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Numerical Solution of Lienard Equation Using Hybrid Heuristic Computation

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Abstract: In this paper, a hybrid heuristic computing technique, stochastic in nature, is used for obtaining an approximate numerical solution of the Lienard equation. The proposed technique converts the nonlinear differential equation into an equivalent global error minimization problem. A trial solution is developed using a fitness function with unknown adaptable parameters. The memetic computation or hybrid genetic algorithms (HGAs) combining genetic algorithm (GA) with interior point algorithm (IPA), active set algorithm (ASA) and pattern search (PS) is used to solve the minimization problem and to obtain the unknown adaptable parameters. The accuracy and efficacy of the proposed technique is illustrated by considering the Lienard's equation with two special cases. Comparison of numerical results is made with the exact solution and two important deterministic standard methods, including variational iteration method (VIM) and differential transform method (DTM). The comparison of numerical results validate the effectiveness and viability of the suggested technique. The results obtained by the proposed method are found to be in excellent agreement with the exact solution.

Key words: Lienard equation · Memetic computation · Hybrid genetic algorithms (HGAs) · Interior point algorithm (IPA) · Active set algorithm (ASA).

INTRODUCTION

Nonlinear problems appearing in many physical phenomena, engineering and scientific applications are modeled with nonlinear differential equations. Since most of the nonlinear differential equations are difficult to be solved using analytical techniques, these problems must be tackled using approximate analytical and numerical methods. Many approximate analytical and numerical methods like finite difference method, differential transform method (DTM), homotopy perturbation method (HPM), adomian decomposition method (ADM), variational iteration method (VIM) etc. have been vastly used for solving nonlinear differential equations. In this paper, we consider the Lienard equation [1-5]

$$u'' + f(u)u' + g(u) = h(x) \tag{1}$$

Which is regarded as a generalization of damped pendulum equation or damped spring-mass system (where $f(u)$, $g(u)$ and $h(x)$ represent the damping force, restoring force and external force respectively). Moreover it is used as nonlinear models in many physically significant fields when taking different choices of $f(u)$, $g(u)$ and $h(x)$. For example, the choices $f(u) = \varepsilon(u^2-1)$, $g(u) = u$ and $h(x) = 0$ lead (1) to the well-known Van der Pol equation of nonlinear electronic oscillator [1-5]. Some nonlinear evolution equations such as Burgers-KdV equation can also be transformed to (1) [5]. Therefore the study of (1) is of great importance.

In the general case, it is commonly believed that it is very difficult to find the exact solution of (1) by usual ways [1-5], the following special case was investigated by authors in [1-7] and references there in.

$$u''(x) + lu(x) + mu^3(x) + nu^5(x) = 0 \tag{2}$$

where l, m and n are real coefficients.

Many methods including differential transform method (DTM) [1], variational iteration method (VIM) [2], variational homotopy perturbation method

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(VHPM) [3], Adomian decomposition method (ADM) [4], $\left(\frac{G'}{G}\right)$ –expansion method [6] etc. have been utilized for the exact and numerical solutions of (2).

In recent years evolutionary computation (EC) based techniques have been applied with success to solve numerous nonlinear ordinary differential equations (ODEs) arising in engineering and science [8-15]. Khan *et al.* [9] used PSO based artificial neural network (ANN) method for solving Wessinger's equation. Zahoor *et al.* [10] used PSO based ANN method for the solution of fractional order Riccati differential equation. Arqub *et al.* [11] used genetic algorithm (GA) based technique for solving linear and nonlinear singular boundary value problems. Behrang *et al.* [12] applied PSO based ANN for solving the nonlinear ODE arising from vertical full cone embedded in porous media. Only recently, Malik *et al.* [14-15] successfully employed heuristic computation technique, based on the hybrid genetic algorithm (HGA) for solving the Troesch'c problem and the nonlinear singular boundary value problems (BVPs) arising in physiology [14-15].

The main goal of this work is to employ the heuristic computation based technique as an alternative to existing standard numerical methods for solving the Lienard equation of the form (2). The contribution of this research work is that a heuristic computation based technique is for the first time used for solving Lienard equation (2) as per our literature survey. Moreover the proposed technique is stochastic in nature as compared to the standard methods [1-4, 6] utilized for solving this equation, which are deterministic in nature.

In the proposed method the nonlinear ODE is converted into an equivalent minimization problem. The method employs the hybridization approach of genetic algorithm (GA) with three local search algorithms such as interior point algorithm (IPA), pattern search (PS) and active set algorithm (ASA) for the minimization of the fitness function. The three hybrid genetic algorithm (HGA) schemes used in this work are called here as GA-IPA (GA hybridized with IPA), GA-PS (GA hybridized with PS) and GA-ASA (GA hybridized with ASA). The effectiveness and viability of the proposed method are illustrated by solving Lienard's equation with two special cases. The comparisons of our results are made with the exact solutions and the standard numerical methods including DTM [1] and VIM [2].

MATERIALS AND METHODS

Brief Introduction of Hybrid Genetic Algorithms (HGAs): In past few decades evolutionary algorithms (EAs) have received remarkably great attention and these techniques have been massively used for solving diverse optimization problems due to their simplicity and robustness [16]. Genetic algorithm (GA) invented by John Holland in 1960s is one of the most popular and widely used global search method in EAs. In GA a population of individuals called chromosome represents a candidate solution to the given problem. A problem exclusive fitness function is used to compute the fitness of each individual in a population. The algorithm evolves populations towards the global best solution over the successive generations employing selection and genetic operators of crossover and mutation [17].

Hybrid genetic algorithms (HGAs) s) have been investigated by many authors [16, 18]. It has been proved that HGAs can improve the performance and quality of the solution [16, 18]. A Hybrid genetic algorithm (HGA) combines GA with local search algorithms such as interior point algorithm (IPA), active set algorithm (ASA) etc.

In this work, we have utilized HGAs combining GA with IPA (GA-IPA), ASA (GA-ASA) and PS (GA-PS). GA has been used as global optimizer, which finds global optimal chromosome, while IPA, ASA and PS have been utilized for the local search fine-tuning. The procedural steps of the HGA approach are given as follows, while the parameter settings for the implementation of these algorithms for Lienard equation are given in Table 1 and Table 2.

Algorithm: Hybrid Genetic Algorithm (HGA)

Step 1: (Population Initialization)

- A population of N individuals or chromosomes (C_1, C_2, …,C_N) is generated using random number generator. Each chromosome consists of M number of genes, which represent the number of unknown adaptable parameters to be optimized.

Step 2: (Fitness Evaluation)

- The fitness of each individual or chromosome in the current population is determined using a problem exclusive fitness function.

Table 1: Parameter Settings of GA and IPA

GA		IPA	
Parameters	Settings	Parameters	Settings
Population size	[120 120]	Start point	best chromosome from GA
Chromosome size	30	Maximum iterations	1000
Selection function	Stochastic uniform	Maximum function evaluations	200000
Mutation function	Adaptive feasible	Function tolerance	1e-15
Crossover function	Heuristic	Nonlinear constraint tolerance	1e-15
Function tolerance	1e-15	Derivative type	Central differences
Nonlinear constraint tolerance	1e-15	Hessian	BFGS
No. of generations	1000	Subproblem algorithm	ldl factorization
Bounds	-15, +15	Bounds	-15, +15

Table 2: Parameter Settings of PS and ASA

PS		ASA	
Parameters	Settings	Parameters	Settings
Start point	Best chromosome from GA	Start point	Best chromosome from GA
Poll method	GPS positive basis 2N	Maximum iterations	400
Maximum iterations	3000	Maximum function evaluations	200000
Maximum function evaluation	200000	Function tolerance	1e-15
Function tolerance	1e-15	Nonlinear constraint tolerance	1e-15
Nonlinear constraint tolerance	1e-15	SQP constraint tolerance	1e-6
Bounds	-15, +15	Bounds	-15, +15

Step 3: (Stoppage Criterion)

- The algorithm stops, if a certain number of cycles has reached or fitness reaches a certain value. If the stoppage criterion is satisfied then go to step 6 for local search fine-tuning, else continue and repeat steps 2 to 5.

Step 4: (Reproduction)

- The chromosomes from the current population are chosen on the basis of their fitness which acts as parents for new generation. These parents produce children (offsprings) with a probability to their fitness through crossover operation.

Step 5: (Mutation)

- Mutation operation introduces random alterations in the genes to maintain the genetic diversity to find a good solution.

Step 6: (Local Search Fine-Tuning)

- The optimal chromosome found by GA is taken over by IPA, ASA and PS for fine- tuning and improvement.

Interior Point Algorithm (IPA) is a local search algorithm that reaches and optimal solution by computing iterates that lie in the feasible interior region. The algorithm applies a direct step also called Newton step or a conjugate gradient step to solve a system of Karush-Kuhn-Tucker (KKT) equations at each iteration [19-20].

Active Set Algorithm (ASA) is an iterative method that solves constrained optimization problems by searching solutions in the feasible sets. The main objective of the algorithm is to estimate the active set at the solution of the problem. Generally these methods work in two separate phases such as feasibility phase and optimality phase. In the feasibility phase the method attempts to find a feasible point for the constraints while the objective function is ignored. In the optimality phase the method preserves the feasibility while it attempts to find an optimal point [21].

Pattern Search (PS) belongs to the direct search optimization methods that explore a series of points that may reach to the optimal point. The algorithm initiates the search by creating a mesh from a set of points, around the current point. The algorithm looks for a point in the mesh that gives improvement in the objective function value. If the PS algorithm discovers such a point in the mesh that point turns into the current point in the next step. This process continues until the optimal value of the objective function is achieved by the algorithm [22-23].

Methodology for Lienard Equation: We may assume that the approximate numerical solution $u(x)$ and its first and second derivatives $u'(x)$ and $u''(x)$ of Lienard equation can be represented by a linear combination of some basis functions as follows.

$$u(x)=\sum_{i=1}^{k}\alpha_i\varphi(b_ix+c_i) \tag{3}$$

$$u'(x)=\sum_{i=1}^{k}\alpha_ib_i\varphi'(b_ix+c_i) \tag{4}$$

$$u''(x)=\sum_{i=1}^{k}\alpha_ib_i^2\varphi''(b_ix+c_i) \tag{5}$$

where a_i, b_i and c_i are real valued unknown adaptable parameters, k is the number of basis functions and is taken as the log sigmoid function which is given by

$$\varphi(x)=\frac{1}{1+e^{-x}} \tag{6}$$

The values of unknown adaptable parameters ( ) existing in (3) - (5) are determined by formulating a trial solution of the given problem using a fitness function given by

$$\varepsilon_j=\varepsilon_1+\varepsilon_2 \tag{7}$$

where j is the cycle index.

The fitness function given by (7) consists of the sum of two parts. The first part represents the mean square error (ε_1) of the given ODE without initial conditions and the second part represents the mean square error (ε_2) associated with the initial conditions.

The fitness function contains unknown adaptable parameters (a_i,b_i and c_i). The optimal values of unknown adaptable parameters are achieved by performing the minimization of the fitness function using the heuristic search algorithms described above. Consequently the approximate numerical solution $u(x)$ of the given problem is achieved.

RESULTS AND DISCUSSION

In this section the proposed heuristic technique is employed to the Lienard equation (2) with two special cases. To prove viability and effectiveness of the proposed technique, comparisons of the results are made with the exact solution and two standard numerical methods including DTM [1] and VIM [2].

Example 1: We consider the Lienard equation (2), with m = 4, n = -3 and l = -1 subject to the following initial conditions [1-2]

$$u(0)=\sqrt{\frac{-2l}{m}},u(0)=-\frac{l\sqrt{-l}}{m\sqrt{\frac{-2l}{m}}} \tag{8}$$

The exact solution of (2) with initial conditions (8) is given by (9) [1-2].

$$u(x)=\sqrt{\frac{-2l\left(1+tanh\left(\sqrt{-lx}\right)\right)}{m}} \tag{9}$$

To obtain the approximate numerical solution of (2) with given initial conditions (8), we construct a trial solution using the fitness function as follows

$$\varepsilon_1=\frac{1}{k+1}\sum_{i=0}^{k}\left[u''(x_i)-u(x_i)+4u^3(x_i)\right]^2 \tag{10}$$

$$\varepsilon_2=\frac{1}{2}\left\{\left(u(0)=\sqrt{\frac{-2l}{m}}\right)+\left(u(0)=-\frac{l\sqrt{-l}}{m\sqrt{\frac{-2l}{m}}}\right)^2\right\} \tag{11}$$

where $u(x)$, u (x) and $u''(x)$ are given by (3) - (5) respectively.

Therefore the fitness function is given by

$$\varepsilon_j=\varepsilon_1+\varepsilon_2 \tag{12}$$

The fitness function given by (12) is subject to minimization for achieving the unknown adaptable parameters. The minimization is performed using heuristic algorithms GA, PS, IPA, ASA and three hybrid genetic algorithm (HGA) schemes such as GA-IPA, GA-ASA and GA-PS. For the implementation Matlab has been utilized in this work.

The parameter settings used for the implementation of the algorithms are given in Table 1 and Table 2. The number of basis functions is taken equal to 10. The length of chromosome i.e. the number of unknown adaptable

Table 3: Optimal values of unknown adaptable parameters achieved by HGAs

	GA-IPA			GA-ASA			GA-PS		
i	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	1.4765	1.6435	0.0319	-1.1873	-0.9927	4.3567	1.5718	1.4311	0.5615
2	-0.4076	0.1844	0.7842	0.0532	2.8125	1.3375	-0.4871	-0.3624	0.8345
3	0.0571	1.8190	0.8479	3.3774	0.0092	-2.8024	-0.5389	1.2069	1.7976
4	-0.8915	1.6377	-0.3568	-2.3967	-1.4546	-0.3605	-0.1163	2.0438	-1.3707
5	-0.1359	-0.0050	0.6930	-1.9747	-0.2414	1.5959	-0.0205	0.2205	0.3142
6	-0.8117	0.4175	0.4188	0.4933	-0.7706	2.8568	-0.9034	0.4790	0.5725
7	-0.0811	0.5103	0.6673	-2.7251	1.2393	-1.7020	-0.1764	0.4371	0.2237
8	1.3478	0.2121	0.5170	-2.0717	1.5492	3.2516	1.6670	-0.0544	0.5038
9	0.5038	1.0990	0.7797	2.6098	-1.4604	3.9947	0.3111	0.7741	0.7144
10	0.0126	-0.3347	0.5364	3.0735	2.0118	3.1644	-0.0605	-0.1227	-0.0943

Table 4: Comparison of numerical results of example 1 between exact and proposed heuristic computation technique

x	Exact	GA	IPA	ASA	PS	GA-IPA	GA-ASA	GA-PS
0.0	0.70710678	0.70708157	0.70710049	0.70709981	0.70689442	0.70709992	0.7070993	0.70709729
0.1	0.74150792	0.74148195	0.74150168	0.74150062	0.74125573	0.74150067	0.74149966	0.74149750
0.2	0.77374909	0.77371815	0.77374349	0.77374154	0.77345350	0.77374120	0.77373869	0.77373622
0.3	0.80352741	0.80348785	0.80352226	0.80351966	0.80319280	0.80351891	0.80351477	0.80351196
0.4	0.83064703	0.83059890	0.83064195	0.83063915	0.83027477	0.83063819	0.8306333	0.83063026
0.5	0.85501964	0.85496522	0.85501464	0.85501169	0.85460429	0.85501066	0.85500567	0.85500252
0.6	0.87665545	0.87659658	0.87665085	0.87664739	0.87619110	0.87664623	0.87664089	0.87663761
0.7	0.89564719	0.89558353	0.89564323	0.89563887	0.89513198	0.89563743	0.89563087	0.89562736
0.8	0.91215042	0.91207956	0.91214703	0.91214169	0.91158491	0.91213991	0.91213155	0.91212773
0.9	0.92636328	0.92628212	0.92636013	0.92635410	0.92574466	0.92635202	0.92634223	0.92633803
1.0	0.9385079	0.93841447	0.93850472	0.93849825	0.93782349	0.93849589	0.93848536	0.93848068

parameters (a_i,b_i and c_i)are chosen equal to 30. The values of these unknown adaptable parameters are restricted between -15 and + 15. This was observed by several simulations that by using the parameter settings as prescribed in Tables 1 and 2, we get better results.

The algorithms are executed according to the prescribed settings. The optimal values of the unknown adaptable parameters are achieved. In Table 3 we provide the optimal values of a_i, b_i and c_i achieved using three HGA schemes including GA-IPA, GA-ASA and GA-PS, while we have omitted here the values achieved using other heuristic schemes (GA, IPA, ASA and PS).

The approximate numerical solution $u(x)$ of the Lienard equation (2) with initial conditions given by (8) is consequently achieved by using the values of a_i, b_i and c_i in (3). The numerical results achieved by the proposed heuristic technique are presented in Table 4 and compared with the exact solution.

The absolute errors by the proposed heuristic technique have been calculated relative to the exact solution and presented in Table 5. For the effectiveness and the accuracy of the proposed heuristic technique comparisons are made with two standard numerical methods including DTM [1] and VIM [2]. Comparison of the absolute errors reveals that the proposed heuristic technique yields the solution of the Lienard equation (2) with initial conditions (8) with the remarkably greater accuracy. Moreover it is established from the comparison that the absolute errors relative to the exact solutions obtained from the proposed technique are significantly smaller as compared to the standard methods DTM [1] and VIM [2]. Furthermore the better performances of hybrid genetic algorithms (HGAs) are quite evident from Table 5.

Example 2: We consider the Lienard equation (2), with subject to the following initial conditions [1-2]

$$u(0)=\sqrt{\frac{K}{2+D}},u(0)=0 \tag{13}$$

where $K=4\sqrt{\frac{3l_2}{3m^2-16nl}}$ and $D=-1+\frac{\sqrt{3}m}{\sqrt{3m^2-16nl}}$

The exact solution of (2) with initial conditions (13) is given by (14) [1-2]

$$u(x)=\sqrt{\frac{K\sec h^2\left(\sqrt{-l}x\right)}{2+D\sec h^2\left(\sqrt{-l}x\right)}} \tag{14}$$

To apply the proposed heuristic technique for the approximate numerical solution of (2) with given initial conditions (13), we construct a trial solution using the fitness function described in example 1, given by

$$\varepsilon_1 = \frac{1}{k+1}\sum_{i=0}^{k}\left[u''(x_i) - u(x_i) + 4u^3(x_i) - 3u^5(x_i)\right]^2 \tag{15}$$

$$\varepsilon_2 = \frac{1}{2}\left\{\left(u(0) = \sqrt{\frac{K}{2+D}}\right) + \left(u(0)\right)^2\right\} \tag{16}$$

where $u(x)$, u (x) and u'' (x) are given by (3) - (5) respectively.

Therefore the fitness function is given by

$$\varepsilon_j = \varepsilon_1 + \varepsilon_2 \tag{17}$$

The minimization of (17) is performed using heuristic algorithms GA, PS, IPA, ASA and three hybrid genetic algorithm (HGA) schemes such as GA-IPA, GA-ASA and GA-PS.

The algorithms are executed according to the prescribed settings in Table 1 and Table 2. The optimal values of the unknown adaptable parameters are achieved. The optimal values of a_i, b_i and c_i achieved using three HGA schemes including GA-IPA, GA-ASA and GA-PS are provided in Table 6.

The approximate numerical solution of the Lienard equation (2) with initial conditions given by (13) is consequently achieved by using the values of a_i, b_i and c_i in (3). The numerical results achieved by the proposed heuristic technique are presented in Table 7 and compared with the exact solution.

In Table 8 we present the absolute errors for example 2 by the proposed heuristic technique. Comparisons of results are carried with standard numerical methods including DTM [1] and VIM [2]. It is evident from the comparison of the absolute errors in Table 8 that the proposed heuristic technique provides the approximate solution of the Lienard equation (2) with initial conditions (13) with the significantly greater accuracy. The comparison shows that the proposed technique gives

Table 5: Comparison of absolute errors for example 1 between proposed heuristic computation technique and standard numerical methods given in [1-2]

	Proposed Heuristic Technique							Standard Methods	
x	GA	IPA	ASA	PS	GA-IPA	GA-ASA	GA-PS	DTM[1]	VIM[2]
0.1	2.60E-05	6.24E-06	7.30E-06	2.52E-04	7.25E-06	8.26E-06	1.04E-05	2.26E-06	8.83E-07
0.2	3.09E-05	5.60E-06	7.55E-06	2.96E-04	7.89E-06	1.04E-05	1.29E-05	2.52E-06	1.29E-05
0.3	3.96E-05	5.15E-06	7.75E-06	3.35E-04	8.50E-06	1.26E-05	1.55E-05	1.44E-05	5.61E-05
0.4	4.81E-05	5.08E-06	7.88E-06	3.72E-04	8.84E-06	1.37E-05	1.68E-05	6.18E-05	1.37E-04
0.5	5.44E-05	5.00E-06	7.95E-06	4.15E-04	8.98E-06	1.40E-05	1.71E-05	2.21E-04	2.10E-04
0.6	5.89E-05	4.60E-06	8.06E-06	4.64E-04	9.22E-06	1.46E-05	1.78E-05	6.52E-04	1.24E-04
0.7	6.37E-05	3.96E-06	8.32E-06	5.15E-04	9.76E-06	1.63E-05	1.98E-05	1.64E-03	4.46E-04
0.8	7.09E-05	3.39E-06	8.73E-06	5.66E-04	1.05E-05	1.89E-05	2.27E-05	3.65E-03	2.06E-03
0.9	8.12E-05	3.15E-06	9.18E-06	6.19E-04	1.13E-05	2.10E-05	2.52E-05	7.35E-03	5.63E-03
1.0	9.34E-05	3.18E-06	9.65E-06	6.84E-04	1.20E-05	2.25E-05	2.72E-05	1.37E-02	1.24E-02

Table 6: Optimal values of unknown adaptable parameters achieved by HGAs

	GA-IPA			GA-ASA			GA-PS		
i	a_i	b_i	c_i	a_i	b_i	c_i	a_i	b_i	c_i
1	1.2149	-0.3727	-2.8408	1.2149	-0.3727	-2.8408	1.4406	0.1609	-2.6743
2	1.3944	2.1279	0.8475	1.3944	2.1279	0.8475	1.1331	1.9884	1.16
3	-0.2826	2.3517	-0.4081	-0.2826	2.3517	-0.4081	0.0527	-3.5244	0.507
4	-0.3524	1.2247	-0.4775	-0.3524	1.2247	-0.4775	-0.3858	1.289	-0.2388
5	0.1226	1.9992	-1.0056	0.1226	1.9992	-1.0056	0.4586	1.7161	-1.077
6	-0.9897	1.6812	-0.5169	-0.9897	1.6812	-0.5169	-0.8535	2.0503	-1.0667
7	0.0182	3.8165	-0.087	0.0182	3.8165	-0.087	0.0841	3.3504	0.4037
8	0.2249	-0.7396	2.2311	0.2249	-0.7396	2.2311	0.0445	-1.2291	2.6335
9	2.3882	-0.0617	-1.9094	2.3882	-0.0617	-1.9094	2.5068	-0.1073	-2.1426
10	-1.4067	-0.067	-1.1636	-1.4067	-0.067	-1.1636	-1.6696	0.3519	-1.0648

Table 7: Comparison of numerical results of example 2 between exact and proposed heuristic computation technique

x	Exact	GA	IPA	ASA	PS	GA-IPA	GA-ASA	GA-PS
0.0	0.64358868	0.64360617	0.64362394	0.64357366	0.64348008	0.64357999	0.64357999	0.64360959
0.1	0.63983909	0.63986344	0.6398824	0.63982271	0.63973451	0.63983065	0.63983065	0.63985751
0.2	0.62883007	0.62883763	0.62888322	0.62880788	0.62874109	0.62882255	0.62882255	0.62883941
0.3	0.61124131	0.61124014	0.61129975	0.61121583	0.61118077	0.61123551	0.61123551	0.61124073
0.4	0.58808625	0.58809884	0.58814998	0.58806418	0.58805891	0.58808261	0.58808261	0.58808171
0.5	0.56056993	0.56059993	0.56063482	0.56055453	0.56057095	0.56056877	0.56056877	0.56056493
0.6	0.52994636	0.52997467	0.53000645	0.52993526	0.52996741	0.52994800	0.52994800	0.52993642
0.7	0.49740243	0.4974092	0.49745516	0.49739153	0.49743961	0.49740702	0.49740702	0.49738027
0.8	0.463981	0.46396764	0.4640277	0.46396966	0.46403852	0.46398843	0.46398843	0.46394651
0.9	0.43054277	0.43053227	0.43058554	0.43053530	0.43062939	0.43055292	0.43055292	0.43050625
1.0	0.3977589	0.3977683	0.39779717	0.39775980	0.39787793	0.39777179	0.39777179	0.39773005

Table 8: Comparison of absolute errors for example 2 between proposed heuristic computation technique and standard numerical methods given in [1-2]

	Proposed Heuristic Computation Technique							Standard Method	
x	GA	IPA	ASA	PS	GA-IPA	GA-ASA	GA-PS	DTM[1]	VIM[2]
0.1	2.43E-05	4.33E-05	1.64E-05	1.05E-04	8.44E-06	8.44E-06	1.84E-05	4.07E-06	2.04E-05
0.2	7.56E-06	5.31E-05	2.22E-05	8.90E-05	7.52E-06	7.52E-06	9.34E-06	3.70E-06	3.22E-04
0.3	1.17E-06	5.84E-05	2.55E-05	6.05E-05	5.80E-06	5.80E-06	5.80E-07	7.18E-06	1.58E-03
0.4	1.26E-05	6.37E-05	2.21E-05	2.73E-05	3.64E-06	3.64E-06	4.54E-06	4.50E-05	4.82E-03
0.5	3.00E-05	6.49E-05	1.54E-05	1.02E-06	1.16E-06	1.16E-06	5.00E-06	2.47E-04	1.12E-02
0.6	2.83E-05	6.01E-05	1.11E-05	2.11E-05	1.64E-06	1.64E-06	9.94E-06	9.99E-04	2.21E-02
0.7	6.77E-06	5.27E-05	1.09E-05	3.72E-05	4.59E-06	4.59E-06	2.22E-05	3.22E-03	3.85E-02
0.8	1.34E-05	4.67E-05	1.13E-05	5.75E-05	7.43E-06	7.43E-06	3.45E-05	8.74E-03	6.17E-02
0.9	1.05E-05	4.28E-05	7.47E-06	8.66E-05	1.01E-05	1.01E-05	3.65E-05	2.08E-02	9.23E-02
1.0	9.40E-06	3.83E-05	9.00E-07	1.19E-04	1.29E-05	1.29E-05	2.89E-05	4.49E-02	1.31E-01

numerical results that are in excellent agreement with the exact solution as compared to the standard methods DTM [1] and VIM [2].

CONCLUSION

A heuristic computation based technique as an alternative to the existing deterministic standard numerical methods has been applied for solving numerically the Lienard equation. On the basis of the numerical results and comparisons made with standard numerical methods DTM and VIM and exact solutions, it can be concluded that the proposed heuristic computing technique is effective for solving the Lienard equation. The viability and the accuracy of the proposed technique is demonstrated by solving two special cases of the Lienard equation. It is observed that the proposed technique gives better performance than DTM and VIM methods in comparison with the exact solution for the Lienard equation. Furthermore the proposed technique can provide the approximate numerical solution of the given nonlinear ODE conveniently and on the continuous grid of time once the unknown adaptable parameters have been acquired.

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