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Application of Projected Differential Transform Method on Nonlinear Partial Differential Equations with Proportional Delay in One Variable

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Abstract: In this paper we solve nonlinear partial differential equations with proportional delay in the variable t using the projected differential transform method. The purpose of the method is obtained analytical or approximate solutions of some nonlinear partial differential equations with proportional delay. This method is more efficient and easy to handle such partial differential equations with proportional delay in comparison to other methods. The result showed the efficiency, accuracy and validation of the projected differential transform method.

Keywords: Projected Differential Transform Method · Nonlinear Partial Differential Equations with Proportional Delay · Exact Solutions.

INTRODUCTION

Several numerical and analytical techniques including the Adomian's decomposition method [1], differential transform method [2-9], homotopy perturbation method [10-13] and projected differential transform method [14,15], have been developed for solving nonlinear delay differential equations

Delay partial differential equations arise from various applications, like biology, medicine, control theory, climate models and many others. Their independent variables are time t and one or more dimensional variable x, which usually represents position in space but may also represent relative DNA content, size of cells, or their maturation level, or other values. The solutions (dependent variables) of delay partial differential equations may represent temperature, voltage, or concentrations or densities of various particles, for example cells, bacteria, chemicals, animals and so on.

Delay partial differential equations depend on their solutions at retarded variables or/and they depend on smooth averages of the solutions over some retarded intervals. Although delay partial differential equations are also called partial functional differential equations, the class of functional equations is wider than the class of delay equations.

In this work we will present the projected differential transform method and some theorems with their proofs to solve some nonlinear partial differential equations with proportional delay in the variable t.

In this paper we consider the following nonlinear partial differential equations with proportional delay in the variable t.

$$f\left(\begin{array}{l} t,x,\dfrac{\partial}{\partial x}u(x,t),......,\dfrac{\partial^m}{\partial x^m}u(x,t),u\left(x,\dfrac{t}{a_0}\right) \\ ,\dfrac{\partial}{\partial t}u\left(x,\dfrac{t}{a_1}\right),......,\dfrac{\partial^n}{\partial t^n}u\left(x,\dfrac{t}{a_n}\right) \end{array}\right)=0 \qquad (1)$$

$$t,x\geq 0\ ,\ a_0,a_1,......,a_n\ ,\ m,n\in N$$

Projected Differential Transform Methods: In this section we introduce the projected differential transform method [14, 15] which is modified method of the differential transform method.

Definition: The basic definition of projected differential transform method of function $u(x_1,x_2,\cdots\cdots,x_n)$ is defined as

$$U(x_1,x_2,\cdots\cdots,x_{n-1},k)=\frac{1}{k!}\left[\frac{\partial^k u(x_1,x_2,\cdots\cdots,x_n)}{\partial x_n^k}\right]_{x_n=0} \qquad (2)$$

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Such that $u(x_1, x_2, \cdots, x_n)$ is the original function and $U(x_1, x_2, \cdots, x_{n-1}, k)$ is projected transform function.

And the differential inverse transform of $U(x_1, x_2, \cdots, x_{n-1}, k)$ is defined as

$$u(x_1, x_2, \cdots, x_n) = \sum_{k=o}^{\infty} U(x_1, x_2, \cdots, x_{n-1}, k)(x - x_0)^k \qquad (3)$$

The fundamental theorems of the projected differential transform are

Theorems 1:

(1) If $z(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n) \pm v(x_1, x_2, \ldots, x_n)$
Then $z(x_1, x_2, \ldots, x_{n-1}, k) = u(x_1, x_2, \ldots, x_{n-1}, k) \pm v(x_1, x_2, \ldots, x_{n-1}, k)$

(2) If $z(x_1, x_2, \ldots, x_n) = cu(x_1, x_2, \ldots, x_n)$
Then $z(x_1, x_2, \ldots, x_{n-1}, k) = cu(x_1, x_2, \ldots, x_{n-1}, k)$

(3) If $z(x_1, x_2, \ldots, x_n) = \dfrac{du(x_1, x_2, \ldots, x_n)}{dx_n}$
Then $z(x_1, x_2, \ldots, x_{n-1}, k) = (k+1)u(x_1, x_2, \ldots, x_{n-1}, k+1)$

(4) If $z(x_1, x_2, \ldots, x_n) = \dfrac{d^n u(x_1, x_2, \ldots, x_n)}{dx_n^n}$
Then $z(x_1, x_2, \ldots, x_{n-1}, k) = \dfrac{(k+n)!}{k!} u(x_1, x_2, \ldots, x_{n-1}, k+n)$

(5) If $z(x_1, x_2, \ldots, x_n) = u(x_1, x_2, \ldots, x_n)\, v(x_1, x_2, \ldots, x_n)$
Then $z(x_1, x_2, \ldots, x_{n-1}, k) =$
$\sum_{m=0}^{k} u(x_1, x_2, \ldots, x_{n-1}, m)\, v(x_1, x_2, \ldots, x_{n-1}, k-m)$

(6) If $z(x_1, x_2, \ldots, x_n) = u_1(x_1, x_2, \ldots, x_n)$
$u_2(x_1, x_2, \ldots, x_n) \ldots u_n(x_1, x_2, \ldots, x_n)$ Then
$z(x_1, x_2, \ldots, x_{n-1}, k) =$
$\sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} u_1(x_1, x_2, \ldots, x_{n-1}, k_1)$
$u_2(x_1, x_2, \ldots, x_{n-1}, k_2 - k_1)$
$\times \ldots u_{n-1}(x_1, x_2, \ldots, x_{n-1}, k_{n-1} - k_{n-2})$
$u_n(x_1, x_2, \ldots, x_{n-1}, k - k_{n-1})$

Note that c is a constant and n is a nonnegative integer.

Application of Projected Differential Transform Method on Equation (1): In this section we introduce some theorems in projected differential transform method for approximating equation (1).

Theorem 2: Let $Z(x,k), U(x,k)$ a re the projected differential transform with respect to t of $z(x,t), u(x,t)$ and $a \in N$, then

(I) If $z(x,t) = u\left(x, \frac{t}{a}\right)$, then $Z(x,k) = \dfrac{U(x,k)}{a^k}$

(ii) If $z(x,t) = \dfrac{\partial^m}{\partial t^m} u\left(x, \frac{t}{a}\right)$, then

$$z(x,k) = \frac{(k+m)!}{k!} \frac{U(x,k+m)}{a^{k+m}}$$

Proof:

(I) From (2), we have

$$\frac{\partial^k}{\partial t^k} z(x,t) = \frac{\partial^k}{\partial t^k}\left[u\left(x, \frac{t}{a}\right)\right] = \frac{1}{a^k} \frac{\partial^k}{\partial t^k}\left[u\left(x, \frac{t}{a}\right)\right]$$

Therefore

$$\left[\frac{\partial^k z(x,t)}{\partial t^k}\right]_{t=0} = \frac{1}{a^k}\left[\frac{\partial^k}{\partial t^k} u\left(x, \frac{t}{a}\right)\right]_{t=0} = \frac{k!\,U(x,k)}{a^k}$$

Then by (2), we get

$$Z(x,k) = \frac{1}{k!}\left[\frac{\partial^k z(x,t)}{\partial t^k}\right]_{t=0} = \frac{1}{k!}\left[\frac{k!\,U(x,k)}{a^k}\right] = \frac{U(x,k)}{a^k}$$

Such that k = 0,1,2,....,∞

(ii) From (i), we have

$$\left[\frac{\partial^k z(x,t)}{\partial t^k}\right]_{t=0} = \frac{1}{a^{k+m}}\left[\frac{\partial^{k+m}}{\partial t^{k+m}} u\left(x, \frac{t}{a}\right)\right]_{t=0} = \frac{(k+m)!\,U(x,k+m)}{a^{k+m}}$$

Then

$$Z(x,k) = \frac{1}{k!}\left[\frac{\partial^k z(x,t)}{\partial t^k}\right]_{t=0}$$
$$= \frac{1}{k!}\left[\frac{(k+m)!\,U(x,k+m)}{a^{k+m}}\right] = \frac{(k+m)!\,U(x,k+m)}{k!\,a^{k+m}}$$

Theorem 3: Let $Z(x,k), U(x,k), V(x,k)$ are the projected differential transform with respect to *t* of $z(x,t), u(x,t), v(x,t)$ respectively and *a, b,* $\in N$, then

If $z(x,t) = u\left(x,\frac{t}{a}\right)v\left(x,\frac{t}{b}\right)$, then

$$Z(x,k) = \sum_{r=0}^{k} \frac{U(x,r)V(x,k-r)}{a^r b^{k-r}} \text{ , } k = 0,1,2,.....,\infty$$

Proof: From theorem (2) and Leibnitz, we get

$$\frac{\partial^k}{\partial t^k} z(x,t) = \frac{\partial^k}{\partial t^k}\left[u\left(x,\frac{t}{a}\right)v\left(x,\frac{t}{b}\right)\right]$$

$$= \sum_{r=0}^{k} \binom{k}{r} \frac{1}{a^r}\frac{\partial^k}{\partial t^k}\left[u\left(x,\frac{t}{a}\right)\right]\frac{1}{b^{k-r}}\frac{\partial^{k-r}}{\partial x^{k-r}}\left[v\left(x,\frac{t}{a}\right)\right]$$

Therefore

$$\left[\frac{\partial^k z(x,t)}{\partial t^k}\right]_{t=0} = \sum_{r=0}^{k}\binom{k}{r}\left[\frac{r!U(x,r)}{a^r}\right]\left[\frac{(k-r)!V(x,k-r)}{b^{k-r}}\right] = \sum_{r=0}^{k}\frac{k!U(x,r)V(x,k-r)}{a^r\, b^{k-r}}$$

And from equation (2), we get

$$Z(x,k) = \sum_{r=0}^{k} \frac{U(x,r)V(x,k-r)}{a^r b^{k-r}} \text{ , } k = 0,1,2,.....,\infty$$

Theorem 4: Let $Z(x,k), U(x,k), V(x,k)$ are the projected differential transform with respect to *t* of $z(x,t), u(x,t), v(x,t)$ respectively and $a,b,n,m \in N$, then

If $z(x,t) = \frac{\partial^n}{\partial t^n}u\left(x,\frac{t}{a}\right)\frac{\partial^m}{\partial x^m}v\left(x,\frac{t}{a}\right)$, then

$$Z(k,x) = \sum_{r=0}^{k} \frac{(r+n)!(k-r+m)!\,U(x,r+n)V(x,k-r+m)}{r!(k-r)!a^{r+n}b^{k-r+m}} \text{ , } k = 0,1,2,.....,\infty$$

Proof: From theorem (7) and Leibnitz, we get

$$\frac{\partial^k}{\partial t^k} z(x,t) = \frac{\partial^k}{\partial t^k}\left[\frac{\partial^n}{\partial t^n}u\left(x,\frac{t}{a}\right)\frac{\partial^m}{\partial t^m}v\left(x,\frac{t}{a}\right)\right]$$

$$= \sum_{r=0}^{k}\binom{k}{r}\frac{1}{a^{r+n}}\frac{\partial^{r+n}}{\partial t^{r+n}}\left[u\left(x,\frac{t}{a}\right)\right]\frac{1}{b^{k-r+m}}\frac{\partial^{k-r+m}}{\partial t^{k-r+m}}\left[v\left(x,\frac{t}{a}\right)\right]$$

Therefore

$$\left[\frac{\partial^k z(x,t)}{\partial t^k}\right]_{t=0} = \sum_{r=0}^{k}\binom{k}{r}\left[\frac{(r+n)!U(x,r+n)}{a^{r+n}}\right]\left[\frac{(k-r+m)!V(x,k-r+m)}{b^{k-r+m}}\right]$$

$$= \sum_{r=0}^{k}\frac{k!(r+n)!(k-r+m)!U(x,r+n)V(x,k-r+m)}{r!(k-r)!a^{r+n}\,b^{k-r+m}}$$

And from equation (2), we get

$$Z(k,x) = \sum_{r=0}^{k} \frac{(r+n)!(k-r+m)!\,U(x,r+n)V(x,k-r+m)}{r!(k-r)!a^{r+n}b^{k-r+m}} \text{ , } k = 0,1,2,.....,\infty$$

Applications: In this section we illustrate some examples to explain the presented method. We choose examples have exact solutions.

Example 1: In the first example, consider the following nonlinear first-order partial differential equation with proportional delay in the variable *t*.

$$2u\left(x,\frac{t}{2}\right).\frac{\partial}{\partial t}u\left(x,\frac{t}{2}\right) = xu(x,t) \text{ ,} x,t \geq 0,\; u(x,0) = x \tag{4}$$

Using the projected differential transform method with respect to *t* of Eq. (4), yields

$$2\sum_{r=0}^{k}\frac{(k-r+1)U(x,r)U(x,k-r+1)}{2^r.2^{k-r+1}} = xU(x,k) \tag{5}$$

The projected differential transform method with respect to *t* of initial condition is $U(x,0) = x$, then by this equation and Eq. (5), we have

$$U(x,1) = x \text{ , } U(x,2) = \frac{x}{2} \text{ , } U(x,3) = \frac{x}{6},$$
$$U(x,4) = \frac{x}{24} \text{ ,........,} U(x,n) = \frac{x}{n!}$$

Substituting in Eq.(3), to get the exact solution of Eq. (4), in the form

$$u(x,t) = \sum_{k=0}^{\infty}\frac{x}{k!}t^k = xe^t$$

Example 2: Consider the following nonlinear second-order partial differential equation with proportional delay in the variable *t*.

$$\frac{\partial^2 u(x,t)}{\partial t^2} - 4\frac{\partial}{\partial x}u\left(x,\frac{t}{2}\right).\frac{\partial}{\partial t}u\left(x,\frac{t}{2}\right) + u(x,t) = 0,\ x,t \geq 0 \quad (6)$$

With the initial conditions

$$u(x,0) = x,\ \frac{\partial u(x,0)}{\partial t} = x \quad (7)$$

Taking the projected differential transform method with respect to t of Eq. (6), we have

$$(k+2)(k+1)U(x,k+2) + U(x,k)$$
$$-4\sum_{r=0}^{k}\frac{\frac{\partial}{\partial x}U(x,r)(k-r+1)U(x,k-r+1)}{2^r.2^{k-r+1}} = 0 \quad (8)$$

And the projected differential transform with respect to t of initial conditions $u(x,0) = x, \frac{\partial u(x,0)}{\partial t} = x$ will be

$$U(x,0) = U(x,1) = x \quad (9)$$

Where $U(x,k)$ is the projected differential transform method with respect to t of $U(x,t)$

Using Eqs. (8) and (9), we have

$$U(x,2) = \frac{x}{2},\ U(x,3) = \frac{x}{6},\ U(x,4) = \frac{x}{24},\ ...., U(x,n) = \frac{x}{n!}$$

Substituting in Eq.(3), to get the exact solution of Eq. (6), in the form

$$u(x,t) = \sum_{k=0}^{\infty}\frac{x}{k!}t^k = xe^t$$

Example 3: Consider the following nonlinear second-order partial differential equation with proportional delay in the variable t.

$$16\frac{\partial}{\partial t}u\left(x,\frac{t}{2}\right).\frac{\partial^2}{\partial t^2}u\left(x,\frac{t}{2}\right) + \cos x\frac{\partial^2 u(x,t)}{\partial x^2} = 0,\ x,t \geq 0 \quad (10)$$

With the conditions

$$u(x,0) = 0,\ \frac{\partial u(x,0)}{\partial t} = \cos x \quad (11)$$

Taking the projected differential transform with respect to t of Eq. (10), we have

$$16\sum_{r=0}^{k}\frac{(r+1)(k-r+2)(k-r+1)U(x,r+1)U(x,k-r+2)}{2^{r+1}.2^{k-r+2}} + \cos x\frac{\partial^2 U(x,k)}{\partial x^2} = 0 \quad (12)$$

And the projected differential transform with respect to t of initial condition $u(x,0) = 0, \frac{\partial u(x,0)}{\partial t} = \cos x$ will be

$$U(x,0) = 0,\ U(x,1) = \cos x \quad (13)$$

Using Eqs. (12) and (13), we have

$$U(x,2) = 0,\ U(x,3) = \frac{\cos x}{3!}, U(x,4) = 0, U(x,5) = \frac{\cos x}{5!}, ...$$

Substituting in Eq.(3), to get the exact solution of Eq. (10), in the form

$$u(x,t) = \sinh t \cos x$$

CONCLUSION

In this paper we applied the projected differential transformation technique to solve nonlinear partial differential equation with proportional delay in the variable t. The obtained results show that this method is powerful and meaningful for solving partial differential equations with proportional delays. In fact, PDTM is very efficient methods to find the numerical and analytic solutions of partial differential-difference equations, delay partial differential equations as well as integral equations.

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