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Bayesian Estimation of Change-Point in Unobserved-ARCH Models

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Abstract: Change-point problem deals with sudden change in the distribution of a set of given data. Change in financial time series is a common event, because many factors for example some news, etc. may affect the series and cause change. In this work, we intend to detect the time of change-point, using Bayesian methods in Unobserved-ARCH models. We estimate the model and the time of the change-point.

Key words: Bayesian change-point . Gibbs and Metropolis-Hastings algorithms . Monte Carlo Markov chain . Unobserved-ARCH Models

INTRODUCTION

We focus on a member of a class of models, introduced by Harvey *et al*. [1] viz the Unobserved ARCH model, first explicitly presented by Shephard [2]. In this model, the ARCH component is observed with error, or it may be seen as a latent process. Fiorentini *et al*. [3] extensively discuss the need to study models in which an ARCH process is used as a latent process. In economic applications, for instance, common sense suggests that the behavior of economic agents may face abrupt changes under the effect of economic policy, political events, etc. We consider a uniform prior for the "change-point time" and some improper priors for other parameters. We estimate the model and time of change-point, as well. Normally, our inference is based on posterior distribution of parameters. As the posterior distribution is complex and not tractable MCMC methods, particulary Gibbs and Metropolis-Hastings algorithms, are used.

The remainder of the paper is organized as follows. In Section 2, we present the Unobserved ARCH model and some of its theoretical and the change-point time problem. In Section 3, we propose a change-point problem in Unobserved-ARCH model and a method to capture the time of the change-point. In Section 4, we illustrate this algorithm with a simulation and a real data set. In Section 5, we give some concluding remarks.

THE UNOBSERVED-ARCH MODEL

The Unobserved ARCH model has been presented by Shephard [2]. The ARCH components in this model are observed with errors. Using the following hierarchical structure of the conditional densities, the model can be written as:

$$\begin{aligned} &(y_t \mid x_t, \sigma^2) \sim N(x_t, \sigma^2); \\ &(x_t \mid x_{t-1}, \alpha, \beta, x_0) \sim N(0, h_t); \\ &h_t = \alpha + \beta x_{t-1}^2; \end{aligned} \tag{1}$$

Where $y_1, y_2, \ldots y_T$ is a realization of the process under study, x_t is the Unobserved ARCH component at time t, x_0 is the initial state or the "history" of the unobserved components and $N(\cdot,\cdot)$ is the Normal distribution. To obtain $h_t>0$, the parameters α and β are resstricted to be positive. The additional restriction $0<\beta\le 1$ is imposed so that the ARCH component of the model is covariance stationary.

Giakoumatos *et al*. [4] introduced Bayesian approaches for estimating the model. In this paper, we used their method to sample from the joint posterior distribution of all the unknown parameters and the latent variables. The Unobserved-ARCH models, considering a single change-point, can be generalized in the following way:

$$y_t = x_t + \sigma\varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$
$$x_t = \sqrt{\alpha_t + \beta_t x_{t-1}^2}\; u_t, \quad u_t \sim N(0,1)$$

$$(\alpha_t, \beta_t) = \begin{cases} (\alpha_1, \beta_1), & t < \tau \\ (\alpha_2, \beta_2), & t \ge \tau \end{cases} \tag{2}$$

We consider a change-point in the hidden process. In this work, also we consider a model with a change-point in intercept and slope of the hidden process.

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BAYESIAN CHANGE-POINT TIME

The posterior density of the parameters of the Unobserved-ARCH model with a change-point time can be extracted via Bayes theorem, by

$$[\alpha_1,\alpha_2,\beta_1,\beta_2,\sigma^2,x_0,x\mid y]\propto\prod_{t=1}^{\tau}([y_t\mid x_t,\sigma^2][x_t\mid x_{t-1},\alpha_1,\beta_1])$$
$$\times\prod_{t=\tau+1}^{\tau}([y_t\mid x_t,\sigma^2][x_t\mid x_{t-1},\alpha_2,\beta_2])\times[\alpha_1,\alpha_2,\beta_1,\beta_2,\sigma^2,\tau,x_0]$$

(Throughout the paper the usual square bracket notation is used for joint, conditional and marginal densities.) The first four terms in the above product are derived from the hierarchical structure in (1) and the last term, $[\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma^2, \tau, x_0]$, is the joint prior density of α_1, α_2, β_1, β_2, σ^2, τ and x_0. These parameters are assumed to be a priori independent. Improper priors are used for α_1, α_2, β_1, β_2, σ^2 and discrete uniform prior for τ and a vague Normal density N(0,v) for x_0, so that the joint prior density takes the form $[\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma^2, \tau, x_0]\propto(a_1a_2\sigma^2)^{-1}\exp\{-0.5x_0^2/v\}$. The same transformation given in [4] for Bayesian inference for parameters are used,

$$g_1=\sqrt{\alpha_1/\beta_1},\quad g_2=\sqrt{\alpha_2/\beta_2},$$
$$w_{t,1}=\sqrt{\beta_1/\alpha_1}\,x_t,\quad t=1,2,\ldots,\tau,$$
$$w_{t,2}=\sqrt{\beta_2/\alpha_2}\,x_t,\quad t=\tau+1,\ldots,T$$

Therefore, the posterior density is given by

$$[g_1,g_2,\sigma^2,\tau,w_1,w_2\mid y]\propto\frac{1}{\prod_{t=1}^{\tau}\sqrt{1+w_{t-1,1}^2}}\exp\{-\frac{1}{2\beta_1}\sum_{t=1}^{\tau}\frac{w_{t,1}^2}{1+w_{t-1,1}^2}\}$$
$$\times\frac{1}{\sigma^{2\frac{T+2}{2}}\beta_1^{\tau/2}\beta_2^{\frac{T-\tau}{2}}}\exp\{-\frac{1}{2}(\frac{1}{\sigma^2}\sum_{t=1}^{\tau}(y_t-g_1w_{t,1})^2+\frac{g_1w_0^2}{\nu})\}$$
$$\times\frac{1}{\prod_{t=\tau+1}^{T}\sqrt{1+w_{t-1,2}^2}}\exp\{-\frac{1}{2\beta_2}\sum_{t=\tau+1}^{T}\frac{w_{t,2}^2}{1+w_{t-1,2}^2}\}$$
$$\times\exp\{-\frac{1}{2}(\frac{1}{\sigma^2}\sum_{t=\tau+1}^{T}(y_t-g_2w_{t,2})^2)\},$$

where g_1, $g_2>0$, $0<\beta_1$, $\beta_2\le 1$. Regarding the joint posterior density of the parameters above, the full conditional distribution is given by:

$$[\sigma^2\mid\cdot]\equiv IG\left(\frac{T}{2},\frac{1}{2}\left[\begin{array}{l}\sum_{t=1}^{\tau}(y_t-g_1w_{t,1})^2+\\ \sum_{t=\tau+1}^{T}(y_t-g_2w_{t,2})^2\end{array}\right]\right)$$

where IG(a, b) denotes the Inverse Gamma density with mean b/(a-1); the notation |. implies conditioning on all the remaining parameters.

$$[\beta_1\mid\cdot]\equiv IG(\frac{\tau}{2}-1,\frac{1}{2}\sum_{t=1}^{\tau}\frac{w_{t,1}^2}{1+w_{t-1,1}^2})I_{(\beta_1\le 1)}$$

$$[\beta_2\mid\cdot]\equiv IG(\frac{T-\tau}{2}-1,\frac{1}{2}\sum_{t=\tau+1}^{T}\frac{w_{t,2}^2}{1+w_{t-1,2}^2})I_{(\beta_2\le 1)}$$

Where $I(\cdot)$ is the indicator function.

$$[g_1\mid\cdot]\equiv N(m_1,s_1)I_{(g_1\ge 0)}$$

Where,

$$m_1=\frac{(\nu\sum_{t=1}^{\tau}w_{t,1}y_t)}{(\sigma^2w_0^2+\nu\sum_{t=1}^{\tau}w_{t,1}^2)}$$

and

$$s_1=\frac{\sigma^2\nu}{(\sigma^2w_0^2+\nu\sum_{t=1}^{\tau}w_{t,1}^2)}$$

$$[g_2\mid\cdot]\equiv N(m_2,s_2)I_{(g_2\ge 0)}$$

Where,

$$m_2=\frac{(\nu\sum_{t=\tau+1}^{T}w_{t,2}y_t)}{\sum_{t=\tau+1}^{T}w_{t,2}^2}$$

and

$$s_2=\frac{\sigma^2}{\sum_{t=\tau+1}^{T}w_{t,2}^2}$$

$$[w_0\mid\cdot]\propto N(0,\frac{\nu}{g_1^2})\frac{1}{\sqrt{1+w_0^2}}\exp\{-\frac{1}{2\beta_1}\frac{w_{1,1}^2}{(1+w_0^2)}\}$$

$$[w_{t,1}\mid\cdot]\propto N(m_{t,1},s_{t,1}^2)\frac{1}{\sqrt{1+w_{t,1}^2}}\exp\{-\frac{1}{2\beta_1}\frac{w_{t+1,1}^2}{(1+w_{t,1}^2)}\},$$

For t = 1, 2,…, τ, where $m_{t,1}$ and $s_{t,1}^2$ are given by

$$m_{t,1}=\frac{y_tg_1\beta_1(1+w_{t-1,1}^2)}{g_1^2\beta_1(1+w_{t-1,1}^2)+\sigma^2}$$

$$s_{t,1}^2=\frac{\sigma^2\beta_1(1+w_{t-1,1}^2)}{g_1^2\beta_1(1+w_{t-1,1}^2)+\sigma^2}$$

$$[w_{t,2}\mid\cdot]\propto N(m_{t,2},s_{t,2}^2)\frac{1}{\sqrt{1+w_{t,2}^2}}\exp\{-\frac{1}{2\beta_2}\frac{w_{t+1,2}^2}{(1+w_{t,2}^2)}\},$$

For t = τ+1,2,…,T, where $m_{t,2}$ and $s_{t,2}^2$ are given by:

$$m_{t,2} = \frac{y_t g_2 \beta_2 (1 + w^2_{t-1,2})}{g_2^2 \beta_1 (1 + w^2_{t-1,2}) + \sigma^2}$$

$$s^2_{t,1} = \frac{\sigma^2 \beta_2 (1 + w^2_{t-1,2})}{g_2^2 \beta_2 (1 + w^2_{t-1,2}) + \sigma^2}$$

$$[w_T \mid \cdot] \equiv N(m_T, s_T^2)$$

To sample from w_0, $w_{t,1}$ and $w_{t,2}$ we use Metropolis-Hastings algorithm [5,6]. The full conditional distribution for the "change-point time", τ, is:

$$[\tau \mid \cdot] \propto \frac{1}{\prod_{t=1}^{\tau} \sqrt{1 + w^2_{t-1,1}}} \exp\{-\frac{1}{2\beta_1} \sum_{t=1}^{\tau} \frac{w^2_{t,1}}{1 + w^2_{t-1,1}}\}$$
$$\times \frac{1}{\beta_1^{\tau/2} \beta_2^{\frac{T-\tau}{2}}} \exp\{-\frac{1}{2}(\frac{1}{\sigma^2}[\sum_{t=1}^{\tau}(y_t - g_1 w_{t,1})^2 + \sum_{t=\tau+1}^{T}(y_t - g_2 w_{t,2})^2])\}$$
$$\times \frac{1}{\prod_{t=\tau+1}^{T} \sqrt{1 + w^2_{t-1,2}}} \exp\{-\frac{1}{2\beta_2} \sum_{t=\tau+1}^{T} (\frac{w^2_{t,2}}{1 + w^2_{t-1,2}})\}.$$

For details the MCMC methods see [7].

APPLICATION

In this section, we demonstrate how our method is used in practice. We simulate y_1, y_2,…., y_{1000} from the following model of the form:

$$y_t = x_t + 0.1\varepsilon_t, \qquad \varepsilon_t \sim N(0,1)$$
$$x_t = \sqrt{\alpha_t + \beta_t x^2_{t-1}}\ u_t \qquad u_t \sim N(0,1)$$

$$(\alpha_t, \beta_t) = \begin{cases} (0.004, 0.80), & t < 400 \\ (0.008, 0.99), & t \geq 400 \end{cases}$$

We ran the algorithm for N = 20000 iterations. Figure 1 gives the approximated posterior density for the parameters, which are obtained from simulated values of full conditionals. Regarding Fig. 1, the histogram of τ, or the approximate posterior density of change-point, it is clear that almost all probability mass are distributed from t = 390 to 410 Depending on the loss function, the estimate of "change-point time" can be computed.

Relative to the squared loss function, the Bayesian estimates of the parameters are given in

Now, we illustrate our proposed methodology using T = 730 the daily exchange rate of the Germany Marc (DEM) with respect to the Greek Drachma. To elaborate, let c_t be the exchange rate of a currency with respect to the Drachma on day t; then data series is given by:

$$y_t = \log(\frac{c_t}{c_t - 1}).100$$

that represents the daily relative (percentage) change of the exchange rate since

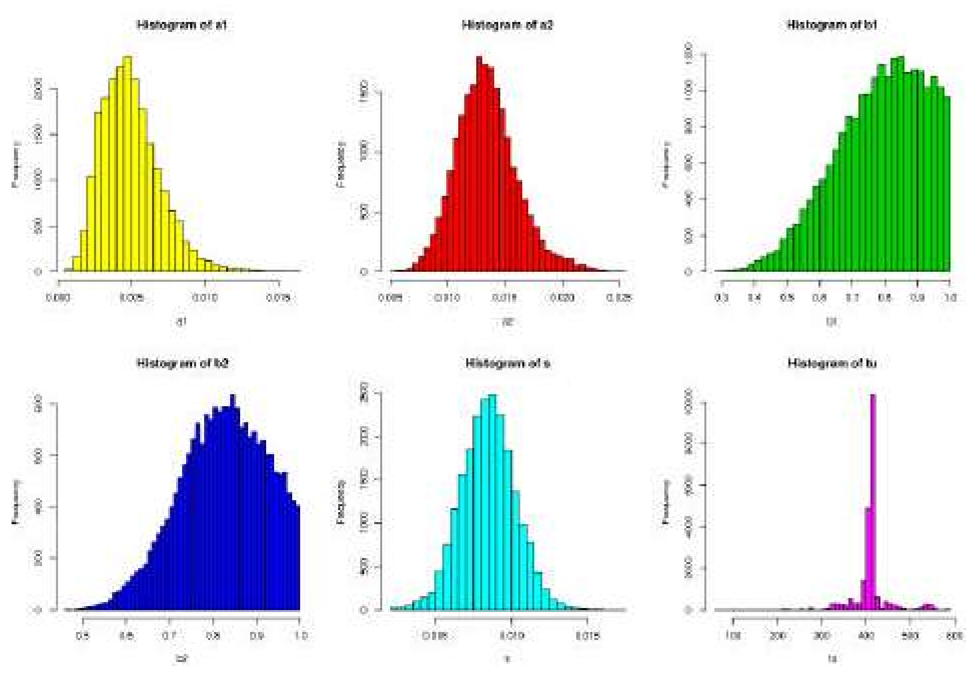

Fig. 1: Histograms of parameters, over 20000 iterations

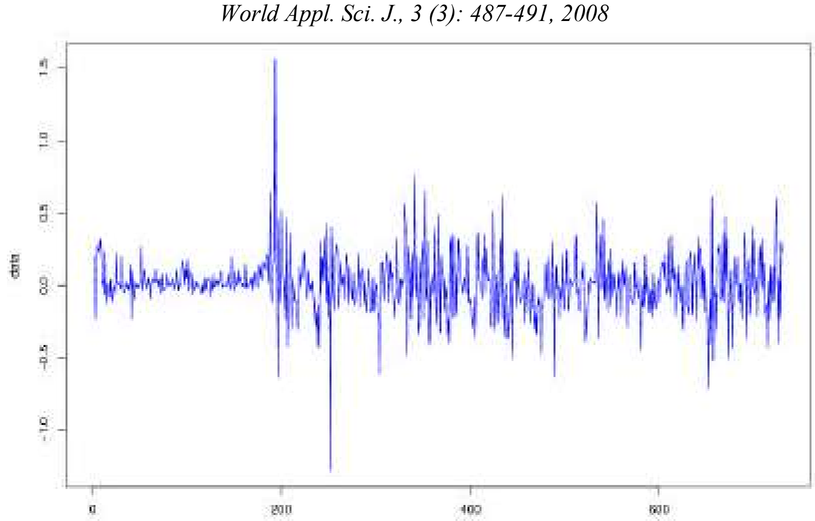

Fig. 2: DEM/Drachma exchange rates

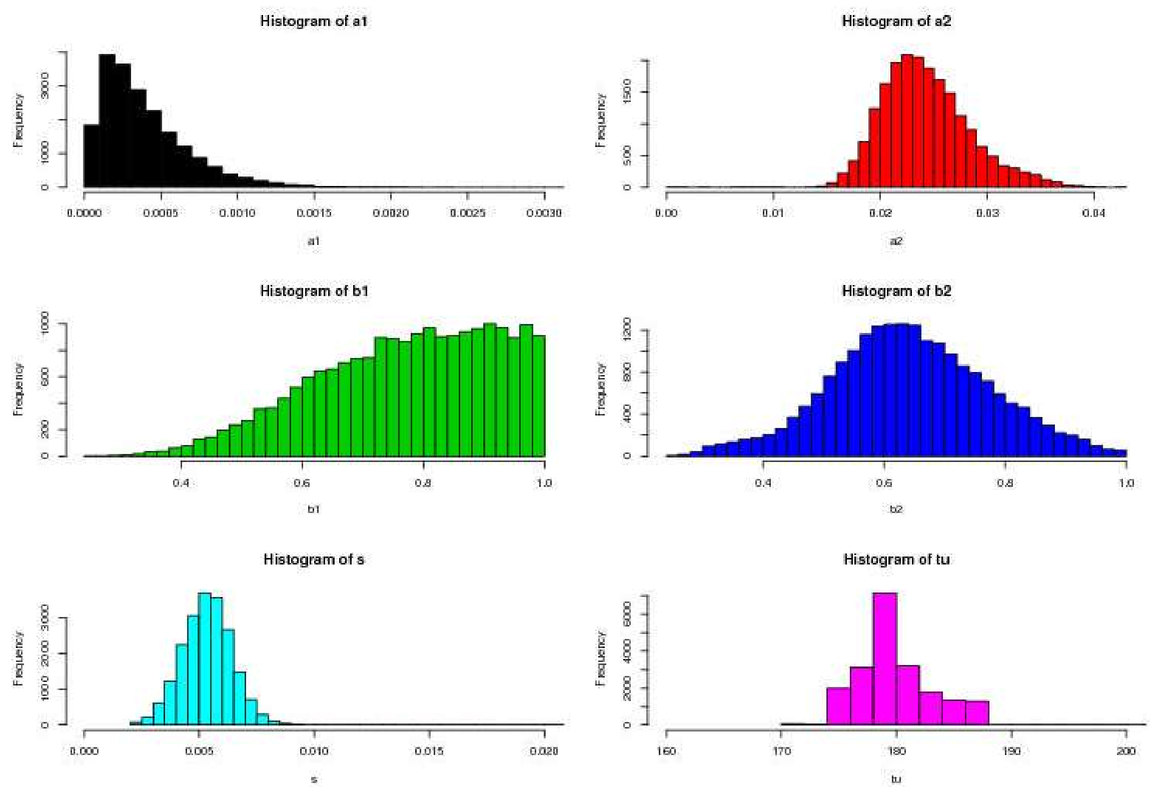

Fig. 3: Histograms of the posterior sample of the parameters, over 20000 iterations

$$\log(\frac{c_t}{c_{t-1}}) \approx (\frac{c_t}{c_{t-1}}) - 1 = \frac{(c_t - c_{t-1})}{c_{t-1}}$$

for

$$\frac{(c_t - c_{t-1})}{c_{t-1}} \approx 1$$

Our data set (Fig. 2) consists of 730 observations taken in the period 16/6/94--2/5/97. Using our proposed algorithm of last Section to capture the time of change and estimating the unknown parameters, we obtain a sample from the posterior density of the parameters.

We ran the algorithm for N = 20000 iterations to obtain samples from the marginal densities of the parameters of interest.

We present in Fig. 3 the histograms of the posterior sample of the parameters of the Unobserved-ARCH model with a change-point time. As seen in Fig. 3, the histogram of the posterior sample of the change-point, τ, almost all probability mass distributes from t = 170 to 190. Table 2, presents the Bayesian estimates of the parameters and especially change-point time relative to the squared loss function.

Table 1: The Bayes estimates of the parameters along with true values

Parameters	True value	Bayesian estimation
α_1	0.004	0.0049
α_2	0.008	0.0134
β_1	0.800	0.7857
β_2	0.990	0.8225
σ^2	0.010	0.0084
τ	400.000	411.4879

Table 2: The Bayes estimates of the parameters for the daily exchange rate of the Germany Mark (DEM) with respect to the Greek Drachma

Parameters	Bayesian estimation
α_1	0.00038
α_2	0.02424
β_1	0.77172
β_2	0.64112
σ^2	0.00536
τ	180.423

CONCLUDING REMARKS

Unobserved-ARCH models can describe some financial time series and change in these models is a common event. In this paper, we estimated the model and the time of change point. We assumed improper priors for unknown parameters and discrete uniform distribution for the change point.

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