World Applied Sciences Journal 23 (12):59-64, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.2013.23.12.2889

Generalized Derivations on Prime Γ-Rings

Abdul Rauf Khan, Muhammad Anwar Chaudhry and Imran Javaid

Center for Advanced Studies in Pure and Applied Mathematics,
Bahauddin Zakariya University Multan, Pakistan

Abstract: Let M be a prime Γ-ring with characteristic not equal to 2, I a non zero ideal of M and f: $M \to M$ a generalized derivation of M, with associated non zero derivation d on M. If $f(x) \in Z(M)$ for all $x \in I$, then M is a commutative Γ-ring.

Mathematics subject classification (2010): 16W25 . 16N60 . 47B47 . 16U80

Key words: Gamma ring . prime gamma ring . derivation . generalized derivation . commutators

INTRODUCTION

Nobusawa [1] introduced the notion of a Γ-ring, a notion more general than a ring. Barnes [2] slightly weakened the conditions in the definition of a Γ-ring in the sense of Nobusawa. After the study of Γ-rings by Nobusawa [1] and Barnes [2], many researchers have done a lot of work and have obtained some generalizations of the corresponding results in ring theory [3, 4]. Barnes [2] and Kyuno [3] studied the structure of Γ-rings and obtained various generalizations of the corresponding results of ring theory.

If M and Γ are additive abelian groups and there exists a mapping $M \times \Gamma \times M \to M$ which satisfies the following conditions:
For all $a,b \in M$ and $\alpha, \beta \in \Gamma$,

(i) (a,β,b), denoted by $a\beta b$, is an element of M
(ii) $(a+b)\beta c = a\beta c + b\beta c$, $a(\alpha+\beta)b = a\alpha b + a\beta b$, $a\beta(b+c) = a\beta b + a\beta c$
(iii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

then M is called a Γ- ring [2]. It is known that from (i)-(iii) the following follows:

(*) $0\beta b = a0b = a\beta 0 = 0$

for all a and b in M and all β in Γ [2].

Every ring is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Let M be a Γ-ring, then M is called a prime Γ-ring if $a\Gamma M\Gamma b = 0 \Rightarrow a = 0$ or $b = 0$, $a, b \in M$ and M is called a semiprime Γ-ring if $a\Gamma M\Gamma a = 0 \Rightarrow a = 0$, $a \in M$. Every prime Γ-ring is obviously semiprime. If M is a Γ-ring, then M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An additive subgroup I of M is called a left (right) ideal of M if $M\Gamma I \subseteq I$ $(I\Gamma M \subseteq I)$. If I is both left and right ideal of M, then we say I is an ideal of M. Moreover, the set

$$Z(M) = \{x \in M : x\beta y = y\beta x \quad \forall \beta \in \Gamma, \quad y \in M\}$$

is called the centre of the Γ-ring M. We shall write $[x,y]_\beta = x\beta y - y\beta x$ for all $x, y \in M$ and $\beta \in \Gamma$. We shall make use of the basic commutator identities:

Corresponding Author: Abdul Rauf Khan, Center for Advanced Studies in Pure and Applied Mathematics,
Bahauddin Zakariya University Multan, Pakistan.

$$[x\beta y, z]_\alpha = [x,z]_\alpha \beta y + x[\beta, \alpha]_z\, y + x\beta[y, z]_\alpha \text{ and } [x, y\beta z]_\alpha = [x,y]_\alpha \beta z + y[\alpha, \beta]_x\, z + y\beta[x,z]_\alpha$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. If Γ-ring satisfies the assumption (**) $a\alpha b\beta c = a\beta b\alpha c$ for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$, then the above identities reduce to $[x\beta y, z]_\alpha = [x,z]_\alpha \beta y + x\beta[y,z]_\alpha$ and $[x, y\beta z]_\alpha = [x,y]_\alpha \beta z + y\beta[x,z]_\alpha$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Derivations have been generalized as generalized derivations by Bresar [5] and Hvala [6] and they have investigated some properties of such derivation in the context of prime and semiprime rings. Recently the notion of a generalized derivation is introduced in semiprime Γ-rings by Dey, Paul and Rakhimov [7]. Yilmaz and Ozturk [8] have also proved some results in Γ-rings for Jordan generalized derivations.

Let M be a Γ-ring. An additive mapping D: M $\to$ M is called a derivation on M if $d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. An additive mapping D: M $\to$ M is called a right generalized derivation if there exists a derivation D: M $\to$ M such that $D(x\gamma y) = D(x)\gamma y + x\gamma d(y)$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. An additive mapping D: M $\to$ M is called a left generalized derivation if there exists a derivation D: M $\to$ M such that $D(x\gamma y) = d(x)\gamma y + x\gamma D(y)$ holds for all $x, y \in M$ and $\gamma \in \Gamma$. D is said to be a generalized derivation, with associated a derivation d, if it is both a right and a left generalized derivation. A derivation of the form $x \to a\alpha x + x\alpha b$ where a, b are fixed elements of M and $\alpha \in \Gamma$ is called generalized inner derivation. An additive mapping T: M $\to$ M is called a left (right) centralizer if $T(x\alpha y) = T(x)\alpha y$ $(T(x\alpha y) = x\alpha T(y))$ for all $x, y \in M$, $\alpha \in \Gamma$. Obviously the concept of a generalized derivation covers concepts of a derivation and a left centralizer.

RESULTS

In this section we prove our results.

Lemma 1: Let M be a prime Γ-ring with characteristic not equal to 2 and I a non zero ideal of M. Let f: M $\to$ M be a generalized derivation of M, with associated derivation d. If $f(x) = 0$ for all $x \in I$, then $f = 0$.

Proof: For all $x, y \in I$ and $\beta \in \Gamma$, $f(x\beta y) = 0$. That is, $f(x)\beta y + x\beta d(y) = 0$, which implies $x\beta d(y) = 0$. Let $z \in M$, $\alpha \in \Gamma$. The last relation alongwith (*) gives, $x\alpha z\beta d(y) = 0$. Since M is prime Γ-ring and I is a nonzero ideal, so $d(y) = 0$ for all $y \in I$. Hence, by hypothesis, $f(r\beta y) = 0$ for all $y \in I$, $\beta \in \Gamma$ and $r \in M$. That is, $f(r)\beta y + r\beta d(y) = 0$, which gives $f(r)\beta y = 0$. Let $w \in M$, $g \in \Gamma$. The last relation alongwith (*), implies $f(r)\beta w\gamma y = 0$. Since I is nonzero and primeness of M, gives $f = 0$.

Lemma 2: Let I be a non zero ideal of a prime Γ-ring M, $a \in M$ and $f \neq 0$ is a generalized derivation of M, with associated non zero derivation d, then

(i) If $a\beta f(x) = 0$ for all $x \in I$ and $\beta \in \Gamma$, then $a = 0$,
(ii) If $f(x)\beta a = 0$ for all $x \in I$ and $\beta \in \Gamma$, then $a = 0$.

Proof

(i) For any $x \in I$, $r \in M$ and $\beta \in \Gamma$, $a\beta f(x\alpha r) = 0$. That is, $a\beta f(x)\alpha r + a\beta x\alpha d(r) = 0$, which implies, $a\beta x\alpha d(r) = 0$. Since I is a non zero ideal of M and $d \neq 0$, we get $a = 0$.
(ii) Proof is similar to (i).

Theorem 1: Let M be a prime Γ-ring with characteristic not equal to 2 and I a non zero ideal of M. Let f: M $\rightarrow$ M be a generalized derivation of M, with associated non zero derivation d on M. If $f(x) \in Z(M)$ for all $x \in I$, then M is a commutative Γ-ring.

Proof: Using hypothesis, we have $[f(x\beta y), y]_{\alpha} = 0$ for all $x, y \in I$, $\alpha, \beta \in \Gamma$, which gives $[f(x)\beta y + x\beta d(y), y]_{\alpha} = 0$, which implies $[f(x)\beta y, y]_{\alpha} + [x\beta d(y), y]_{\alpha} = 0$. Using hypothesis, we get $x\beta[d(y), y]_{\alpha} + [x, y]_{\alpha}\beta d(y) = 0$, which gives

$$x\beta d(y)\alpha y - x\beta y\alpha d(y) + x\alpha y\beta d(y) - y\alpha x\beta d(y) = 0.$$

Using (**), from the last equation we get

$$x\beta d(y)\alpha y - x\beta y\alpha d(y) + x\beta y\alpha d(y) - y\alpha x\beta d(y) = 0,$$

which gives

(1) $x\beta d(y)\alpha y - y\alpha x\beta d(y) = 0$,

for all $x, y \in I$, $\alpha, \beta \in \Gamma$. Let $z \in I$. Replacing x by $x\beta z$ in (1), we get $x\beta z\beta d(y)\alpha y - y\alpha x\beta z\beta d(y) = 0$, which alongwith (1) and (**) gives $x\alpha y\beta z\beta d(y) - y\alpha x\beta z\beta d(y) = 0$. That is, $[x, y]_{\alpha}\beta z\beta d(y) = 0$, for all $x, y \in I$, $\alpha, \beta \in \Gamma$. Since I is a non zero ideal of M and $d \neq 0$, therefore M is a commutative Γ-ring.

Theorem 2: Let M be a prime Γ-ring with characteristic not equal to 2 and I a non zero ideal of M. Let f: M $\rightarrow$ M be a generalized derivation of M, with associated derivation d on M. If $a \in M$ and $[f(x), a]_{\alpha} = 0$ for all $x \in I$, $\alpha \in \Gamma$, then either $a \in Z(M)$ or $d(a) = 0$.

Proof: Using hypothesis, we have $[f(x\beta y), a]_{\alpha} = 0$ for any $x \in M$, $y \in I$ and $\alpha, \beta \in \Gamma$, which gives $[d(x)\beta y + x\beta f(y), a]_{\alpha} = 0$.

That is,

$$[d(x)\beta y, a]_{\alpha} + [x\beta f(y), a]_{\alpha} = 0.$$

The last equation gives

$$d(x)\beta[y, a]_{\alpha} + [d(x), a]_{\alpha}\beta y + x\beta[f(y), a]_{\alpha} + [x, a]_{\alpha}\beta f(y) = 0.$$

Using hypothesis, from the last equation we get

$$d(x)\beta[y, a]_{\alpha} + [d(x), a]_{\alpha}\beta y + [x, a]_{\alpha}\beta f(y) = 0,$$

which implies

$$d(x)\beta y\alpha a - d(x)\beta a\alpha y + d(x)\alpha a\beta y - a\alpha d(x)\beta y + x\alpha a\beta f(y) - a\alpha x\beta f(y) = 0.$$

Using (**), from the last equation we get

$$d(x)\beta y\alpha a - d(x)\alpha a\beta y + d(x)\alpha a\beta y - a\alpha d(x)\beta y + x\alpha a\beta f(y) - a\alpha x\beta f(y) = 0,$$

which gives

(2) $d(x)\beta y\alpha a - a\alpha d(x)\beta y + x\alpha a\beta f(y) - a\alpha x\beta f(y) = 0$.

Let $z \in M$. Replacing y by $y\gamma z$ in (2), we get

$$d(x)\beta y\gamma z\alpha a - a\alpha d(x)\beta y\gamma z + x\alpha a\beta f(y\gamma z) - a\alpha x\beta f(y\gamma z) = 0.$$

That is,

$$d(x)\beta y\gamma z\alpha a - a\alpha d(x)\beta y\gamma z + x\alpha a\beta f(y)\gamma z + x\alpha a\beta y\gamma d(z) - a\alpha x\beta f(y)\gamma z - a\alpha x\beta y\gamma d(z) = 0,$$

which implies

$$d(x)\beta y\gamma z\alpha a - a\alpha d(x)\beta y\gamma z + (x\alpha a\beta f(y) - a\alpha x\beta f(y))\gamma z + (x\alpha a - a\alpha x)\beta y\gamma d(z) = 0.$$

Using (2), from the last equation we get

$$d(x)\beta y\gamma z\alpha a - a\alpha d(x)\beta y\gamma z + (a\alpha d(x)\beta y - d(x)\beta y\alpha a)\gamma z + [x,a]_\alpha\beta y\gamma d(z) = 0.$$

This implies

$$d(x)\beta y\gamma z\alpha a - a\alpha d(x)\beta y\gamma z + a\alpha d(x)\beta y\gamma z - d(x)\beta y\alpha a\gamma z + [x,a]_\alpha\beta y\gamma d(z) = 0.$$

Using (**), from the last equation we get

$$d(x)\beta y\gamma z\alpha a - d(x)\beta y\gamma a\alpha z + [x,a]_\alpha\beta y\gamma d(z) = 0,$$

which gives

$$d(x)\beta y\gamma[z,a]_\alpha + [x,a]_\alpha\beta y\gamma d(z) = 0.$$

Replacing x by a from the last equation, we get $d(a)\beta y\gamma[z,a]_\alpha = 0$ for all $y \in I$, $z \in M$, and $\alpha, \beta, \gamma \in \Gamma$. Since I is non zero ideal of prime Γ-ring M, therefore either $d(a) = 0$ or $a \in Z(M)$.

Corollary 1: Let M be a prime Γ-ring with characteristic not equal to 2 and I a non zero ideal of M. Let $f: M \to M$ be a generalized derivation of M, with associated derivation d on M. If $[f(x), f(y)]_\beta = 0$ for all $x, y \in I$, $\beta \in \Gamma$, then M is a commutative Γ-ring.

Proof: Using Theorem 2, we have $f(I) \subset Z(M)$. Then using Theorem 2, we get the corollary 2.

Theorem 3: Let M be a prime Γ-ring with characteristic not equal to 2 and I a non zero ideal of M. Let $f: M \to M$ be a generalized derivation of M, with associated derivation d on M. If $f(x\beta y) = f(x)\beta f(y)$ for all $x, y \in I$, $\beta \in \Gamma$, then $d = 0$.

Proof: For any $x, y \in I$, $\beta \in \Gamma$, $f(x\beta y) = f(x)\beta y + x\beta d(y)$, which implies

(3) $f(x)\beta f(y) = f(x)\beta y + x\beta d(y)$.

Let $w \in I$, $\gamma \in \Gamma$. Then replacing x by $x\gamma w$ in (3), we get

$$f(x\gamma w)\beta f(y) = f(x\gamma w)\beta y + x\gamma w\beta d(y),$$

which gives

$$f(x)\gamma f(w)\beta f(y) = f(x)\gamma f(w)\beta y + x\gamma w\beta d(y).$$

That is,

$$f(x)\gamma f(w\beta y) = f(x)\gamma f(w)\beta y + x\gamma w\beta d(y),$$

which implies

$$f(x)\gamma f(w)\beta y + f(x)\gamma w\beta d(y) = f(x)\gamma f(w)\beta y + x\gamma w\beta d(y).$$

That is, $f(x)\gamma w\beta d(y) = x\gamma w\beta d(y)$, which gives $(f(x)-x)\gamma w\beta d(y)=0$ for all $x\in I$ and $\gamma, \beta\in\Gamma$. Since I is a non zero ideal of the prime Γ-ring M, therefore either $f(x)-x=0$ for all $x\in I$ or $d(y) = 0$ for all $y\in I$.
If $f(x)-x=0$ for all $x\in I$, then $f(x) = x$ for all $x\in I$. For all $y\in I$. Replacing x by $x\beta y$ in the last equation, we get $f(x\beta y)=x\beta y$, which implies $d(x)\beta y + x\beta f(y) = x\beta y$, which gives $d(x)\beta y + x\beta y = x\beta y$. That is, $d(x)\beta y = 0$ for all $x, y\in I$, $\beta\in\Gamma$. Thus $d(x) = 0$ for all $x\in I$ for both cases. So $d = 0$.

Theorem 4: Let M be a prime Γ-ring with characteristic not equal to 2 and I a non zero ideal of M. Let $f\colon M \to M$ be a generalized derivation of M, with associated derivation d on M. If $f(x\beta y) = f(y)\beta f(x)$ for all $x, y\in I$, $\beta\in\Gamma$, then $d = 0$.

Proof: Let $x, y\in I$, $\beta\in\Gamma$. Then $f(x\beta y) = d(x)\beta y + x\beta f(y)$, which implies

(4) $f(y)\beta f(x) = d(x)\beta y + x\beta f(y)$.

Let $x\in I$, $\gamma\in\Gamma$. Then replacing y by $x\gamma y$ in (4), we get

$$f(x\gamma y)\beta f(x) = d(x)\beta x\gamma y + x\beta f(x\gamma y).$$

That is,

$$d(x)\gamma y\beta f(x) + x\gamma f(y)\beta f(x) = d(x)\beta x\gamma y + x\beta f(y)\gamma f(x).$$

Using (**), from the last equation we get

$$d(x)\gamma y\beta f(x) + x\beta f(y)\gamma f(x) = d(x)\beta x\gamma y + x\beta f(y)\gamma f(x),$$

which gives

(5) $d(x)\gamma y\beta f(x) = d(x)\beta x\gamma y$.

Let $w\in I$, $\alpha\in\Gamma$. Then replacing y by $y\alpha w$, we get $d(x)\gamma y\alpha w\beta f(x) = d(x)\beta x\gamma y\alpha w$, which alongwith (5) gives

$$d(x)\gamma y\alpha w\beta f(x) = d(x)\gamma y\beta f(x)\alpha w.$$

Using (**), from the last equation we get

$$d(x)\gamma y\alpha f(x)\beta w - d(x)\gamma y\alpha w\beta f(x) = 0.$$

That is, $d(x)\gamma y\alpha[f(x), w]_\beta = 0$. Since I is a non zero ideal of the prime Γ-ring M, therefore either $d(x) = 0$ for all $x \in I$ or $[f(x), w]_\beta = 0$ for all $x, w\in I$ and $\beta\in\Gamma$. Let $A = \{x\in I : d(x) = 0\}$ and $B = \{x\in I : [f(x), w]_\beta = 0, \forall w\in I\}$. Obviously A and B are additive subgroups of I. Moreover I is the set theoretic union of A and B. But a group can not be set theoretic union of two proper subgroups. Hence either $A = I$ or $B = I$. If $A = I$, we have $d(R) = 0$, which completes the proof. If $B = I$, then $0 = [f(x), w]_\beta = w\alpha[f(x), r]_\beta$ for all $x, w\in I$, $r\in M$ and $\alpha, \beta\in\Gamma$. Thus, we obtain $f(I)\subset Z(M)$. Using Theorem 2, we get $d = 0$.

ACKNOWLEDGEMENT

The authors are thankful to the Bahauddin Zakariya University, Multan and Higher Education Commission of Pakistan for the support and facilities provided during this research.

REFERENCES

1 Nobusawa, N., 1964. On the generalization of the ring theory. Osaka J. Math., 1: 81-89.
2. Barnes, W. E., 1966. On the Γ-rings of Nobusawa. Pacific J. Math., 18 (3): 411-422.
3. Kyuno, S., 1978. On prime gamma rings. Pacific J. Math., 75 (1): 185-190.
4. Khan, A. R., I. Javaid and M. Anwar Chaudhry, 2013. Derivations on semiprime Γ-rings, Utilitas Mathematica, 90: 171-185.
5. Bresar, M., 1991. On the distance of the composition of two derivations to the generalized derivations. Glasgow Math. J., 33 (1): 89-93.
6. Hvala, B., 1998. Generalized derivations in rings. Comm. Algebra, 26 (4): 1147-1166.
7. Dey, K. K., A. C. Paul and I. S. Rakhimov, 2012. Generalized derivations in semiprime gamma rings. IJMMS.
8. Yilmaz, C. and M. A. Ozturk, 2004. On Jordan generalized derivations in gamma rings. Hacet. J. Math. Stat., 33: 11-14.
9. Ozturk, M. A., Y. B. Jun and K. H. Kim, 2002. On derivations of prime gamma rings. Turk. J. Math., 26: 317-327.