \left[ \frac{\lambda}{-\mu - c_1\lambda^2[\cosh(\xi\lambda) - \sinh(\xi\lambda)]^2} \right]^2 \quad (35)$$

$$v(\xi) = \frac{\lambda b_1}{2\mu} + \frac{b_1\lambda}{-\mu + c_1\lambda[\cosh(\xi\lambda) - \sinh(\xi\lambda)]}, \quad (36)$$

where  $\xi = x + \frac{(r+2s)pb_1^2}{12q\mu^2} t$

## CONCLUSIONS

In this work, we have used the  $(1/G')$ -expansion method to derive new exact solutions of the Nonlinear potential Boussinesq and ZKBBM Equations.

We show that the solutions we found in this article are different from the solutions presented by other authors in recent papers. We foresee that our results can be found potentially useful for applications in mathematical physics and engineering. All solutions in this paper have been found by aid of Maple packet program. Thus, we conclude that the proposed method can be extended to solve the nonlinear problems which arise in the theory of solitons and other areas.

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