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A New Non-Tensor Product C^1 Subdivision Scheme for Regular Quad Meshes

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Abstract: This paper presents a new approach to construct a non-tensor product C^1 subdivision scheme for quadrilateral meshes. The approach is based on a quadratic function in $\mathbb{R}^2$ and suggest a methodology for producing a new class of subdivision schemes.

MSC: 65D17 · 65D07 · 65D05
Key words: Subdivision scheme · Continuity · Control mesh · Eigenvalues

INTRODUCTION

Computer Aided Geometric Design (CAGD) is a branch of applied mathematics, which deals with algorithms for free-form curves, surfaces and volumes. It plays a significant role in integrating computers and industry. Smooth curves and surfaces have pivotal importance in the field of air craft manufacturing, movie animation, computer game character design and general product design. The present work focuses on subdivision schemes that operate on a mesh of control points, $P=\{p_{i,j}\}_{i,j\in\mathbb{Z}}$. Subdivision surfaces are used to create new edges and faces within the mesh by complying with insertion rules. All subdivision schemes are iterative and contractive generating a new smaller mesh after each iteration. It is possible to generate a set of meshes $p^1, p^2, \ldots$ by continually applying subdivision rules on the previous mesh. Then the limits of these meshes converge to the mesh P^∞, called the limit surface. This technique of recursive subdivision can be visualised, loosely, as successively trimming the corners of a polyhedron to make them less pointed. In 1978, Catmull-Clark [1] and Doo-Sabin [2] first introduced subdivision surface schemes, which generalised the tensor product of bicubic and biquadratic B-splines respectively. For the univariate case, the binary cubic box spline, which has two insertion rules (stencils), is given as

$$p_{2i}^{k+1}=\frac{4}{8}p_{i-1}^{k}+\frac{4}{8}p_{i}^{k},\ p_{2i+1}^{k+1}=\frac{1}{8}p_{i-1}^{k}+\frac{6}{8}p_{i}^{k}+\frac{1}{8}p_{i+1}^{k}. \quad (1.1)$$

This leads to the Laurent polynomial $\left(1+4z+6z^2+4z^3+z^4\right)/8=2\left((1+z)/2\right)^4$ [14]. In the bivariate case, it is simply the tensor product of the univariate subdivision scheme, as the principle directions are orthogonal to each other. So the cubic product bivariate box spline subdivision schemes can be generated by (1.1) given as

$$2\left(\frac{1+z_1}{2}\right)^4 2\left(\frac{1+z_2}{2}\right)^4. \quad (1.2)$$

The expression (1.2) can be written in z-transform, increasing the power of z_1 in the horizontal direction and z_2 in the vertical direction with the mask coefficients, each divided by 64

$$\begin{array}{cccccccccc}
 & 1z_1^0z_2^4 & + & 4z_1^1z_2^4 & + & 6z_1^2z_2^4 & + & 4z_1^3z_2^4 & + & 1z_1^4z_2^4\\
+ & 4z_1^0z_2^3 & + & 16z_1^1z_2^3 & + & 24z_1^2z_2^3 & + & 16z_1^3z_2^3 & + & 4z_1^4z_2^3\\
+ & 6z_1^0z_2^2 & + & 24z_1^1z_2^2 & + & 36z_1^2z_2^2 & + & 24z_1^3z_2^2 & + & 6z_1^4z_2^2\\
+ & 4z_1^0z_2^1 & + & 16z_1^1z_2^1 & + & 24z_1^2z_2^1 & + & 16z_1^3z_2^1 & + & 4z_1^4z_2^1\\
+ & 1z_1^0z_2^0 & + & 4z_1^1z_2^0 & + & 6z_1^2z_2^0 & + & 4z_1^3z_2^0 & + & 1z_1^4z_2^0.
\end{array} \quad (1.3)$$

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Deriving the four stencils, $p^{k+1}_{2i+l,2j+m}$ from the above z-transform (1.3), where

$$l+m=\begin{cases}0, & \text{corresponds to a } \textit{vertex} \text{ point,}\\ 1, & \text{corresponds to an } \textit{edge} \text{ point,}\\ 2, & \text{corresponds to a } \textit{face} \text{ point.}\end{cases} \qquad l,m=0,1.$$

We have

$$\begin{cases} p^{k+1}_{2i,2j}=\frac{1}{64}\left\{p^k_{i-1,j-1}+p^k_{i-1,j+1}+p^k_{i+1,j-1}+p^k_{i+1,j+1}\right\}+\frac{6}{64}\left\{p^k_{i-1,j}+p^k_{i,j-1}\right. \\ \left.+p^k_{i,j+1}+p^k_{i+1,j}\right\}+\frac{36}{64}p^k_{i,j}, \\ p^{k+1}_{2i+1,2j}=\frac{4}{64}\left\{p^k_{i,j-1}+p^k_{i,j+1}+p^k_{i+1,j-1}+p^k_{i+1,j+1}\right\}+\frac{24}{64}\left\{p^k_{i,j}+p^k_{i+1,j}\right\}, \\ p^{k+1}_{2i,2j+1}=\frac{4}{64}\left\{p^k_{i-1,j-1}+p^k_{i-1,j}+p^k_{i+1,j-1}+p^k_{i+1,j}\right\}+\frac{24}{64}\left\{p^k_{i,j-1}+p^k_{i,j}\right\}, \\ p^{k+1}_{2i+1,2j+1}=\frac{16}{64}\left\{p^k_{i,j-1}+p^k_{i,j}+p^k_{i+1,j-1}+p^k_{i+1,j}\right\}. \end{cases} \tag{1.4}$$

The continuity of tensor product bivariate box spline subdivision schemes [15] has been derived from the z-transform of the scheme. Laurent polynomial analysis [14] shows that the order of continuity of a tensor product scheme is the same as for the univariate case. A general formula of z-transform for N-ary multivariate box spline subdivision scheme for C^m-continuity is given by:

$$N^{2^{s-1}(m+1)}\prod_{j=1}^{s}\left(\frac{1-z_j^N}{1-z_j}\right)^{m+2} \tag{1.5}$$

where s=1 is univariate case and s=2 the bivariate case. In the case $N = m = 2$, by (1.5) we get the cubic box spline (1.2). In expression(1.5), using Laurent polynomial analysis [14], the order of continuity of the subdivision scheme is two fewer than the exponent of $\left(\frac{1-z_j^N}{1-z_j}\right)$.

From (1.5) the order of continuity is simply m. In the case of a binary cubic box spline, it is C^2 for both univariate and bivariate cases. Since Catmull-Clark and Doo-Sabin [1, 2], many subdivision schemes have been proposed, including triangle mesh schemes [3, 4, 8, 9], quad mesh schemes [1, 2, 5, 6, 10, 11] and combined triangle-quad (tri-quad) mesh schemes [12, 13]. This is not a definitive list of tensor product schemes, however all subdivision algorithms, both tensor-product and univariate forms, are the result of modifications and convolvements of the mask of existing schemes. It is interesting to present a new non-tensor product generating insertion rules, which are for quad meshes and independent of all the previous methods. The work in this paper is based on regular quad meshes (all vertices have valence 4) and gives four insertion rules such that each vertex is shared by four quadrilaterals. Due to the nature of the refinement rules, tensor product schemes naturally lead to quad meshes. The proposed method also leads to quad mesh but its generating method is different from tensor product schemes and hence will generate a new class of subdivision scheme. The work is organized as follows:

In Section 2, some basic definitions and preliminary concepts are reviewed and discussed that will be used in our work. Construction of the subdivision insertion rules is presented in Section 3. In Section 4, the scheme is analyzed via its eigenvalues and its continuity in the limit is established to be C^1. Conclusions and summary of the results including future research directions are discussed in the last section.

Preliminaries and Basic Concepts: In this section, some basic notation and concepts regarding bivariate subdivision schemes defined on a regular quad mesh are presented.

Given a mesh of control points $p^k_{i,j}\in\mathbb{R}^N, i,j\in\mathbb{Z}, N\geqslant 2$, $p^k_{i,j}=\left(x^k_{i,j},y^k_{i,j},z^k_{i,j}\right)$ where $p^k_{i,j}=\left(x^k_{i,j},y^k_{i,j},z^k_{i,j}\right)$ and $k\geq 0$ indicates the subdivision level. In the bivariate case, consider the four subdivision rules for a quad mesh

$$p^{k+1}_{2i+\alpha,2j+\beta}=\sum_{r,s}a_{2r+\alpha,2s+\beta}\,p^k_{i-r,j-s}, \quad \alpha,\beta=0,1, \tag{2.1}$$

where a necessary condition for the convergence of the subdivision process for arbitrary initial data is that

$$\sum_{r,s}a_{2r+\alpha,2s+\beta}=1, \quad \alpha,\beta=0,1. \tag{2.2}$$

Given initial values $p^0_{i,j}\in\mathbb{R}^N, i,j\in\mathbb{Z}$, then in the limit $k\to\infty$ the process defines an infinite set of points in $\mathbb{R}^N$. The sequence of values $\{p^k_{i,j}\}$ is related, in a natural way, with the diadic mesh points $\left(\frac{i}{2^k},\frac{j}{2^k}\right), i,j\in\mathbb{Z}$. The process (2.1) then defines a scheme whereby $p^{k+1}_{2i+l,2j+m}$ replaces the values $p^k_{i+l,j+m}$ at the mesh points $\left(\frac{i+l}{2^k},\frac{j+m}{2^k}\right)$ ($(l,m=0,1)$) respectively. The values $p^{k+1}_{2i,2j+1}, p^{k+1}_{2i+1,2j+1}$, $p^{k+1}_{2i+2,2j+1}$ and $p^{k+1}_{2i+1,2j+2}$ are inserted at the new mesh points $\left(\frac{i+1}{2^{k+1}},\frac{j}{2^{k+1}}\right),\left(\frac{i}{2^{k+1}},\frac{j+1}{2^{k+1}}\right),\left(\frac{i+1}{2^{k+1}},\frac{j+1}{2^{k+1}}\right),\left(\frac{i+2}{2^{k+1}},\frac{j+1}{2^{k+1}}\right)$

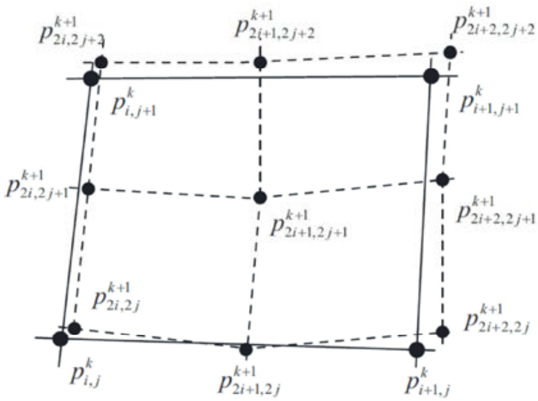


Fig. 1: Solid lines show one face of coarse polygon; doted lines are refined polygon

and $\left(\frac{i+1}{2^{k+1}}, \frac{j+2}{2^{k+1}}\right)$ respectively. *Figure* 1 gives the structure of the subdivision scheme (2.1) along with the labelling scheme for new and old points.

Smoothness analysis for regular quadrilateral meshes follows directly from the univariate case. This is because, in general, the tensor product has the same order of continuity as for the univariate case. So for a non-tensor product scheme, Reif's [7] sufficient condition for smoothness of a stationary subdivision scheme can be used. This method applies to the matrix of the subdivision algorithm, which relates the values on the localization set of the topological 1-neighbourhood of zero in the k-regular complex to the values on the similar neighborhood on the next subdivision level. If the number of initial values is equal to the value of an eigenvector of the subdivision matrix, the subdivision will produce a limit function, which is called the eigenbasis function.

In order to guarantee affine invariance of the subdivision algorithm, the sum of each row of the subdivision matrix must be 1. Thus 1 is always an eigenvalue of a subdivision matrix with associated eigenvector [1,....,1]. Reif's sufficient criterion for a subdivision algorithm to generate a smooth limit surface is given as:

Suppose the eigenvectors of a subdivision matrix form the basis, the largest three eigenvalues except $1 = \lambda_0$, are real and satisfy

$$\lambda := \lambda_1 = \lambda_2, \quad 1 > \lambda > |\lambda_3| \tag{2.3}$$

If the characteristic map is regular and injective, then the limit surface P^∞ is a regular C^1-manifold for almost every choice of initial data.

Scheme Construction: The construction of the non-tensor product scheme for quad meshes is based on a general quadratic polynomial function $f \in R^2$. In this case, an insertion rule can be obtained by interpolation with a function from the space spanned by $\{1, x, y, xy, x^2, y^2\}$.

Consider the Function:

$$f(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \tag{3.1}$$

such that the set of mesh points $p^k_{i,j}, i,j \in \mathbb{Z}$ at kth level satisfies $f\left(\frac{i}{2^k}, \frac{j}{2^k}\right) = p^k_{i,j}$. Now interpolating the data points $p^k_{i+h,j+r} = f\left(\frac{ht}{2^k}, \frac{rt}{2^k}\right), h,r = -1,0,1$ in (3.1), results in the following system of equations:

$$\begin{cases} f\left(\frac{-t}{2^k}, \frac{-t}{2^k}\right) = p^k_{i-1,j-1} = A\frac{t^2}{2^{2k}} + B\frac{t^2}{2^{2k}} + C\frac{t^2}{2^{2k}} - D\frac{t}{2^k} - E\frac{t}{2^k} + F, \\ f\left(\frac{-t}{2^k}, \frac{0}{2^k}\right) = p^k_{i-1,j} = A\frac{t^2}{2^{2k}} - D\frac{t}{2^k} + F, \\ f\left(\frac{-t}{2^k}, \frac{t}{2^k}\right) = p^k_{i-1,j+1} = A\frac{t^2}{2^{2k}} - B\frac{t^2}{2^{2k}} + C\frac{t^2}{2^{2k}} - D\frac{t}{2^k} + E\frac{t}{2^k} + F, \\ f\left(\frac{0}{2^k}, \frac{-t}{2^k}\right) = p^k_{i,j-1} = C\frac{t^2}{2^{2k}} - E\frac{t}{2^k} + F, \\ f\left(\frac{0}{2^k}, \frac{0}{2^k}\right) = p^k_{i,j} = F, \\ f\left(\frac{0}{2^k}, \frac{t}{2^k}\right) = p^k_{i,j+1} = C\frac{t^2}{2^{2k}} + E\frac{t}{2^k} + F, \\ f\left(\frac{t}{2^k}, \frac{-t}{2^k}\right) = p^k_{i+1,j-1} = A\frac{t^2}{2^{2k}} - B\frac{t^2}{2^{2k}} + C\frac{t^2}{2^{2k}} + D\frac{t}{2^k} - E\frac{t}{2^k} + F, \\ f\left(\frac{t}{2^k}, \frac{0}{2^k}\right) = p^k_{i+1,j} = A\frac{t^2}{2^{2k}} + D\frac{t}{2^k} + F, \\ f\left(\frac{t}{2^k}, \frac{t}{2^k}\right) = p^k_{i+1,j+1} = A\frac{t^2}{2^{2k}} + B\frac{t^2}{2^{2k}} + C\frac{t^2}{2^{2k}} + D\frac{t}{2^k} + E\frac{t}{2^k} + F. \end{cases} \tag{3.2}$$

To obtain the four stencils, first solve the above system of equations for the coefficients given in (3.1) and compute the value of the interpolation function $f(x,y)$ at the grid points $\left(\frac{0\cdot t}{2^{k+1}}, \frac{0\cdot t}{2^{k+1}}\right), \left(\frac{t}{2^{k+1}}, \frac{0\cdot t}{2^{k+1}}\right), \left(\frac{0\cdot t}{2^{k+1}}, \frac{t}{2^{k+1}}\right)$ and $\left(\frac{t}{2^{k+1}}, \frac{t}{2^{k+1}}\right)$. The coefficients of $f(x,y)$ are

$$A=\frac{2^{2k}}{6t^2}\begin{cases}-2(p_{i,j-1}^k+p_{i,j}^k+p_{i,j+1}^k)+p_{i+1,j-1}^k+p_{i-1,j-1}^k\\+p_{i-1,j}^k+p_{i-1,j+1}^k\end{cases}$$

$$+p_{i+1,j+1}^k+p_{i+1,j}^k\Big\},$$

$$B=\frac{2^{2k}}{4t^2}\left\{p_{i-1,j-1}^k-p_{i+1,j+1}^k-p_{i+1,j-1}^k+p_{i+1,j+1}^k\right\},$$

$$C=\frac{2^{2k}}{6t^2}\left\{-2(p_{i,j}^k+p_{i-1,j}^k+p_{i+1,j}^k)+p_{i,j-1}^k+p_{i,j+1}^k+p_{i+1,j-1}^k+p_{i-1,j-1}^k+p_{i-1,j+1}^k+p_{i+1,j+1}^k\right\},$$

$$D=-\frac{2^k}{6t}\left\{p_{i-1,j-1}^k+p_{i-1,j}^k+p_{i-1,j+1}^k-p_{i+1,j-1}^k-p_{i+1,j}^k-p_{i+1,j+1}^k\right\},$$

$$E=-\frac{2^k}{6t}\left\{p_{i-1,j-1}^k-p_{i-1,j}^k+p_{i,j-1}^k-p_{i,j+1}^k+p_{i+1,j-1}^k-p_{i+1,j+1}^k\right\},$$

$$F=\frac{2}{9}\left\{p_{i,j-1}^k+p_{i,j+1}^k+p_{i-1,j}^k+p_{i+1,j}^k\right\}$$

$$-\frac{1}{9}\left\{p_{i+1,j-1}^k+p_{i-1,j-1}^k+p_{i-1,j+1}^k+p_{i+1,j+1}^k\right\}+\frac{5}{9}p_{i,j}^k.$$

Thus, the new subdivided points $p_{2i,2j}^{k+1}, p_{2i+1,2j}^{k+1}, p_{2i,2j+1}^{k+1}, p_{2i+1,2j+1}^{k+1}$ are a linear combination of the nine support points $p_{i-1,j-1}^k, p_{i-1,j}^k, p_{i-1,j+1}^k, p_{i,j-1}^k, p_{i,j}^k, p_{i,j+1}^k, p_{i+1,j-1}^k, p_{i+1,j}^k, p_{i+1,j+1}^k$ given in (3.2). The resulting four insertion rules for the $(k+1)^{th}$ level are given by:

$$\begin{cases}p_{2i,2j}^{k+1}=\frac{1}{144}\left\{-16p_{i-1,j-1}^k+32p_{i-1,j}^k-16p_{i-1,j+1}^k+32p_{i,j-1}^k+80p_{i,j}^k\right.\\ \left.+32p_{i,j+1}^k-16p_{i+1,j-1}^k+32p_{i+1,j}^k-16p_{i+1,j+1}^k\right\},\\ p_{2i+1,2j}^{k+1}=\frac{1}{144}\left\{-22p_{i-1,j-1}^k+26p_{i-1,j}^k-22p_{i-1,j+1}^k+20p_{i,j-1}^k+68p_{i,j}^k\right.\\ \left.+20p_{i,j+1}^k+2p_{i+1,j-1}^k+50p_{i+1,j}^k+2p_{i+1,j+1}^k\right\},\\ p_{2i,2j+1}^{k+1}=\frac{1}{144}\left\{-22p_{i-1,j-1}^k+20p_{i-1,j}^k+2p_{i-1,j+1}^k+26p_{i,j-1}^k+68p_{i,j}^k\right.\\ \left.+50p_{i,j+1}^k-22p_{i+1,j-1}^k+20p_{i+1,j}^k+2p_{i+1,j+1}^k\right\},\\ p_{2i+1,2j+1}^{k+1}=\frac{1}{144}\left\{-19p_{i-1,j-1}^k+14p_{i-1,j}^k-13p_{i-1,j+1}^k+14p_{i,j-1}^k+56p_{i,j}^k\right.\\ \left.+38p_{i,j+1}^k-13p_{i+1,j-1}^k+38p_{i+1,j}^k+29p_{i+1,j+1}^k\right\}.\end{cases}$$

The masks for each of the four insertion rules are shown in *Figure* 2, where $p_{2i,2j}^{k+1}$ is called the *vertex* rule, $p_{2i+1,2j}^{k+1}, p_{2i,2j+1}^{k+1}$ are called *edge* rules and $p_{2i+1,2j+1}^{k+1}$ is the *face* rule.

Smoothness Analysis: In order to determine the continuity of the proposed scheme, it is reformulated in terms of subdivision matrices. This enables the eigenvalues to be determined and hence compared to (2.3). To calculate the eigenvalues of the proposed subdivision scheme, consider the polygonal mesh of the sixteen control points of the new mesh $\mathcal{P}_{k+1}=p_{2i+l,2j+m}^{k+1}$ corresponding to old mesh $\mathcal{P}_k=p_{i+l,j+m}^k$, where $l,m=-2,-1,0,1$ (Figure 3). The following matrix expression is showing operation to generate new control mesh $\mathcal{P}_{k+1}$.

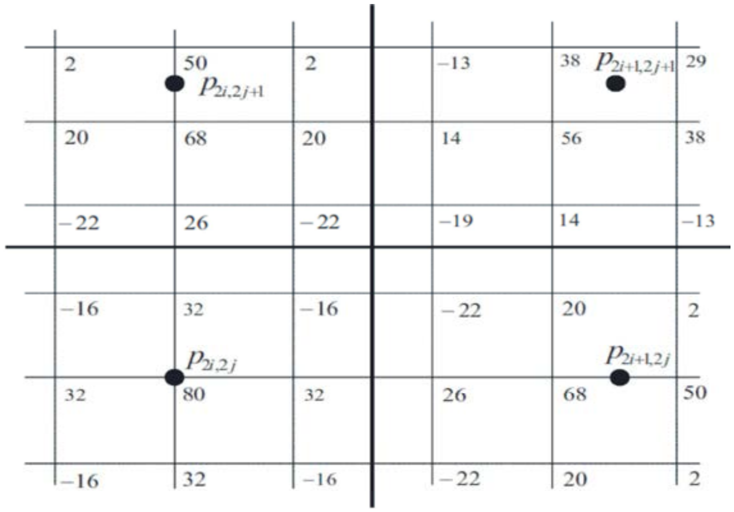


Fig. 2: Mask configuration of the scheme (3.3) using nine grid points.

Combining the four masks that generate the new vertices, edges and face points into a single matrix operator, $[M]$ enables the subdivision scheme (3.3) at level k to be expressed as:

$$[\mathcal{M}][\mathcal{P}^k]=[\mathcal{P}^{k+1}], \tag{4.1}$$

where

$$\frac{1}{144}\begin{pmatrix}
-16 & 31 & -16 & 0 & 32 & 80 & 32 & 0 & -16 & 32 & -16 & 0 & 0 & 0 & 0 & 0\\
-22 & 20 & 2 & 0 & 26 & 68 & 50 & 0 & -22 & 20 & 2 & 0 & 0 & 0 & 0 & 0\\
0 & -16 & 32 & -16 & 0 & 32 & 80 & 32 & 0 & -16 & 32 & -16 & 0 & 0 & 0 & 0\\
0 & -22 & 20 & 2 & 0 & 26 & 68 & 50 & 0 & -22 & 20 & 2 & 0 & 0 & 0 & 0\\
-22 & 26 & -22 & 0 & 20 & 68 & 20 & 0 & 2 & 50 & 2 & 0 & 0 & 0 & 0 & 0\\
-19 & 14 & -13 & 0 & 14 & 56 & 38 & 0 & -13 & 38 & 29 & 0 & 0 & 0 & 0 & 0\\
0 & -22 & 26 & -22 & 0 & 20 & 68 & 20 & 0 & 2 & 50 & 2 & 0 & 0 & 0 & 0\\
0 & -19 & 14 & -13 & 0 & 14 & 56 & 38 & 0 & -13 & 38 & 29 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -16 & 32 & -16 & 0 & 32 & 80 & 32 & 0 & -16 & 32 & -16 & 0\\
0 & 0 & 0 & 0 & -22 & 20 & 2 & 0 & 26 & 68 & 50 & 0 & -22 & 20 & 2 & 0\\
0 & 0 & 0 & 0 & 0 & -16 & 32 & -16 & 0 & 32 & 80 & 32 & 0 & -16 & 32 & -16\\
0 & 0 & 0 & 0 & 0 & -22 & 20 & 2 & 0 & 26 & 68 & 50 & 0 & -22 & 20 & 2\\
0 & 0 & 0 & 0 & -22 & 26 & -22 & 0 & 20 & 68 & 20 & 0 & 2 & 50 & 2 & 0\\
0 & 0 & 0 & 0 & -19 & 14 & -13 & 0 & 14 & 56 & 38 & 0 & -13 & 38 & 29 & 0\\
0 & 0 & 0 & 0 & 0 & -22 & 26 & -22 & 0 & 20 & 68 & 20 & 0 & 2 & 50 & 2\\
0 & 0 & 0 & 0 & 0 & -19 & 14 & -13 & 0 & 14 & 56 & 38 & 0 & -13 & 38 & 29
\end{pmatrix}
[\mathcal{P}^k]=\begin{pmatrix}
p^k_{i-2,j-2}\\ p^k_{i-1,j-2}\\ p^k_{i,j-2}\\ p^k_{i+1,j-2}\\
p^k_{i-2,j-1}\\ p^k_{i-1,j-1}\\ p^k_{i,j-1}\\ p^k_{i+1,j-1}\\
p^k_{i-2,j}\\ p^k_{i-1,j}\\ p^k_{i,j}\\ p^k_{i+1,j}\\
p^k_{i-2,j+1}\\ p^k_{i-1,j+1}\\ p^k_{i,j+1}\\ p^k_{i+1,j+1}
\end{pmatrix}
[\mathcal{P}^{k+1}]=\begin{pmatrix}
p^{k+1}_{2i-2,2j-2}\\ p^{k+1}_{2i-1,2j-2}\\ p^{k+1}_{2i,2j-2}\\ p^{k+1}_{2i+1,2j-2}\\
p^{k+1}_{2i-2,2j-1}\\ p^{k+1}_{2i-1,2j-1}\\ p^{k+1}_{2i,2j-1}\\ p^{k+1}_{2i+1,2j-1}\\
p^{k+1}_{2i-2,2j}\\ p^{k+1}_{2i-1,2j}\\ p^{k+1}_{2i,2j}\\ p^{k+1}_{2i+1,2j}\\
p^{k+1}_{2i-2,2j+1}\\ p^{k+1}_{2i-1,2j+1}\\ p^{k+1}_{2i,2j+1}\\ p^{k+1}_{2i+1,2j+1}
\end{pmatrix}$$

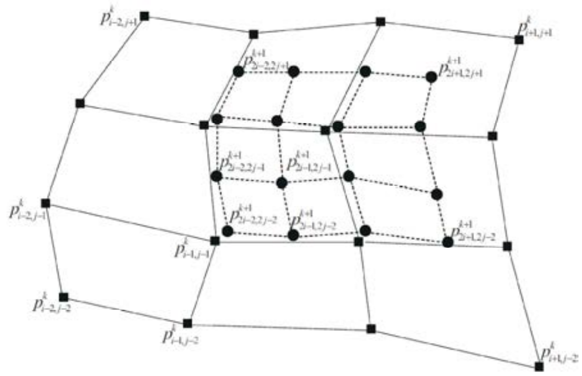
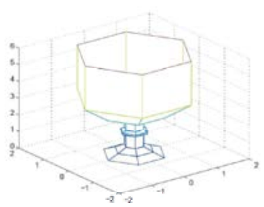
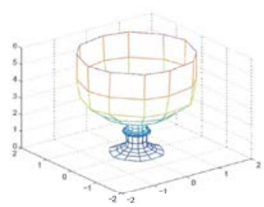


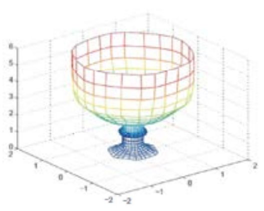
Fig. 3: Configuration of new mesh (round dots) corresponding to the old mesh (square dots).



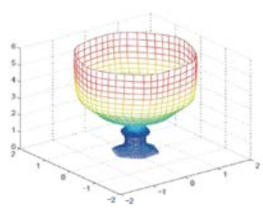
(a) Level 0(Control mesh)



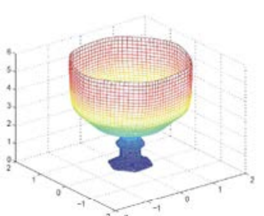
(b) Level 1



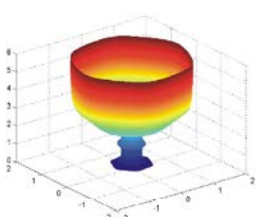
(c) Level 2



(d) Level 3



(e) Level



(f) Level

Fig. 4: Results of proposed non-tensor product scheme (3.3).

The polygonal mesh of new sixteen points $\mathcal{P}^{k+1} = p^{k+1}_{2i+l,2j+m}$ corresponding to old mesh $\mathcal{P}^{k} = p^{k}_{i+l,j+m}$, where $l,m = -2,-1,0,1$, is illustrated in *Figure* 3.

According to Reif's condition (2.3) the eigenvalues of [M] determine the continuity of the subdivision scheme. The eigenvalues of [M] are

$$[\lambda]=\Big[1,\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{8},-\frac{(46450+18\sqrt{1510431})^{1/3}}{432}-\frac{593}{216(46450+18\sqrt{1510431})^{1/3}}+\frac{25}{216},\frac{1}{16},\frac{1}{24},-\frac{1}{16}\pm\frac{\sqrt{5}}{48}i,$$
$$\frac{(46450+18\sqrt{1510431})^{1/3}}{864}+\frac{593}{432(46450+18\sqrt{1510431})^{1/3}}+\frac{25}{216}\pm\frac{\sqrt{3}}{8}\left(-\frac{(46450+18\sqrt{1510431})^{1/3}}{108}+\frac{593}{54(46450+18\sqrt{1510431})^{1/3}}\right)i\Big].$$

Since the characteristic map is regular (i.e. it is 1-1 and onto everywhere) and satisfy Reif's condition (2.3), the proposed non-tensor product scheme (3.3) is C^1-continuous for regular quad meshes. *Figure* 4 illustrates the performance of proposed scheme at different levels. *Figure* 4 (a) is the initial control polygon while *Figure* 4(b), (c), (d), (e) & (f) show the result after subdivision level 1, 2, 3, 4 and 8 respectively.

DISCUSSION

A new non-tensor product binary C^1 subdivision scheme for regular quad meshes has been introduced. The construction method is direct, with no reliance on any existing schemes. The subdivision mask comprises four stencils each of support 9, which generate new vertex, edge and face points from previous level mesh points. The scheme has been demonstrated to generate visually smooth surfaces, however it is at best only C^1 continuous. Future research direction will focus on selecting an optimal function which gives high smoothness and small support. However, care needs to be taken to avoid the growth of the support to ensure the scheme is as local as possible. The idea of using a cubic function in the generation of subdivision rules is worthwhile. That is, if we use cubic function $A+Bx+Cy+Dxy+Ex^2+Fy^2+Gxy^2+Hyx^2+Jx^3+Ky^3$, we get the following insertion rules

$$p^{k+1}_{2i,2j}=\frac{1}{800}\Big\{-92p^k_{i-1,j-1}+176(p^k_{i-1,j}+p^k_{i,j-1})-16(p^k_{i-1,j+1}+p^k_{i+1,j-1})$$
$$-68(p^k_{i-1,j+2}+p^k_{i+1,j+1}+p^k_{i+2,j-1})+372p^k_{i,j}+148(p^k_{i,j+1}+p^k_{i+1,j})$$
$$+104(p^k_{i,j+2}+p^k_{i+2,j})-64(p^k_{i+1,j+2}+p^k_{i+2,j+1})+28p^k_{i+2,j+2}\Big\},$$

$$p^{k+1}_{2i+1,2j}=\frac{1}{800}\Big\{100(-p^k_{i-1,j-1}+p^k_{i,j-1}+p^k_{i+1,j-1}-p^k_{i+2,j-1})+125(p^k_{i-1,j}$$
$$+p^k_{i+2,j})+275(p^k_{i,j}+p^k_{i+1,j})+50(-p^k_{i-1,j+1}+p^k_{i,j+1}+p^k_{i+1,j+1}+p^k_{i+2,j+1})$$
$$+25(-p^k_{i-1,j+2}+p^k_{i,j+2}+p^k_{i+1,j+2}-p^k_{i+2,j+2})\Big\},$$

$$p^{k+1}_{2i,2j+1} = \frac{1}{800}\Big\{100(-p^k_{i-1,j-1} + p^k_{i-1,j} + p^k_{i-1,j+1} - p^k_{i-1,j+2}) + 125(p^k_{i,j-1} + p^k_{i,j+2}) + 275(p^k_{i,j} + p^k_{i,j+1}) + 50(-p^k_{i+1,j-1} + p^k_{i+1,j} + p^k_{i+1,j+1} + p^k_{i+1,j+2}) + 25(-p^k_{i+2,j-1} + p^k_{i+2,j} + p^k_{i+2,j+1} - p^k_{i+2,j+2})\Big\},$$

$$p^{k+1}_{2i+1,2j+1} = \frac{1}{800}\Big\{-75(p^k_{i-1,j-1} + p^k_{i-1,j+2} + p^k_{i+2,j-1} + p^k_{i+2,j+2}) + 175(p^k_{i,j} + p^k_{i,j+1} + p^k_{i+1,j} + p^k_{i+1,j+1}) + 50(p^k_{i-1,j} + p^k_{i-1,j+1} + p^k_{i,j-1} + p^k_{i,j+2} + p^k_{i+1,j-1} + p^k_{i+1,j+2} + p^k_{i+2,j} + p^k_{i+2,j+1})\Big\}.$$

Unfortunately, its support size is 16, which is higher than the scheme (3.3), which makes it less local.

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