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D-Bounded Sets in Generalized Probabilistic 2-Normed Spaces

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Abstract: In this paper, we defined bounded sets in generalized probabilistic 2-normed space and we studied the relationship among D-bounded sets and proved some result in this space.

Key words: Probabilistic 2-normed spaces, Triangle functions, D-boundedness, 2-normed spaces

1. INTRODUCTION

In [1] K. Menger introduced the notion of probabilistic metric spaces. The idea of Menger was to use distribution function instead of nonnegative real numbers as values of the metric. The concept of probabilistic normed spaces were introduced by Serstnev [2]. Then I. Golet defined generalized probabilistic 2-normed space (briefly GP-2-N space) [3]. New definition of probabilistic normed spaces was studied by Alsina, Schweizer and Sklar [4-6]. D-bounded sets in probabilistic normed spaces were studied by C. Sempi, B. Lafuerza-Guillen and A. Rodriguez-Lallena [7, 8]. The concept of probabilistic and Menger normed space may have very important applications in quantum particle physics particularly in connections with both string and ε^{∞} theory which were given and studied by El Naschie [9, 10]. A distribution function (briefly a d.f.) is a function F from the extended real line $\overline{R} = [-\infty, +\infty]$ into the unit interval $I = [0,1]$ that is nondecreasing and satisfies F $(-\infty) = 0$, F $(+\infty) = 0$. The set of all d.f.'s will be denoted by Δ and the subset of those d.f.'s such that F $(0) = 0$, will be denoted by Δ^+.$D^+ \subseteq \Delta^+$ is defined as follows:

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\},$$

Where $l^- f(x)$ denotes the left limit of the function f at the point x. By setting $F \le G$ whenever $F(x) \le G(x)$ for all x in R, the maximal element for Δ^+ in this order is the d.f. given by:

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \le 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A t-norm T is a two-place function $T: I \times I \to I$ which is associative, commutative, nondecreasing in each place and such that T(a,1)=a, for all $a \in [0,1]$. A triangle function is a binary operation on Δ^+, namely a function $\tau: \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, non-decreasing and which has ε_0 as unit. That is, for all $F, G, H \in \Delta^+$, we have

$$\tau(\tau(F,G),H) = \tau(F, \tau(G,H)),$$

$$\tau(F,G) = \tau(G,F),$$

$$F \le G \Rightarrow \tau(F,H) \le \tau(G,H),$$

$$\tau(F, \varepsilon_0) = F.$$

A 2-normed space is a pair (L, ||.,.||), where L is a linear space of a dimension greater than one and ||.,.|| is a real valued mapping on $L \times L$ such that the following conditions be satisfied:

(N1) $\|x, y\| = 0$ if and only if,

x and y are linearly dependent;

(N2) $\|x,y\| = \|y,x\|$

for all $x,y \in L$;

(N3) $\|\alpha.x,y\| = |\alpha|\,\|x,y\|$

when ever $x, y \in L$ and $\alpha \in R$;

(N4) $\|x + y,z\| \le \|x,z\| + \|y,z\|$

for all $x, y, z \in L$

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Definition 1.1: A probabilistic normed (briefly PN) space is a quadruple (V, ν, τ, τ^*), where V a real vector space, τ and τ^* are continuous triangle functions and ν is a mapping from V into Δ^+ such that, for all p, q in V, the following conditions hold:

$$(PN1)\ \nu_p = \varepsilon_0 \text{ if and only if}, p = \theta,$$

where θ is the null vector in V;

$$(PN2)\ \nu_{-p} = \nu_p, \text{for each } p \in V;$$

$$(PN3)\ \nu_{p+q} \geq \tau(\nu_p, \nu_q), \text{for all } p, q \in V;$$

$$(PN4)\ \nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p}),$$

for all α in [0,1]

Definition 1.2: Let L be a linear space of a dimension greater than one, τ a triangle function and let F be a mapping from L×L into D^+. If the following conditions are satisfied:

$$(P2-N1)\ F_{x,y} = \varepsilon_0$$

if x and y are linearly dependent;

$$(P2-N2)\ F_{x,y} \neq \varepsilon_0$$

if x and y are linearly independent;

$$(P2-N3) F_{x,y} = F_{y,x}$$

for every x, y in L;

$$(P2-N4)\ F_{\alpha x,y}(t) = F_{x,y}\left(\frac{t}{|\alpha|}\right),$$

for every t>0, $\alpha \neq 0$ and $x,y \in L$;

$$(P2\text{-}N5)\ F_{x+y,z} {}^3t(F_{x,z}, F_{y,z})$$

Whenever x, y, z $\in$L

Then F is called a probabilistic 2-norm on L and the triple (L, F, τ) is called a probabilistic 2-normed space.

Definition 1.3: Let L, M be two real linear spaces of dimension greater than one and let F be a function defined on the Cartesian product L×M into Δ^+ satisfying the following properties:

$$(GP2-N1)\ F_{\alpha x,y}(t) = F_{x,\alpha y}(t) = F_{x,y}\left(\frac{t}{|\alpha|}\right)$$

or every t>0, $\alpha \in R - \{0\}$

and $(x, y) \in L \times M$;

$$(GP2-N2)\ F_{x+y,z} \geq \tau(F_{x,z}, F_{y,z})$$

for every x,y $\in$ L and z$\in$M;

$$(GP2-N3)\ F_{x,y+z} \geq \tau(F_{x,y}, F_{x,z})$$

for every x $\in$ L and y, z$\in$M;

The function F is called a generalized probabilistic 2-norm on L×M and the triple (L×M, F, τ) is called a generalized probabilistic 2-normed space (briefly GP-2-N space).

2. MAIN RESULTS

Definition 2.1: Let a nonempty set A×B be in a GP-2-N space (L×M, F, τ), then its probabilistic radius $R_{A\times B}$ define by

$$R_{A\times B}(x) = \begin{cases} 1^- \varphi_{A\times B}(x), & \text{if } x \in [0, +\infty), \\ 1, & \text{if } x = +\infty \end{cases}$$

Where

$$\varphi_{A\times B}(x) := \inf\{F_{p,q}(x) : p \in A, q \in B\}.$$

Definition 2.2: A nonempty set A×B in a GP-2-N space (L×M, F, τ) is said to be:

- Certainly bounded, if $R_{A\times B}(x_0)=1$, for some $x_0 \in (0,+\infty)$,
- Perhaps bounded, if one has $R_{A\times B}(x)<1$, for every $x \in (0,+\infty)$ and $1^- R_{A\times B}(+\infty)=1$,
- perhaps unbounded, if $R_{A\times B}(x_0)>0$, for some $x_0 \in (0,+\infty)$ and $1^- R_{A\times B}(+\infty) \in (0,1)$,
- Certainly unbounded, if $1^- R_{A\times B}(+\infty) \in (0)$.

Moreover, A will be said to be D-bounded if either (a) or (b) holds.

Lemma 2.3: Let $(L\times M, F, \tau)$ be a GP-2-N space and $A\times B\subseteq L\times M$. Then $A\times B$ is a D-bounded set if and only if,

$$\lim_{x\to+\infty}\varphi_{(A\times B)}(x)=1.$$

Proof: If $(A\times B)$ is a D-bounded set then it is clear that

$$\lim_{x\to+\infty}\varphi_{(A\times B)}(x)=1\ 1$$

Conversely, if

$$\lim_{x\to+\infty}\varphi_{(A\times B)}(x)=1$$

Then we have

$$\forall\ \delta>0\quad \exists M\ >0;\quad \forall x_0>M\Rightarrow 1-\delta<\varphi_{(A\times B)}(x_0)\le 1$$

Therefore

$$\exists\ y\ ;\ x_0>y>M\Rightarrow 1-\delta<\varphi_{(A\times B)}(y)\le 1,$$

This implies that

$$1-\delta<\lim_{y\to x_0}\varphi_{(A\times B)}(y)\le 1,$$

This implies that

$$1-\delta<R_{(A\times B)}(x_0)\le 1,$$

Therefore

$$\lim_{x_0\to+\infty}R_{(A\times B)}(x_0)=1.$$

It shows that $(A\times B)$ is a D-bounded set.

Theorem 2.4: A set $(A\times B)$ in the GP-2-N space $(L\times M, F, \tau)$ is D-bounded if and only if, there exists a d.f. $G\in D^+$ such that $F_{a,b}\ge G$ for every $a\in A$, $b\in B$.

Proof: Let $G=\varphi_{(A\times B)}$, therefore $G(0)=0$ and by previous lemma, if $A\times B$ is a D-bounded set, then we have:

$$\lim_{x\to+\infty}G(x)=1$$

This shows that $G\in D^+$ and $G(x)=\varphi_{(A\times B)}(x)\le F_{a,b}(x)$, for every $a\in A$, $b\in B$. Conversely, if there is a d.f. G with the above properties, then $G(x)\le F_{a,b}(x)$ for every $a\in A$, $b\in B$. If $G(x)=F_{a,b}(x)$, for every $a\in A$, $b\in B$, it implies that $A\times B$ is D-bounded set. Suppose that $G(x)<F_{a,b}(x)$, therefor $G(x)\le\varphi_{(A\times B)}(x)\le F_{a,b}(x)$. By assumption $G\in D^+$ then:

$$\lim_{x\to+\infty}G(x)=1$$

This implies that:

$$\lim_{x\to+\infty}\varphi_{(A\times B)}(x)=1$$

So the set $A\times B$ is D-bounded.

Proposition 2.5: Let $(L\times M, F, \tau)$ be a GP-2-N space. If $|\alpha|\le|\beta|$, then $F_{\beta p,q}\le F_{\alpha p,q}$, for every $(p,q)\in L\times M$ and $\alpha,\beta\in R-\{0\}$.

Proof: By (GP2-N1), we have:

$$F_{\alpha p,q}(x)=F_{p,q}\left(\frac{x}{|\alpha|}\right)$$

And

$$F_{\beta p,q}(x)=F_{p,q}\left(\frac{x}{|\beta|}\right).$$

Hence $|\alpha|\le|\beta|$, this implies that

$$\frac{x}{|\beta|}\le\frac{x}{|\alpha|}$$

Therefore

$$F_{p,q}\left(\frac{x}{|\beta|}\right)\le F_{p,q}\left(\frac{x}{|\alpha|}\right)$$

Then

$$F_{\beta p,q}\le F_{\alpha p,q}.$$

If $A\times B$ is a D-bounded set then $\alpha A\times B$ need not be D-bounded set, but this will hold under suitable conditions, as is shown in the next theorem.

Theorem 2.6: Let $(L\times M, F, \tau)$ be a GP-2-N space and $A\times B$ be a D-bounded subset in $L\times M$. The set $\alpha A\times B:=\{(aa,b):a\in A,b\in B\}$ is also D-bounded for every fixed $\alpha\in R-\{0\}$ if D^+ is a closed set under τ, i.e. $\tau(D^+\times D^+)\subseteq D^+$.

Proof: Because of (GP2-N1), it suffices to consider the case $\alpha>0$. If $\alpha\in(0,1)$ then by proposition 2.5, we have:

$$F_{\alpha a,b}(x) = F_{a,b}\left(\frac{x}{|\alpha|}\right) \geq F_{a,b}(x)$$

But $F_{a,b}\geq R_{A\times B}$, therefore $F_{\alpha a,b}\geq R_{A\times B}$. This shows that $\alpha A\times B$ is a D-bounded set. If $\alpha=1$, then it is clear that $\alpha A\times B$ is a D-bounded set. If $\alpha>1$, let $k = [\alpha]+1$, then by proposition 2.5, we have $F_{\alpha a,b}\geq F_{ka,b}$. Now let:

$$G_\alpha = \tau^{k-1}(R_{A\times B}, R_{A\times B}, \ldots, R_{A\times B}),$$

One has by induction

$$F_{ka,b}(x) \geq \tau(F_{(k-1)a,b}F_{a,b})(x) \geq \tau(\tau(F_{(k-2)a,b}F_{a,b}), F_{a,b})(x) \geq \ldots \geq$$

$$\geq \tau^{k-1}(F_{a,b}, F_{a,b}, \ldots, F_{a,b})(x) \geq \tau^{k-1}(R_{A\times B}, R_{A\times B}, \ldots, R_{A\times B})(x)$$

And hence $F_{\alpha a,b}\geq G_\alpha$. Finally, one can say that $R_{\alpha A\times B}\geq G_\alpha$ and since $G_\alpha\in D^+$, then $\alpha A\times B$ is D-bounded.

Theorem 2.7: Let $(L\times M, F, \tau)$ be a GP-2-N space. Suppose $A\times B$ and $B\times C$ be two nonempty and D-bounded sets in $L\times M$. Then $(A+C)\times B$ is a D-bounded set if D^+ is a closed set under τ, i.e. $\tau(D^+\times D^+)\subseteq D^+$.

Proof: For every $(a,b)\in A\times B$ and $(c,b)\in C\times B$, we have $(a+c,b)\in(A+C)\times B$. Therefore

$$\begin{aligned} F_{a+c,b} &\geq \tau(F_{a,b}, F_{c,b}) \\ &\geq \tau(F_{a,b}, R_{C\times B}) \\ &\geq \tau(R_{A\times B}, R_{C\times B}) \end{aligned}$$

This implies that

$$R_{(A+C)\times B} \geq \tau(R_{A\times B}, R_{C\times B}).$$

Now by this fact that $\tau(D^+\times D^+)\subseteq D^+$ we have $\tau(R_{A\times B}, R_{C\times B})\subseteq D^+$ is in D^+. It means $1^- R_{(A\times C\times)B}(+\infty)=1$. This completes the proof.

Theorem 2.8: Let $(L\times M, F, \tau)$ be a GP-2-N space. Suppose $A\times B$ and $A\times D$ be two nonempty and D-bounded sets in $L\times M$. Then $A\times(B+D)$ is a D-bounded set if D^+ is a closed set under τ, i.e. $\tau(D^+\times D^+)\subseteq D^+$.

Corollary 2.9: If $A\times B$ and $C\times B$ be two nonempty and D-bounded sets in $L\times M$ and $\tau(D^+\times D^+)\subseteq D^+$, then $\alpha(A+C)\times B$ is a D-bounded set for every $\alpha\in R-\{0\}$.

Corollary 2.10: If $A\times B$ and $A\times D$ be two nonempty and D-bounded sets in $L\times M$ and $\tau(D^+\times D^+)\subseteq D^+$, then $\alpha A\times(B+D)$ is a D-bounded set for every $\alpha\in R-\{0\}$.

REFERENCES

1. Menger, K., 1942. Statistical metrics. Proc. Nat. Acad. Sci., USA, 28: 535-537.
2. Serstnev, A.N., 1963. On the motion of a random normed space. Dokl. Akad. Nauk SSSR, 149 (2): 280-283. English translation in Soviet Math. Dokl., 4: 388-390.
3. Golet, I., 2005. On probabilistic 2-normed spaces. Novi Sad J. Math., 35: 95-102.
4. Alsina, C., B. Schweizer and A. Sklar, 1993. On the definition of a probabilistic normed space. Aequationes Math, 46: 91-98.
5. Alsina, C., B. Schweizer, C. Sempi and A. Sklar, 1997. On the definition of a probabilistic inner product space. Rend Mat. Appl., 17: 115-127.
6. Alsina, C., B. Schweizer and A. Sklar, 1997. Continuity properties of probabilistic norms. J. Math. Anal. Appl, 208: 446-452.
7. Lafuerza-Guillen, B., A. Rodriguez-Lallena and C. Sempi, 1999. A study of boundedness in probabilistic normed spaces. J. Math. Anal. Appl, 232: 183-196.
8. Lafuerza-Guillen, B., 2001. D-bounded sets in probabilistic normed spaces and their products. Rendiconti di Mathematica, Serie, VII 21: 17-28.
9. E1 Naschie M.S., 1998. On the uncertainty of Cantorian geometry and two-slit experiment. Chaos, Solitons; Fractals, 9: 517-529.
10. E1 Naschie, M.S., 2000. On the verifications of heterotic string theory and ε^∞ theory. Chaos, Solitons; Fractals, 11: 379-407.