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Claim Dependence with Common Effects in Credibility Models with Error Uniform Dependence

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Abstract: In classical Bühlmann credibility models, observations are made of a group of risks selected from a population and claims are assumed to be independent between different risks. Further, for each individual, past claims have the same mean and variance and are independent and identically distributed conditional on the risk parameter. However, there are situations in practical applications that these assumptions may be violated including the possibility of relationship among the risks and conditional dependence over time. Thus, this warrants a more appropriate approach to handle such situations. In this paper, we extend the Bühlmann and Bühlmann-Straub credibility models to account for not only a certain uniform conditional dependence for daim amounts for each individual risk, but also a special type of dependence between risks induced by common effects. We further give illustrative examples to show the influence of the error uniform dependence when these common effects have Normal distributions.

Key words: Credibility models . common effects . influence of dependence . error uniform dependence . Bühlmann model . Bühlmann-straub models

INTRODUCTION

In insurance practice, a common object is to determine insurance premium and its use in credibility theory. Under credibility techniques, one can calculate the future premiums for a risk or group of risks based on past experience. In insurance premium determination, it is a familiar practice to group individual risks in order to ensure homogeneity in reaching a fair and equitable premium across the individuals. Thus the risks within each group are presented as homogeneous as possible in terms of certain observable risk characteristics as well as underwriting characteristics. However, it is also known that not all risks in the group are truly homogeneous. Some unobservable factors always affect possible presence of heterogeneity among the individuals.

A collective premium, also known as the "manual premium", is then calculated and charged for the group. The collective premium is designed to reflect the expected experience of the entire rating class and implicitly provided that the risks are homogeneous. Based on the experience and the collective premium, the credibility theory determines the credibility premium by the following credibility form:

$$\text{Credibility premium} = Z\times(\text{experience})+(1-Z)\times(\text{collective premium}) \qquad (1)$$

where Z, value between 0 and 1, is the "credibility factor" and needs to be chosen. There are many suggested formulas for Z in the actuarial literature and they are usually justified on intuitive rather than theoretical grounds. Note that there are two possibilities for data: In the first possibility, the group's collective experience might be large enough so that the law of large numbers is applicable and therefore ignores the existence of heterogeneity. In the second, however, the individual's own experience may contain useful information about the risk characteristics of the individual but may not be subject to random fluctuation due to lack of volume. The credibility models therefore are able to lead an evidently attractive formula by allowing for larger credibility for longer number of years of individual experience. If the body of observed data is large and not likely to vary much from one period to another or if there is a high degree of heterogeneity in the overall experience then Z, credibility factor, will be closer to one. If the observation consists of limited data such that individual risk experience is lacking or unreliable or possibly the

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population is fairly homogeneous with respect to the individual risk experience, then Z is closer to zero.

The credibility theory was introduced by Mowbray [1] where he attempted to derive credibility formula on purely classical statistics arguments by using confidence bounds to arrive at "full credibility" (i.e. giving weight 1 to the individual estimate). Similarly, Whitney [2] suggested using a weighted average between the individual and the collective experience. The world of Bailey [3] might be the first actuarial world science that recognizes the general Bayesian structure. However, an excellent introduction to credibility theory can be found, e.g., in [4-8].

Bühlmann [9] and in the sequel Bühlmann and Straub [10] established the theoretical foundation of modern credibility theory which presented a distribution free credibility estimation. The method was extended in Hachemeister [11] regression model, where the credibility premium depends linearly on a number of risk characteristics ([12] for time series regression models with multicollinearity). Jewell, Taylor and Norberg presented hierarchical models assuming that the obtainable portfolio can be split into sub-portfolios (sectors) and each of these sectors contains individual contracts [13-15]. Klugman [16] gave an introduction to the use of Bayesian methods covering some particular aspects of credibility theory. The papers by Gerber and Jones [17] and Frees *et al*. [18] discussed credibility models with time dependence of claims. Purcaru and Denuit [19, 20] provided a kind of dependence induced by introducing the common latent variables in the annual numbers of claims reported by several policyholders. Recently, Yeo and Valdez [21] addressed two-level common effects by using a simultaneous dependence of claims across individuals for a fixed time period and across time periods for a fixed individual. They investigated the corresponding credibility premiums under normally distributed claim amounts. Wen *et al*. extended the Bühlmann and Bühlmann-Straub credibility models to account for a special type of dependence between risks induced by common stochastic effects. They built the Bühlmann credibility model with uniform dependence and derived the corresponding credibility estimators and the model was extended to Bühlmann-Straub credibility in which the natural weights among contracts were introduced [22, 23].

In the present paper, we extend the Bühlmann and Bühlmann-Straub credibility models to account for not only a certain uniform conditional dependence for claim amounts each individual risk, but also a special type of dependence between risks induced by common effects.

PRELIMINARIES

The purpose of this paper is to study the two level common effects model of dependence with error uniform dependence. The model structure with two level common effects of claim dependence is explained as follows.

Consider a portfolio of insurance contracts consisting of K insured individuals and suppose that each individual has available a history of a total of n time periods. Denote by X_{ij}, the claim amount for individual i during period j. To simplify our exposition, the same time periods will be applied to all individuals. Let $X_i=(X_{i,1}, X_{i,2},..., X_{i,T})'$, i=1,2,...,K.

The model of dependence being proposed in this section will allow for both the dependence among the individual risks as well as the dependence of experience for a particular individual risk over time. The risk quality of an individual i is characterized by a risk parameter T_i and the common effect is represented by a random variable Λ. Formally, the assumptions of the model are stated as below which are similar to the assumptions in [23].

A1. The common effect random variable Λ has known expectation $E(\Lambda)] = \mu_\lambda$ and variance $Var(\Lambda)= \sigma_\lambda^2$.

A2. Given Λ, the random vectors (X_i, Θ_i), i=1,2,...,K, are mutually independent and identically distributed.

A3. For fixed contract i, given Λ, the claims $X_{i,1}$, $X_{i,2}$,..., $X_{i,n}$, ... are characterized by a risk parameter T_i and T_i itself is random variable with structure distribution $p(\theta_i)$.

A4. For fixed contract i, given Λ and T_i, the X_{ij} follows the linear model: $X_{ij} = \mu(X_i, \Theta_i)+\varepsilon_{ij}$ and the errors are conditionally uniformly dependent, ie, $corr(\varepsilon_{ij}, \varepsilon_{im}) = \rho$, $j\neq m$ and $\rho<1$, where "corr" indicates correlation coefficient. We also assume that $E(\varepsilon_{ij} \mid \Theta_i, \Lambda) = 0$ and $Var(\varepsilon_{ij} \mid \Theta_i, \Lambda) = \sigma_1^2(\Theta_i, \Lambda)$.

We introduce the following notations for future use (which are similar to the notations in [22]):

$$
\begin{aligned}
&E[\mu(\Theta_i,\Lambda) \mid \Lambda] = \mu_1(\Lambda)\\
&Var[\mu(\Theta_i,\Lambda) \mid \Lambda] = \sigma_2^2(\Lambda)\\
&E[\sigma_2^2(\Lambda)] = \sigma_2^2\\
&Var[\mu(\Lambda)] = a_1\\
&E[\mu(\Lambda)] = \mu_1\\
&E[\sigma_x^2(\Theta_i, \Lambda)] = \sigma_1^2(\Lambda)\\
&E[\sigma_x^2(\Lambda)] = \sigma_1^2\\
&Cov(X_{ip}, X_{iq} \mid \Theta_i, \Lambda) = \rho\,\sigma_1^2(\Theta_i, \Lambda)\\
&E[Cov(X_{ip}, X_{iq} \mid \Theta_i, \Lambda)] = \rho\,\sigma_1^2 \qquad (2)
\end{aligned}
$$

The following lemma gives some simple but fundamental features of the dependence structure just specified. The lemma in the case of two-level common effects was proved in [22].

Lemma 1: Under assumptions A1 to A4 and the notation (2), we have

(i) The means of X_i are given by

$$E[X_i] = \mu_1 1_n, \; i = 1, 2, \ldots, K \tag{3}$$

where 1_n is an n-vector with 1 in all of the n entries.

(ii) The covariance of X is given by

$$\Sigma_{XX} \overset{\Delta}{=} Cov(X) = I_K \otimes [S_1 I_n + S_2 1_n 1_n^{'}] + a_1 1_{nK} 1_{nK}^{'} \tag{4}$$

where $\overset{\Delta}{=}$ means "defined by", $\otimes$ indicates the Kronecker product of matrices, $S_1 = (1-\rho)\sigma_1^2$ and $S_2 = \rho\sigma_1^2 + \sigma_2^2$.

(iii) The covariance between X and $\mu(\Theta_i, \Lambda)$ is given by

$$\Sigma_{\mu(\Theta_i,\Lambda),X} \overset{\Delta}{=} Cov(\mu(\Theta_i, \Lambda), X) = (\sigma_2^2 e_i^{'} + a_1 1_K^{'}) \otimes 1_n^{'} \tag{5}$$

where e_i is a vector with 1 in the i^{th} entry and 0 in the other entries.

(iv) The inverse of the variance matrix of X is given by

$$\Sigma_{XX}^{-1} = \frac{1}{S_1} I_K \otimes \left(I_n - \frac{S_2}{S_1 + nS_2} 1_n 1_n^{'} \right) - \frac{a_1}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} 1_{nK} 1_{nK}^{'} \tag{6}$$

Proof: (i) (3) is straightforward.

(ii) It follows from assumption A4 that

$$E(X|\Theta, \Lambda) = [\mu(\Theta_1, \Lambda), \mu(\Theta_2, \Lambda), \ldots, \mu(\Theta_K, \Lambda)]^{'} \otimes 1_n^{'} \tag{7}$$

and

$$Cov(X|\Theta, \Lambda) = diag\{[\rho\sigma^2(\Theta_1, \Lambda)] 1_n 1_n^{'} + (1-\rho)\sigma^2(\Theta_1, \Lambda) I_n, \ldots, [\rho\sigma^2(\Theta_K, \Lambda)] 1_n 1_n^{'} + (1-\rho)\sigma^2(\Theta_K, \Lambda) I_n\} \tag{8}$$

where diag[…] is a (block) diagonal matrix with the elements in the bracket down the diagonal. Using the notation (2) and (8) implies that

$$E[Cov(X|\Theta, \Lambda)] = diag\{\rho\sigma_1^2 1_n 1_n^{'} + (1-\rho)\sigma_1^2 I_n, \ldots, \rho\sigma_1^2 1_n 1_n^{'} + (1-\rho)\sigma_1^2 I_n\} = \rho\sigma_1^2 I_K \otimes 1_n 1_n^{'} + (1-\rho)\sigma_1^2 I_{nK} \tag{9}$$

On the other hand, in view of (7) and the conditional independence among T_i's given Λ, the covariance matrix of $E(X | \Theta_i, \Lambda)$ can be computed as

$$\begin{aligned} Cov[E(X|\Theta, \Lambda)] &= E\{Cov[(\mu(\Theta_1, \Lambda), \mu(\Theta_2, \Lambda), \ldots, \mu(\Theta_K, \Lambda))^{'} \otimes 1_n \,|\, \Lambda]\} \\ &+ Cov\{E[(\mu(\Theta_1, \Lambda), \mu(\Theta_2, \Lambda), \ldots, \mu(\Theta_K, \Lambda))^{'} \otimes 1_n \,|\, \Lambda]\} \\ &= E[\sigma_2^2(\Lambda) | \Lambda] I_K \otimes 1_n 1_n^{'} + Cov[\mu_1(\Lambda) 1_{nK}] = \sigma_2^2 I_K \otimes 1_n 1_n^{'} + a_1 1_{nK} 1_{nK}^{'} \end{aligned} \tag{10}$$

Therefore,

$$\begin{aligned} \Sigma_{XX} &= E[Cov(X|\Theta, \Lambda)] + Cov[E(X|\Theta, \Lambda)] \\ &= I_K \otimes [S_1 I_n + S_2 1_n 1_n^{'}] + a_1 1_{nK} 1_{nK}^{'} \end{aligned}$$

(iii) We know that $Cov[\mu(\Theta_i, \Lambda), X | \Theta, \Lambda] = 0$. Thus,

$$\begin{aligned} \Sigma_{\mu(\Theta_i,\Lambda),X} &= E[Cov(\mu(\Theta_i, \Lambda), X | \Theta, \Lambda)] \\ &+ Cov[E(\mu(\Theta_i, \Lambda) | \Theta, \Lambda), E(X | \Theta, \Lambda)] \\ &= Cov\left\{ \mu(\Theta_i, \Lambda | \Lambda), \begin{pmatrix} E[\mu(\Theta_1, \Lambda | \Lambda], E[\mu(\Theta_2, \Lambda | \Lambda], \ldots, \\ E[\mu(\Theta_K, \Lambda | \Lambda] \end{pmatrix} \otimes 1_n \right\} \\ &= \sigma_2^2 e_i^{'} 1_n^{'} + Cov[\mu_1(\Lambda), \mu_1(\Lambda) 1_{nK}] \\ &= \sigma_2^2 e_i^{'} \otimes 1_n^{'} + a_1 1_{nK}^{'} = (\sigma_2^2 e_i^{'} + a_1 1_K^{'}) \otimes 1_n^{'} \end{aligned}$$

(iv) Using the following formula for matrix inverse [24].

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1} \tag{11}$$

we can check that

$$[I_K \otimes (S_1 I_n + S_2 1_n 1_n^{'})]^{-1} = I_K \otimes \frac{1}{S_1}(I_n - \frac{S_2 1_n 1_n^{'}}{S_1 + nS_2})$$

and then

$$(S_1 I_n + S_2 1_n 1_n^{'})^{-1} 1_n = \frac{1}{S_1 + nS_2} 1_n$$

Note that $1_{nk} = 1_k \otimes 1_n$. Finally, using (11) it follows that

$$\Sigma_{XX}^{-1} = [I_K \otimes (S_1 I_n + S_2 1_n 1_n^{'}) + a_1 1_{nK} 1_{nK}^{'}]^{-1}$$

$$= [I_K \otimes (S_1 I_n + S_2 1_n 1_n^{'})]^{-1} - \frac{[I_K \otimes (S_1 I_n + S_2 1_n 1_n^{'})]^{-1} 1_{nK} 1_{nK}^{'} [I_K \otimes (S_1 I_n + S_2 1_n 1_n^{'})]^{-1}}{\frac{1}{a_1} + 1_{nK}^{'} [I_K \otimes (S_1 I_n + S_2 1_n 1_n^{'})]^{-1} 1_{nK}}$$

$$= I_K \otimes \frac{1}{S_1}(I_n - \frac{S_2 1_n 1_n^{'}}{S_1 + nS_2}) - \frac{(\frac{1}{S_1 + nS_2})^2 1_{nK} 1_{nK}^{'}}{\frac{1}{a_1} + \frac{1}{S_1 + nS_2} 1_{nK} 1_{nK}^{'}}$$

$$= I_K \otimes \frac{1}{S_1}(I_n - \frac{S_2 1_n 1_n^{'}}{S_1 + nS_2}) - \frac{a_1}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} 1_{nK} 1_{nK}^{'}$$

Thus the lemma is proved.

Now, we present the following lemma on projections of random variables for later use. The lemma has been proved in [22].

Lemma 2: Let $(X'_{1\times p}, Y'_{1\times q})'$ be a random vector with expectation (μ'_X, μ'_Y) and covariance matrix $\begin{pmatrix} \Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY} \end{pmatrix}$. Then, Y can be optimally predicted in the class of inhomogeneous linear functions of X by

$$\mathrm{proj}(Y|X,1) = \mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(X - \mu_X) \tag{12}$$

where proj (Y|X,1) represents the projection of Y on the linear space spanned by X, Σ_{YX} is the covariance matrix of Y and X and Σ_{XX} indicates the covariance matrix of X.

BÜHLMANN MODEL WITH COMMON EFFECTS

We now proceed to calculate a premium for the future claim, denoting by $\mu(\Theta_i, \Lambda)$, i = 1, 2, …, K, with error uniform dependence in Bühlmann model with dependence induced by common effects.

Theorem 1: Under assumptions A1 to A4 and the notations in previous section, the optimal credibility estimator of $\mu(\Theta_i, \Lambda)$, i = 1, 2, …, K, is given by

$$\hat{\mu}(\Theta_i, \Lambda) = z_1\bar{X}_i + z_2\bar{\bar{X}} + (1 - z_1 - z_2)\mu_1 \tag{13}$$

where

$$\bar{X}_i = \frac{1}{n}\sum_{j=1}^{n} X_{ij}, \quad \bar{\bar{X}} = \frac{1}{K}\sum_{i=1}^{K}\bar{X}_i,$$

$$z_1 = \frac{n\sigma_2^2}{S_1 + nS_2}, \quad z_2 = \frac{nKa_1(S_1 + nS_2 + n\sigma_2^2)}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)},$$

$S_1 = (1-\rho)\sigma_1^2$ and $S_2 = \rho\sigma_1^2 + \sigma_2^2$

Proof: On using Lemma 2,

$$\hat{\mu}(\Theta_i, \Lambda) = \mathrm{proj}(\mu(\Theta_i, \Lambda)|X,1)$$

Thus we need to get the following term by using Lemma 1.

$$\Sigma_{\mu(\Theta_i,\Lambda),X}\Sigma_{XX}^{-1} = [(\sigma_2^2 e_i' + a_1 1_K') \otimes 1_n'] \times \left[I_K \otimes \frac{1}{S_1}\left(I_n - \frac{S_2 1_n 1_n'}{S_1 + nS_2}\right) - \frac{a_1}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} 1_{nK}1_{nK}' \right]$$

$$= [(\sigma_2^2 e_i' + a_1 1_K') \otimes 1_n'] \left[I_K \otimes \frac{1}{S_1}\left(I_n - \frac{S_2 1_n 1_n'}{S_1 + nS_2}\right) \right] - [(\sigma_2^2 e_i' + a_1 1_K') \otimes 1_n'] \left[\frac{a_1}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} 1_{nK}1_{nK}' \right]$$

$$= \frac{1}{S_1}(\sigma_2^2 e_i' + a_1 1_K') \otimes \left(1_n' - \frac{nS_2 1_n'}{S_1 + nS_2}\right) - \left[\frac{a_1(\sigma_2^2 e_i' + a_1 1_K')1_K}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} \right] \otimes n1_{nK}'$$

$$= \left[\frac{1}{S_1 + nS_2}(\sigma_2^2 e_i' + a_1 1_K') \otimes 1_n' \right] - \left[\frac{na_1(\sigma_2^2 + Ka_1)}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} 1_{nK}' \right] = \frac{1}{S_1 + nS_2}\left[(\sigma_2^2 e_i' + a_1 1_K') \otimes 1_n' - \frac{na_1(\sigma_2^2 + Ka_1)}{S_1 + nS_2 + nKa_1} 1_{nK}' \right]$$

$$= \frac{1}{S_1 + nS_2}\left[\sigma_2^2 e_i' \otimes 1_n' + a_1 1_{nK}' - \frac{na_1(\sigma_2^2 + Ka_1)}{S_1 + nS_2 + nKa_1} 1_{nK}' \right] = \frac{1}{S_1 + nS_2}\left[\sigma_2^2 e_i' \otimes 1_n' + a_1\left(\frac{S_1 + nS_2 - n\sigma_2^2}{S_1 + nS_2 + nKa_1} \right) 1_{nK}' \right].$$

Then

$$\hat{\mu}(\Theta_i, \Lambda) = \mathrm{proj}(\mu(\Theta_i, \Lambda)|X,1) = E[\mu(\Theta_i, \Lambda)] + \Sigma_{\mu(\Theta_i,\Lambda),X}\Sigma_{XX}^{-1}[X - E(X)]$$

$$= \mu_1 + \frac{1}{S_1 + nS_2}\left[\sigma_2^2 e_i' \otimes 1_n' + a_1\left(\frac{S_1 + nS_2 - n\sigma_2^2}{S_1 + nS_2 + nKa_1} \right) 1_{nK}' \right](X - \mu_1 1_{nK})$$

$$= \mu_1 + \frac{1}{S_1 + nS_2}\left[(\sigma_2^2 e_i' \otimes 1_n')(X - \mu_1 1_{nK}) + a_1\left(\frac{S_1 + nS_2 - n\sigma_2^2}{S_1 + nS_2 + nKa_1} \right) 1_{nK}'(X - \mu_1 1_{nK}) \right]$$

$$= \mu_1 + \frac{1}{S_1 + nS_2}\left[n\sigma_2^2(\bar{X}_i - \mu_1) + nKa_1\left(\frac{S_1 + nS_2 - n\sigma_2^2}{S_1 + nS_2 + nKa_1} \right)(\bar{\bar{X}} - \mu_1) \right]$$

$$= \frac{n\sigma_2^2}{S_1 + nS_2}\bar{X}_i + \frac{nKa_1(S_1 + nS_2 - n\sigma_2^2)}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)}\bar{\bar{X}} + \left(1 - \frac{n\sigma_2^2}{S_1 + nS_2} - \frac{nKa_1(S_1 + nS_2 - n\sigma_2^2)}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} \right)\mu_1 = z_1\bar{X}_i + z_2\bar{\bar{X}} + (1 - z_1 - z_2)\mu_1.$$

BÜHLMANN-STRAUB MODEL WITH COMMON EFFECTS

We now turn to the more general Bühlmann-Straub setup developed in [10]. In this section, we extend the Bühlmann model with error uniform dependence described in previous section to the case of the Bühlmann-Straub model. To be more specific, as before, we are given a portfolio of K risks or individuals, with assumptions A1 to A3 and the following A4′:

A4′: For fixed contract i, given Λ and T_i, the X_{ij} follows the linear model: $X_{ij} = \mu(\Theta_i, \Lambda)+\varepsilon_{ij}$ and the errors are conditionally uniformly dependent, i.e, $corr(\varepsilon_{ij}, \varepsilon_{im}) = \rho$, $j\neq m$ and $\rho<1$, where "corr" indicates correlation coefficient. We also assume that $E(\varepsilon_{ij} \mid \Theta_i, \Lambda) = 0$ and

$$Var(\varepsilon_{ij} \mid \Theta_i, \Lambda) = \frac{\sigma^2(\Theta_i, \Lambda)}{\omega_{ij}}$$

where ω_{ij} are known weights.

Moreover, the following will be used to simplify the notations later:

$$\begin{aligned}
\omega_i &= (\omega_{i1}, \omega_{i2}, \ldots, \omega_{in})' \\
\omega_i^{-1} &= \left(\frac{1}{\omega_{i1}}, \frac{1}{\omega_{i2}}, \ldots, \frac{1}{\omega_{in}}\right)' \\
\omega_i^{\frac{1}{2}} &= \left(\sqrt{\omega_{i1}}, \sqrt{\omega_{i2}}, \ldots, \sqrt{\omega_{in}}\right)' \\
\omega_i^{-\frac{1}{2}} &= \left(\frac{1}{\sqrt{\omega_{i1}}}, \frac{1}{\sqrt{\omega_{i2}}}, \ldots, \frac{1}{\sqrt{\omega_{in}}}\right)' \\
W_i^{-\frac{1}{2}} &= \omega_i^{-\frac{1}{2}}\left(\omega_i^{-\frac{1}{2}}\right)' \\
\alpha_i &= \sum_{j=1}^{n} \omega_{ij} \\
\alpha_{ia} &= \sum_{j=1}^{n} \sqrt{\omega_{ij}}
\end{aligned} \tag{14}$$

The following lemma gives some simple but fundamental features of the dependence structure that we just specified. The lemma in the case of two-level common effects was proved in [22].

Lemma 3: Under assumptions A1 to A3 and A4′ and the notations (3) and (14), we have

(i) The means of X_i are given by

$$E[X_i] = \mu_1 1_n, i = 1,2,\ldots,K \tag{15}$$

(ii) The covariance of X is given by

$$\begin{aligned}\Sigma_{XX} &= Cov(X) \\ &= diag\begin{bmatrix}\rho\sigma_1^2 W_i^{-\frac{1}{2}} + (1-\rho)\sigma_1^2 diag(\omega_i^{-1}) \\ +\sigma_2^2 1_n 1_n', i = 1,2,\ldots,K\end{bmatrix} \\ &\quad + a_1 1_{nK} 1_{nK}'\end{aligned} \tag{16}$$

(iii) The covariance between X and $\mu(\Theta_i, \Lambda)$ is given by

$$\Sigma_{\mu(\Theta_i,\Lambda),X} = Cov\left(\mu(\Theta_i, \Lambda), X\right) = (\sigma_2^2 e_i' + a_1 1_K') \otimes 1_n' \tag{17}$$

(iv) The inverse of the variance matrix of X is given by

$$\begin{aligned}\Sigma_{XX}^{-1} &= diag[C_i^{-1}, i = 1,2,\ldots,K] \\ &\quad - \frac{a_1[C_1^{-1}1_n, \ldots, C_K^{-1}1_n]'[C_1^{-1}1_n, \ldots, C_K^{-1}1_n]}{1 + a_1\sum_{i=1}^{K} 1_n' C_i^{-1} 1_n}\end{aligned} \tag{18}$$

where

$$C_i = \rho\sigma_1^2 W_i^{-\frac{1}{2}} + (1-\rho)\sigma_1^2 diag(\omega_i^{-1}) + \sigma_2^2 1_n 1_n', i = 1,2,\ldots,K$$

Proof: (i) (15) is straightforward.

(ii) We have

$$E(X \mid \Theta, \Lambda) = [\mu(\Theta_1, \Lambda), \mu(\Theta_2, \Lambda), \ldots, \mu(\Theta_K, \Lambda)]' \otimes 1_n'$$

and

$$\begin{aligned}Cov(X \mid \Theta, \Lambda) = diag[&\rho\sigma^2(\Theta_i, \Lambda)W_i^{-\frac{1}{2}} \\ &+(1-\rho)\sigma^2(\Theta_i, \Lambda)diag(\omega_i^{-1}), i = 1,2,\ldots,K]\end{aligned}$$

thus

$$\begin{aligned}E[Cov(X \mid \Theta, \Lambda)] = diag[&\rho\sigma_1^2 W_i^{-\frac{1}{2}} + (1-\rho)\sigma_1^2 diag(\omega_i^{-1}), \\ &i = 1,2,\ldots,K]\end{aligned}$$

Then it follows as before

$$Cov[E(X \mid \Theta, \Lambda)] = \sigma_2^2 I_K \otimes 1_n 1_n' + a_1 1_{nK} 1_{nK}'$$

Therefore,

$$\begin{aligned}\Sigma_{XX} &= E[Cov(X \mid \Theta, \Lambda)] + Cov[E(X \mid \Theta, \Lambda)] \\ &= diag\begin{bmatrix}\rho\sigma_1^2 W_i^{-\frac{1}{2}} + (1-\rho)\sigma_1^2 diag(\omega_i^{-1}) \\ +\sigma_2^2 1_n 1_n', i = 1,2,\ldots,K\end{bmatrix} + a_1 1_{nK} 1_{nK}'.\end{aligned}$$

(iii) As part (iii) of Theorem 1.

(iv) Using (11) we can check that

$$B_i^{-1} = \left[\rho\sigma_1^2 W_i^{-\frac{1}{2}} + (1-\rho)\sigma_1^2 \mathrm{diag}(\omega_i^{-1})\right]^{-1} = \left[(1-\rho)\sigma_1^2 \mathrm{diag}(\omega_i^{-1}) + \rho\sigma_1^2 \omega_i^{-\frac{1}{2}}(\omega_i^{-\frac{1}{2}})'\right]^{-1}$$

$$= \frac{1}{(1-\rho)\sigma_1^2}\mathrm{diag}(\omega_i) - \frac{\frac{1}{(1-\rho)\sigma_1^2}\mathrm{diag}(\omega_i)\rho\sigma_1^2\omega_i^{-\frac{1}{2}}(\omega_i^{-\frac{1}{2}})'\frac{1}{(1-\rho)\sigma_1^2}\mathrm{diag}(\omega_i)}{1+(\omega_i^{-\frac{1}{2}})'\frac{1}{(1-\rho)\sigma_1^2}\mathrm{diag}(\omega_i)\rho\sigma_1^2\omega_i^{-\frac{1}{2}}} = \frac{1}{(1-\rho)\sigma_1^2}\left[\mathrm{diag}(\omega_i) - \frac{\rho}{1-\rho+n\rho}W_i^{-\frac{1}{2}}\right]$$

and hence (as in Wen *et al*. [23])

$$B_i^{-1}1_n = \frac{1}{(1-\rho)\sigma_1^2}\left(\omega_i - \frac{\rho\alpha_{ia}}{1-\rho+n\rho}\omega_i^{\frac{1}{2}}\right),\ 1_n' B_i^{-1}1_n = \frac{(1-\rho+n\rho)\alpha_i - \rho\alpha_{ia}^2}{\sigma_1^2(1-\rho)(1-\rho+n\rho)}$$

and

$$C_i^{-1} = [B_i + \sigma_2^2 1_n 1_n']^{-1},\ C_i^{-1}1_n = \left(\omega_i - \frac{\rho\alpha_{ia}}{1-\rho+n\rho}\omega_i^{\frac{1}{2}}\right)\gamma_i,\ 1_n' C_i^{-1}1_n = \left(\alpha_i - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\right)\gamma_i$$

where

$$\gamma_i = \frac{1-\rho+n\rho}{\sigma_1^2(1-\rho)(1-\rho+n\rho) + \sigma_2^2[(1-\rho+n\rho)\alpha_i + \rho\alpha_{ia}^2]}$$

Therefore

$$\Sigma_{XX}^{-1} = \{\mathrm{diag}[C_i, i=1,2,\ldots,K] + a_1 1_{nK} 1_{nK}'\}^{-1} = \mathrm{diag}[C_i^{-1}, i=1,2,\ldots,K] - \frac{a_1[C_1^{-1}1_n,\ldots,C_K^{-1}1_n]'[C_1^{-1}1_n,\ldots,C_K^{-1}1_n]}{1+a_1\sum_{i=1}^{K}1_n' C_i^{-1}1_n}$$

The lemma is thus proved.

Theorem 2: Under assumptions A1 to A3 and A4′ and the notations (3) and (14), the optimal credibility estimator of $\mu(\Theta_i, \Lambda)$, i=1, 2, …, K, is given by

$$\hat{\mu}(\Theta_i, \Lambda) = (z_1\bar{X}_i^{\omega} - z_2\bar{X}_i^{\omega_a}) + (z_3\bar{\bar{X}}^{\omega} - z_4\bar{\bar{X}}^{\omega_a}) + (1 - z_1 + z_2 - z_3 + z_4)\mu_1 \qquad (19)$$

where

$$z_1 = \sigma_2^2\gamma_i\alpha_i,\ z_2 = \frac{\rho\sigma_2^2\gamma_i\alpha_{ia}^2}{1-\rho+n\rho},\ z_3 = (a_1 - \varphi_i)(\sum_{i=1}^{K}\gamma_i\alpha_i),\quad z_4 = \frac{(a_1-\varphi_i)\rho}{1-\rho+n\rho}(\sum_{i=1}^{K}\gamma_i\alpha_{ia}^2),$$

$$\bar{X}_i^{\omega} = \frac{\sum_{j=1}^{n}\omega_{ij}X_{ij}}{\alpha_i},\ \bar{X}_i^{\omega_a} = \frac{\sum_{j=1}^{n}\sqrt{\omega_{ij}}X_{ij}}{\alpha_{ia}},\ \bar{\bar{X}}^{\omega} = \frac{\sum_{i=1}^{K}\gamma_i\alpha_i\bar{X}_i^{\omega}}{\sum_{i=1}^{K}\gamma_i\alpha_i},\ \bar{\bar{X}}^{\omega_a} = \frac{\sum_{i=1}^{K}\gamma_i\alpha_{ia}\bar{X}_i^{\omega_a}}{\sum_{i=1}^{K}\gamma_i\alpha_{ia}^2}$$

and

$$\varphi_i = \frac{a_1}{1+a_1\sum_{i=1}^{K}1_n' C_i^{-1}1_n}\left(a_1\sum_{i=1}^{K}1_n' C_i^{-1}1_n + \sigma_2^2 1_n' C_i^{-1}1_n\right)$$

Proof: Firstly, we need to get the following term.

$$\Sigma_{\mu(\Theta_i,\Lambda),X}\Sigma_{XX}^{-1} = [(\sigma_2^2 e_i' + a_1 1_K')\otimes 1_n'] \times \left\{\mathrm{diag}[C_i^{-1}, i=1,2,\ldots,K] - \frac{a_1[C_1^{-1}1_n,\ldots,C_K^{-1}1_n]'[C_1^{-1}1_n,\ldots,C_K^{-1}1_n]}{1+a_1\sum_{i=1}^{K}1_n' C_i^{-1}1_n}\right\}$$

$$= [a_1 1_n' C_1^{-1},\ldots,(a_1+\sigma_2^2)1_n' C_i^{-1},\ldots,a_1 1_n' C_K^{-1}] - \varphi_i[C_1^{-1}1_n,\ldots,C_K^{-1}1_n]$$

Using Lemma 2 and Lemma 3,

$$\hat{\mu}(\Theta_i,\Lambda) = \mathrm{proj}(\mu(\Theta_i,\Lambda)|X,1) = E[\mu(\Theta_i,\Lambda)] + \Sigma_{\mu(\Theta_i,\Lambda),X}\Sigma_{XX}^{-1}[X - E(X)]$$

$$= \mu_1 + \left\{[a_1 1_n' C_1^{-1},\ldots,(a_1+\sigma_2^2)1_n' C_i^{-1},\ldots,a_1 1_n' C_K^{-1}] - \varphi_i[C_1^{-1}1_n,\ldots,C_K^{-1}1_n]\right\}(X-\mu_1 1_{nk})$$

$$= \mu_1 + [a_1 1_n' C_1^{-1},\ldots,(a_1+\sigma_2^2)1_n' C_i^{-1},\ldots,a_1 1_n' C_K^{-1}](X-\mu_1 1_{nk}) - \varphi_i[C_1^{-1}1_n,\ldots,C_K^{-1}1_n](X-\mu_1 1_{nk})$$

$$= \mu_1 + a_1\sum_{i=1}^{K}\gamma_i\left[\alpha_i\bar{X}_i^{\omega} - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\bar{X}_i^{\omega_a} - \left(\alpha_i - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\right)\mu_1\right] - \varphi_i\sum_{i=1}^{K}\gamma_i\left[\alpha_i\bar{X}_i^{\omega} - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\bar{X}_i^{\omega_a} - \left(\alpha_i - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\right)\mu_1\right]$$

$$+\sigma_2^2\gamma_i\left[\alpha_i\bar{X}_i^{\omega} - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\bar{X}_i^{\omega_a} - \left(\alpha_i - \frac{\rho\alpha_{ia}^2}{1-\rho+n\rho}\right)\mu_1\right]$$

$$= \mu_1 + (a_1-\varphi_i)\times\left[\left(\sum_{i=1}^{K}\gamma_i\alpha_i\right)\bar{\bar{X}}^{\omega} - \frac{\rho\sum_{i=1}^{K}\gamma_i\alpha_{ia}^2}{1-\rho+n\rho}\bar{\bar{X}}^{\omega_a} - \left(\sum_{i=1}^{K}\gamma_i\alpha_i - \frac{\rho\sum_{i=1}^{K}\gamma_i\alpha_{ia}^2}{1-\rho+n\rho}\right)\mu_1\right] + \sigma_2^2\gamma_i\alpha_i\bar{X}_i^{\omega} - \frac{\rho\sigma_2^2\gamma_i\alpha_{ia}^2}{1-\rho+n\rho}\bar{X}_i^{\omega_a} - \left(\sigma_2^2\gamma_i\alpha_i - \frac{\rho\sigma_2^2\gamma_i\alpha_{ia}^2}{1-\rho+n\rho}\right)\mu_1$$

$$= (z_1\bar{X}_i^{\omega} - z_2\bar{X}_i^{\omega_a}) + (z_3\bar{\bar{X}}^{\omega} - z_4\bar{\bar{X}}^{\omega_a}) + (1 - z_1 + z_2 - z_3 + z_4)\mu_1$$

NUMERICAL EXAMPLE

In order to show the influence of the error uniform dependence on credibility models with common effects, we simulate some generated claims data to examine what effect there might be from assuming error uniform dependence, using the two-level common effects framework suggested for Bühlmann model. We compare it with the case of two-level common effects model regardless of the effect of the error uniform dependence.

In this section, we briefly describe the nature of the assumptions used to draw numerical results. First, we generated the observations assuming that assumptions of the two-level common effects model with the error uniform dependence hold in reality and then compare the results based on two different models: the two level common effects model with both considering the effect of the error uniform dependence and regardless of the effect of the error uniform dependence. In the meantime, we use the Normal distribution assumptions of the common effects in both situations models.

A summary of the specification, description, as well as the parameter values used in the simulation is found in Table 1.

We generate R = 1000 different 10-year paths of claims for 10 different individuals assuming the two-level common effects model with the error uniform dependence is the true model. We label the two-level Normal common effects model with considering the effect of the error uniform dependence as Model I and the two-level Normal common effects model regardless of the effect of the error uniform dependence as Model II. For each one of sample paths of claims from 10 individuals, we computed the credibility premium for year 11 (the next period) for individual 1.

Table 1: Summary of model assumptions and parameters used in simulation

Specification	Description
	for i = 1,2,…K and t = 1,2,…,n
Conditional density	$X_{i,t}\|\theta_i,\lambda = N(\theta_i+\lambda, s_1^2)$,
'Individual' common effect	$\theta_I = N(\mu_\theta, s_2^2)$, for i = 1,2,…,K
'Overall' common effect	$\lambda_k = N(\mu_\lambda, s_\lambda^2)$
Assumption	K = 10 Individuals, n = 10 years
Parameter values	$s_1^2 = 6000$
	$\mu_\theta = 100$, $s_2^2 = 1000$
	$\mu_\lambda = 200$, $s_\lambda^2 = 4000$
	$\rho = Corr(X_{ij}, X_{im}\|\theta_i, \lambda) = 0.7$

For Model I, we use the formula presented in Theorem 1 for Bühlmann model which summarized below:

$$\hat{\mu}(\Theta_i,\Lambda) = z_1\bar{X}_i - z_2\bar{\bar{X}} + (1 - z_1 + z_2)\mu_1$$

where

$$z_1 = \frac{n\sigma_2^2}{S_1 + nS_2} = 0.1858736$$

and

$$z_2 = \frac{nKa_1(S_1 + nS_2 - n\sigma_2^2)}{(S_1 + nS_2)(S_1 + nS_2 + nKa_1)} = 0.7176081$$

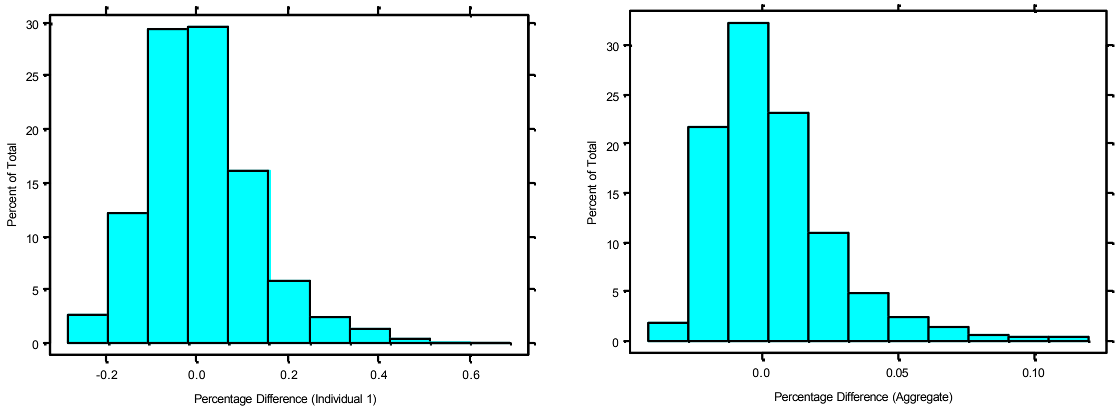
The credibility premium for Model II is well-known and it is given by

$$\hat{\mu}(\Theta_i,\Lambda) = w_1\bar{X}_i - w_2\bar{\bar{X}} + (1 - w_1 + w_2)\mu_1$$

$$w_1 = \frac{n\sigma_2^2}{\sigma_1^2 + n\sigma_2^2} = 0.625$$

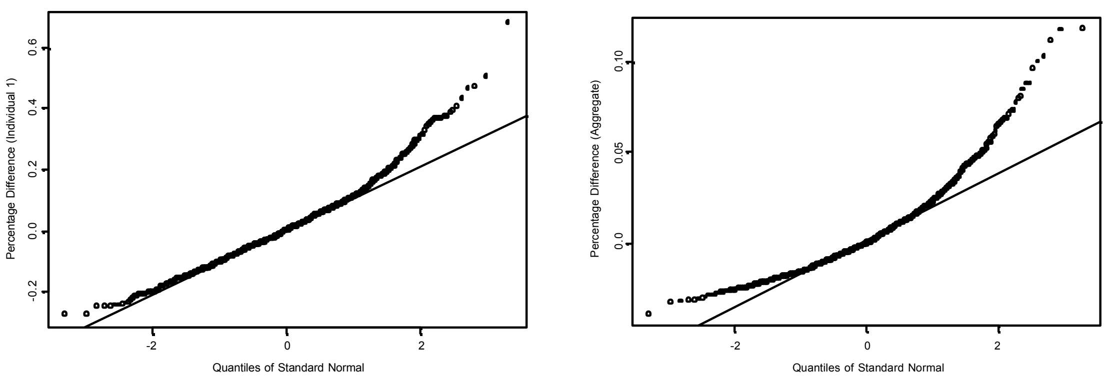
and

$$z_2 = \frac{nKa_1\sigma_1^2}{(\sigma_1^2 + n\sigma_2^2)(\sigma_1^2 + n\sigma_2^2 + nKa_1)} = 0.3605769$$



(a) (b)

Fig. 1: (a) Histogram of the percentage premium differences for individual 1. (b) Histogram of the percentage premium differences for the aggregated portfolio



(a) (b)

Fig. 2: (a) A Normal Q.Q plot for individual 1. (b) A Normal Q.Q plot for the aggregate portfolio

See [19] for discussion of these credibility formulas for Model II.

As a convention, we shall use $\mu^{I}_{j,n+1}$ and $\mu^{II}_{j,n+1}$ to denote the credibility premiums in Model I and Model II, respectively. Now for comparison purposes, we then compute the percentage difference in credibility premiums for individual j between these two models as follows:

$$\Delta_j = \frac{\mu^{I}_{j,n+1} - \mu^{II}_{j,n+1}}{\mu^{I}_{j,n+1}} \times 100$$

For each simulation, we can compute this percentage premium difference and examine the resulted distribution of these premium differences for the entire 1000 simulations. In order not to overwhelm the reader with lots of statistics, we chose to present the results only in terms of individual 1 and the aggregate of all the 10 individuals. The aggregate percentage difference has been computed using:

$$\Delta = \frac{\sum_{j=1}^{K} \mu^{I}_{j,n+1} - \sum_{j=1}^{K} \mu^{II}_{j,n+1}}{\sum_{j=1}^{K} \mu^{I}_{j,n+1}} \times 100$$

In [21], they used Δ_j and Δ for comparing different models.

Some summary of the resulting percentage differences are given in Table 2. Figure 1 provides the histogram of the percentage difference for the case of individual 1 only and also for the aggregate i.e, sum of all the individuals. We also show in Fig. 2 the Normal Q-Q plots of these respective distributions to show the skewness, or the non-symmetry observed from these resulting differences.

Assuming the premium calculated based on the two-level common effects model with error uniform dependent, the two-level common effects Normal model regardless of the effect of the error uniform dependence tends to understate the credibility premium from its true value. This is also evident in the Figures and statistics.

Table 2: Some descriptive statistics of the percentage difference between the credibility premiums in Models I and Model II

Statistics	Individual 1	Aggregate
Mean (%)	1.0802	0.3898
Median (%)	0.0994	-0.0670
Variance ($\%^2$)	142.9794	4.9171
Standard deviation (%)	11.9574	2.2175
Minimum (%)	-27.3317	-3.9713
Maximum (%)	67.9023	11.7920

Table 3: Detailed computation for different ρ_s

ρ	0.00	0.01	…	0.98	0.99
Average of Δ_1 (individual 1)	0.00	0.0395	…	0.9750	1.0292
Average of Δ (aggregate)	0.00	0.0084	…	0.6292	0.4583

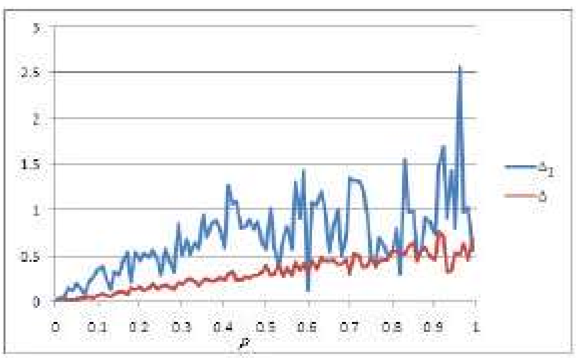


Fig. 3: The plot of the influence of different ρ_s for the case of individual 1 and also for the aggregate

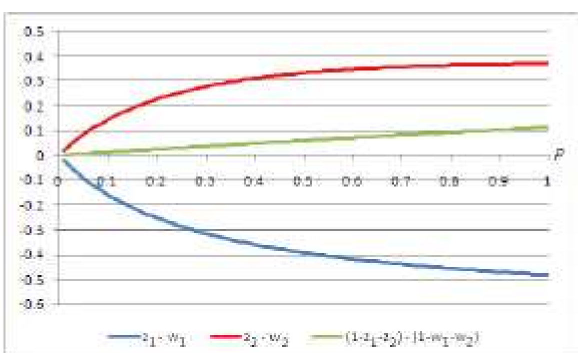


Fig. 4: The plot of the influence of different ρ_s on z1-w1, z2-w2 and (1-z1-z2)-(1-w1-w2)

For the purpose of numerical illustration to show the influence of different ρ_s, we generate claims as previous method for different ρ_s. The summary of the results are given in Table 3.

Figure 3 provides the plot of the influence of different ρ_s for the case of individual 1 only and also for the aggregate. We also show the plots of the differences of weights between these two models for different ρ_s in Fig. 4.

CONCLUSION

In classical Bühlmann credibility models, there are some situations that could drive not only possible relationship among the risks but also certain conditional dependence over time which has been recognized as more appropriate to fit the practice in some circumstances. In this paper, we extended the Bühlmann and Bühlmann-Straub credibility models to account for not only a certain uniform conditional dependence for claim amounts each of individual risk, but also a special type of dependence between risks induced by common effects. We further gave illustrative examples to show the influence of the error uniform dependence when these common effects have Normal distributions.

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