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An Efficient Analytical Algorithm for Wave-type and Time-fractional PDEs

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Abstract: An accurate and efficient Homotopy Analysis Method (HAM) to find the analytical/approximate solutions for wave-type and fractional wave-type equations arising in biology is proposed. This method provides the solutions in rapid convergence series with computable terms. To the best of our knowledge, until now there is no rigorous HAM solutions have been reported for the above mentioned equations. Finally, we have given some numerical examples to demonstrate the validity and applicability of the method. Moreover, the use of HAM is found to be accurate, efficient, simple, low computation costs and computationally attractive.

Key words: Homotopy analysis method . wave-type equations . fractional PDEs

INTRODUCTION

In mathematics a hyperbolic partial differential equation of order n is a partial differential equation (PDE) that has a well-posed initial value problem for the first n-1 derivatives. Many of the equations of mechanics are hyperbolic and so the study of hyperbolic equations is of substantial contemporary interest. In recent years, notable contributions have been made to the applications of fractional differential equations (FDEs). These equations are increasingly applied to efficient model problems in research areas as diverse as machanical systems, dynamical systems, control, chaos,continuous time random walks, anomalous diffusive and subdiffusive systems, wave propagation and son on. The fractional calculus and its applications (that is, the theory of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, mainly due to its applications in diverse fields of science and engineering. Mathematical modeling of complex processes is a major challenge for contemporary scientist. In contrast to simple classical systems, where the theory of integer order differential equations is sufficient to describe their dynamics, fractional derivatives provide an excellent and an efficient instrument for the description of memory and hereditary properties of various complex materials and systems. Recently, the nonlinear oscillation of earthquakes has been modeled with fractional derivatives [1]. The Homotopy analysis method has been implemented for nonlinear equations arising in heat transfer [2], Burgers-Huxley equation [3] and the convergence analysis of HAM [4].

(i) In one spatial dimension, the wave equation is of the form

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \qquad (1)$$

(ii) One dimensional hyperbolic-like (wave-like) equation is of the form

$$\frac{\partial^2 u}{\partial t^2} - f(x)\frac{\partial^2 u}{\partial t^2} = 0 \qquad (2)$$

(iii) The time fractional diffusion equation is of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 (u(x,t))^2}{\partial x^2} + \frac{\partial^2 (u(x,t))^2}{\partial y^2} + u(x,t), t > 0, 0 < \alpha \le 1 \qquad (3)$$

These equations have the property that, if u and its first time derivative are arbitrarily specified initial data on the initial line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all time. Partial differential equations play a crucial role in applied mathematics and physics. The results of solving those equations can guide the authors to know the

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described process deeply. Recently, various approximate methods are applied for getting numerical and analytical solutions of linear and non linear partial differential equations [5-7]. Besides some iterative methods are applied for getting numerical and analytical solutions of hyperbolic and hyperbolic-like equations [8-12]. HAM was introduced by Liao [13, 14]. This proposed method has been used by many mathematicians and engineers to solve various equations based on homotopy, which is a basic concept in topology. In recent years, HAM has been successfully employed to solve many types of nonlinear homogeneous or nonhomogeneous equations and systems of equation as well as problems in science and engineering [15, 16]. Mustafa [17] solved the Laplace equation with Dirichlet and Neumann boundary conditions by Homotopy analysis method. Barari *et al*. [18] used the variational iteration (VIM) and parameterized perturbation (PPM) methods have been used to investigate non-linear vibration of Euler-Bernoulli beams subjected to the axial loads. Yasir Nawaz [19] applied the variational iteration method and homotopy perturbation method for linear and nonlinear boundary value problems for fourth-order fractional integro-differential equations. Kaliji *et al*. [20, 21] have investigated the dynamic behavior of two mechanical structures via analytical Methods and free vibration of Cantilever beams. Recently, Muhammet Kurulay [22] established the homotopy analysis method for solving the fractional nonlinear Klein-Gordon equation.

The validity of the HAM is independent of whether or not there exist small parameters in the considered equation. HAM provides us with a simple way to adjust and control the convergence of solution series..It provides us with freedom to use different base functions to approximate a nonlinear problem. Especially, it provides us with freedom of replace a nonlinear partial differential equation of first order n into an infinite number of linear Differential equations of order k, where the order k is even unnecessarily to be equal to order n. When the base functions are introduced the H(r,t) is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence. By plotting h curves the proper h is chosen. Thus as long as h, H(r,t), $u_0(r,t)$ and linear operator L are properly chosen the solution expression converges to exact solution in the convergence region. Homotopy analysis method provides us the great freedom to choose all of them.

DEFINITIONS OF FRACTIONAL DERIVATIVES AND INTEGRALS

In this section, we have given some notations, definitions and preliminary facts that will be used further in this work. The Caputo fractional derivative allows the utilization of initial and boundary conditions involving integer order derivatives, which have clear physically interpretations. Therefore, in this paper we shall use the Caputo derivative D^{α} proposed by Caputo in his work on the theory of viscoelasticity. In the development of theories of fractional derivatives and integrals, it appears many definitions such as Riemann-Liouville and Caputo fractional differential-integral definition as follows.

(1) Riemann-Liouville definition:

$$
{}_a^R D_t^{\alpha} f(t) = \begin{cases} \dfrac{d^m f(t)}{dt^m}, & \alpha = m \in N \\ \dfrac{d^m}{dt^m} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t \dfrac{f(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \le m-1 < \alpha < m \end{cases}
$$

Fractional integral of order α is as follows:

$$
{}_a^R I_t^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \int_0^t (t-T)^{-\alpha-1} f(T) dT, \quad \alpha < 0
$$

(2) Caputo definition:

$$
{}_a^c D_t^{\alpha} f(t) = \begin{cases} \dfrac{d^m f(t)}{dt^m}, & \alpha = m \in N \\ \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_a^t \dfrac{f^{(m)}(T)}{(t-T)^{\alpha-m+1}} dT, & 0 \le m-1 < \alpha < m \end{cases}
$$

APPLICATIONS

Let us consider the homogeneous hyperbolic equation

$$
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 3u = 0 \tag{4}
$$

With initial conditions

$$
u(x,0)=0,\ u_t(x,0)=2\sin x \tag{5}
$$

Now we will apply the Homotopy analysis method (HAM) to solve Eq. (4) subject to the initial condition Eq.(5)

Here we choose the initial approximation $u_0(x,t)=2t\sin x$

We define the nonlinear operator as:

$$
N[\phi(x,t;q)] = \frac{\partial^2 \phi(x,t;q)}{\partial t^2} - \frac{\partial^2 \phi(x,t;q)}{\partial x^2} + 3\phi(x,t,q) \tag{6}
$$

and linear operator

$$L[\phi(x,t;q)] = \frac{\partial\phi(x,t;q)}{\partial t} \quad (7)$$

With the property

$$L(c_1(x)) = 0 \quad (8)$$

where $c_1(x)$ is the integration constant. Now by using Eq.(63) we have

$$R_m\left[\bar{U}_{m-1}\right] = \frac{\partial^2\phi(x,t;q)}{\partial t^2} - \frac{\partial^2\phi(x,t;q)}{\partial x^2} + 3\phi(x,t,q) \quad (9)$$

and the solution of the m^{th} order deformation Eq.(11) for m=1 becomes

$$u_{m-1}(x,t) = \chi_m U_{m-1}(x,t) + hH(r,t)L^{-1}\left[R_m(\bar{U}_{m-1})\right] \quad (10)$$

Since m=1, χ_m=1, we set h=-1 and H(r,t)=1 and we successively obtain

$$u_1(x,t)=2t\sin x-(2t)^3/3!\sin x$$

$$u_2(x,t)=2t\sin x-(2t)^3/3!\sin x+(2t)^5/5!\sin x$$

....

$$u(x,t) = 2t\sin x-(2t)^3/3!\sin x +(2t)^5/5!\sin x-(2t)^7/7!\sin x+...........$$

The final solution in a closed form is

$$u(x,t) = \sin x\sin 2t \quad (11)$$

Now we consider the inhomogeneous non linear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u - u^2 + x^2 t + x^4 t + 2t = 0 \quad (12)$$

With initial conditions

$$u(x,0)=0,\ u_t(x,0)=x^2 \quad (13)$$

We apply homotopy analysis method to Eq. (12) and Eq.(13), as follows:

Since m=1, χ_m=1 set h=-1 and H(r,t)=1, L== $\frac{\partial^2}{\partial t^2}$ in Eq.(11), then it becomes

$$U_m(x,t)=u_{m-1}(x,t)-L^{-1}(R_m(u_{m-1},x,t))$$

Where

$$R_m(u_{m-1},x,t) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{\partial^2 u_{m-1}}{\partial x^2} - u_{m-1} - u^2_{m-1} + x^2 t + x^4 t + 2t \quad (14)$$

and solution for u_0:
Now we can select

$$u_0 = x^2 t \quad (15)$$

Using HAM, the successive approximations are given by

$$u_1(x,t)=x^2t$$
$$u_2(x,t)=x^2t$$
$$u_3(x,t)=x^2t$$

………..

The final solution in a closed form is

$$u(x,t)=x^2t \quad (16)$$

Consider the one dimensional hyperbolic-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{2}\frac{\partial^2 u}{\partial x^2} = 0 \quad (17)$$

With initial conditions

$$u(x,o)=0,\ u_t(x,0)=x^2 \quad (18)$$

We apply homotopy analysis method to Eq. (17) and Eq.(18) as follows:

Since m=1, χ_m=1 set h = -1 and H(r,t)=1, L == $\frac{\partial^2}{\partial t^2}$ in Eq.(63). It becomes

$$U_m(x,t)=u_{m-1}(x,t)-L^{-1}(R_m(u_{m-1},x,t)) \quad (19)$$

Where

$$R_m(u_{m-1},x,t) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{x^2}{2}\frac{\partial^2 u_{m-1}}{\partial x^2} \quad (20)$$

$$R_m(u_{m-1},x,t) = \partial^2 u_{m-1}/\partial t^2 - x^2/2\ \partial^2 u_{m-1}/\partial x^2$$

and solution for u_0:
Now we can select

$$u_0(x,t)=x^2 \quad (21)$$

Applying Eq.(20) in Eq.(21) we obtain the following successive approximations

$$u_1(x,t)= x^2+x^2t^2/2!$$

$$u_2(x,t)= x^2+x^2t^2/2!+x^2t^4/4!$$
$$\ldots\ldots\ldots\ldots$$
$$u(x,t)= x^2+x^2t^2/2!+x^2t^4/4! + x^2t^6/6!+ \ldots\ldots\ldots$$

The final solution in a closed form is

$$u(x,t)=x^2\cosh t \qquad (22)$$

Consider the two dimensional hyperbolic-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{20}\frac{\partial^2 u}{\partial x^2} - \frac{y^2}{20}\frac{\partial^2 u}{\partial y^2} = 0$$

$$u_{tt}(x,y,t)=x^2/20u_{xx}(x,y,t)+ y^2/20u_{yy}(x,y,t) \qquad (23)$$

With initial conditions

$$u(x,y,0)=x^5, u_t(x,y,0)=y^5 \qquad (24)$$

We apply HAM to Eq. (23) and Eq.(24), as follows:

Since m=1, χ_m=1 set h=-1 and H(r,t)=1, L= $\frac{\partial^2}{\partial t^2}$ in 11 then 11 becomes

$$u_m(x,y,t)=u_{m-1}(x,y,t)-L^{-1}(R_m(u_{m-1},x,y,t)) \qquad (25)$$

Where

$$R_m(u_{m-1},x,t) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{x^2}{20}\frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{y^2}{20}\frac{\partial^2 u_{m-1}}{\partial y^2} \qquad (26)$$

and solution for u_0:
Now we can select

$$u_0(x,t)=x^5+y^5t \qquad (27)$$

Applying Eq. (27) in Eq. (26) we get the following successive approximations

$$u_1(x,t)= x^5+y^5t+x^5t^2/2!+ y^5t^3/3!$$

$$u_2(x,t)= x^5+y^5t+x^5t^2/2!+ y^5t^3/3!+ x^5t^4/4!+ y^5t^5/5!$$
$$\ldots\ldots\ldots\ldots$$

The final solution is

$$u(x,t)= x^5(1+t^2/2!+t^4/4!+\ldots\ldots)+ y^5(t+t^3/3!+ t^5/5!+\ldots\ldots)$$

The final solution in a closed form is

$$u(x,t)= x^5\cosh t+y^5\sinh t \qquad (28)$$

Consider the three dimensional hyperbolic-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{x^2}{6}\frac{\partial^2 u}{\partial x^2} - \frac{y^2}{6}\frac{\partial^2 u}{\partial y^2} - \frac{z^2}{6}\frac{\partial^2 u}{\partial z^2} = 0 \qquad (29)$$

With initial conditions

$$u(x,y,z,0)=0, u_t(x,y,z,0)=(x^3+y^3+z^3) \qquad (30)$$

We apply HAM to Eq. (29) and Eq.(30), as follows:

Since m=1, χ_m=1 set h=-1 and H(r,t)=1, L== $\frac{\partial^2}{\partial t^2}$ in Eq.(63)

$$u_m(x,y,z,t)=u_{m-1}(x,y,z,t)-L^{-1}(R_m(u_{m-1},x,y,z,t)) \qquad (31)$$

Where

$$R_m(u_{m-1},x,t) = \frac{\partial^2 u_{m-1}}{\partial t^2} - \frac{x^2}{6}\frac{\partial^2 u_{m-1}}{\partial x^2} - \frac{y^2}{6}\frac{\partial^2 u_{m-1}}{\partial y^2} - \frac{z^2}{6}\frac{\partial^2 u_{m-1}}{\partial z^2} \qquad (32)$$

and solution for u_0:
Now we can select

$$u_0(x,t)= (x^3+y^3+z^3)+ (x^3+y^3+z^3)t \qquad (33)$$

Now applying Eq.(33) in Eq. (32) we will get the following successive approximations

$$u_1(x,t) = (x^3+y^3+z^3)+ (x^3+y^3+z^3)t +(x^3+y^3+z^3)t^2/2!+ (x^3+y^3+z^3)t^3/3!$$

$$u_2(x,t) = (x^3+y^3+z^3)+ (x^3+y^3+z^3)t +(x^3+y^3+z^3)t^2/2!+ (x^3+y^3+z^3)t^3/3! +(x^3+y^3+z^3)t^4/4!+ (x^3+y^3+z^3)t^5/5!$$
$$\ldots\ldots\ldots\ldots$$

The final solution is given by

$$u(x,t) = (x^3+y^3+Z^3)(1+t+t^2/2!+t^3/3!+\ldots\ldots) = (x^3+y^3+z^3)e^t \qquad (34)$$

We consider another three dimensional hyperbolic-like equation

$$\frac{\partial^2 u}{\partial t^2} - \left(x^2+y^2+z^2\right) - \frac{1}{2}\left(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}\right) = 0 \qquad (35)$$

With initial condition

$$u(x,y,z)=0, u_t(x,y,z) =(x^2+y^2-z^2)t \qquad (36)$$

We apply homotopy analysis method to Eq. (35) and Eq. (36), as follows:

Since m=1, χ_m=1 set h=-1 and H(r,t)=1, L== $\frac{\partial^2}{\partial t^2}$ in Eq.(63).then Eq.(63) becomes

$$u_m(x,y,z,t)=u_{m-1}(x,y,z,t)-L^{-1}(R_m(u_{m-1},x,y,z,t)) \tag{37}$$

Where

$$R_m(u_{m-1},x,t)=\frac{\partial^2 u_{m-1}}{\partial t^2}-(x^2+y^2+z^2) -\frac{x^2}{2}\frac{\partial^2 u_{m-1}}{\partial x^2}-\frac{y^2}{2}\frac{\partial^2 u_{m-1}}{\partial y^2}-\frac{z^2}{2}\frac{\partial^2 u_{m-1}}{\partial z^2} \tag{38}$$

and solution for u_0:
Now we can select

$$u_0(x,t)= (x^2+y^2-z^2)t \tag{39}$$

Applying Eq. (39) in Eq. (38) we will get the following successive approximations

$$u_1(x,t)= (x^2+y^2-z^2)t+ (x^2+y^2+z^2)t+(x^2+y^2-z^2)t^3/3!$$

$$u_2 (x,t)= (x^2+y^2-z^2)t+ (x^2+y^2+z^2)t^2/2! +(x^2+y^2-z^2)t^3/3! + (x^2+y^2+z^2)t^4/4! +(x^2+y^2-z^2)t^5/5!$$

....

The final solution is

$$\begin{aligned} u(x,t) &= (x^2+y^2+z^2)(t^2/2!+t^4/4!+t^6/6!..............) \\ &\quad + (x^2+y^2-z^2)(t+t^3/3!+t^5/5!+......) \\ &= (x^2+y^2+z^2)\cosh t+(x^2+y^2-z^2)\sinh t\text{-})\text{-}(x^2+y^2+z^2) \end{aligned}$$

$$u(x,t) = (x^2+y^2)e^t+z^2e^{-t}-(x^2+y^2+z^2) \tag{40}$$

Consider a nonhomogeneous nonlinear wave equation.The equation of the form

$$\frac{\partial^2 u}{\partial x^2}-u\frac{\partial^2 u}{\partial t^2}-2+2t^2+2x^2=0 \tag{41}$$

With initial conditions

$$\begin{aligned} &u(x,0)=x^2 \\ &u(0,t)=t^2, u_x(0,t)=0 \end{aligned} \tag{42}$$

We apply HAM to Eq. (41) and Eq.(42), as follows:

Since m=1, χ_m=1 set h=-1 and H(r,t)=1,L== $\frac{\partial^2}{\partial x^2}$ in Eq.(63), then it becomes

$$u_m(x,t)=u_{m-1}(x,t)-L^{-1}(R_m(u_{m-1},x,t)) \tag{43}$$

Where

$$R_m(u_{m-1},x,t)=\frac{\partial^2 u_{m-1}}{\partial x^2}-u_{m-1}\frac{\partial^2 u_{m-1}}{\partial t^2}-2+2t^2+2x^2 \tag{44}$$

and solution for u_0:
Now we can select

$$u_0=x^2+t^2-x^2t^2-\frac{x^4}{6} \tag{45}$$

Applying Eq. (45) in Eq. (44) we obtain the following successive approximations:

$$u_1(x,t) = x^2t^2+\frac{x^4}{6}-\frac{x^4t^2}{3}-\frac{7}{90}x^6+\frac{2}{15}x^6t^2+\frac{1}{168}x^8$$

$$u_2(x,t)= \frac{x^4t^2}{3}+\frac{7}{90}x^6-\frac{2}{15}x^6t^2+\frac{1}{168}x^8-.........$$

and so on.

The noise terms appear in the various components will get cancelled themselves.
Then the final solution in a closed form is

$$u(x,t)= x^2+t^2 \tag{46}$$

We consider another time fractional diffusion equation which is known as biological population equation as follows

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2(u(x,t))^2}{\partial x^2}+\frac{\partial^2(u(x,t))^2}{\partial y^2} + u(x,t), t>0, 0<\alpha\le 1 \tag{47}$$

with initial condition

$$u(x,y,0)= \sqrt{x+y} \tag{48}$$

Now we will apply the HAM to solve Eq. (47) subject to the initial condition Eq.(48)
We define the nonlinear operator as

$$N[\phi(x,t;q)]=\frac{\partial^\alpha\phi(x,y,t;q)}{\partial t^\alpha}-\frac{\partial^2(\phi(x,y,t;q))^2}{\partial x^2} -\frac{\partial^2(\phi(x,y,t;q))2}{\partial y^2}-\phi(x,y,t;q) \tag{49}$$

and linear operator

$$L[\phi(x,t;q)]=\frac{\partial^\alpha\phi(x,y,t;q)}{\partial t^\alpha} \tag{50}$$

With the property

$$L(C_1(x))=0 \tag{51}$$

Where C_1 (x) is the integration constant. Now by using Eq. (63) we have

$$R_m\left[\bar{U}_{m-1}\right]=\frac{\partial^\alpha \phi(x,y,t;q)}{\partial t^\alpha}-\frac{\partial^2(\phi(x,y,t;q))^2}{\partial x^2}-\frac{\partial^2(\phi(x,y,t;q))^2}{\partial y^2}-\phi(x,y,t;q) \tag{52}$$

and the solution of the m^{th} order deformation becomes

$$u_{m-1}(x,t)=\chi_m U_{m-1}(x,t)+L^{-1}\left[hH(r,t)R_m(\bar{U}_{m-1})\right] \tag{53}$$

since m=1, χ_m=1,we set h=-1, H(r,t)=1and also the equation has sub diffusive behavior we obtain the final solution as

$$u(x,t)=\sqrt{x+y}\sum_{k=0}^{\infty}\frac{t^\alpha}{\Gamma(k\alpha+1)} \tag{54}$$

CONCLUSION

In this paper, the homotopy analysis method (HAM) has been successfully applied to obtain the approximate/analytical solutions of the various hyperbolic and fractional PDEs arising in various fields. This work shows that HAM has significant advantages over the existing techniques. It avoids the need for calculating the Adomain polynomials which can be difficult in some cases. The reliability of the method and reduction in the size of computational domain give this method wider applicability. The results show that HAM is a powerful mathematical tool for finding the exact and approximate solutions of the nonlinear equations.

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Appendix

Basic idea of Homotopy Analysis method (HAM):

In this section the basic ideas of the homotopy analysis method are presented. Here a description of the method is given to handle the general nonlinear problem.

$$Nu_0(t)=0, \quad t>0 \tag{55}$$

where N is a nonlinear operator and $u_0(t)$ is unknown function of the independent variable t.

Zero-order deformation equation

Let $u_0(t)$ denote the initial guess of the exact solution of Eq. (1)-Eq.(3), h≠0 an auxiliary parameter, H(t)≠0 an auxiliary function and L is an auxiliary linear operator with the property.

$$L(f(t)) = 0, f(t) = 0 \tag{56}$$

The auxiliary parameter h, the auxiliary function H(t) and the auxiliary linear operator L play an important role within the HAM to adjust and control the convergence region of solution series. Liao [15, 16] constructs, using q∈[0,1] as an embedding parameter, the so-called zero-order deformation equation.

$$(1-q)L[(\emptyset(t;q)-u_0(t)]-qhH(t)N[(\emptyset(t;q)], \tag{57}$$

where ∅(t;q) is the solution which depends on h, H(t), L, $u_0(t)$ and q. When q=0, the zero-order deformation Eq. (63) becomes

$$\emptyset(t;0)-u_0(t) \tag{58}$$

and when q = 1, since h≠0 and H(t)≠0, the zero-order deformation Eq.(1) reduces to,

$$N[\emptyset(t;1)]-0, \tag{59}$$

So, ∅(t;1) is exactly the solution of the nonlinear equation. Define the so-called m^{th} order deformation derivatives.

$$u_m(t)=\frac{1}{m!}\frac{\partial^m \emptyset(t;q)}{\partial q^m} \tag{60}$$

If the power series Eq. (57) of ∅(t;q) converges at q=1, then we gets the following series solution:

$$u(t)=u_0(t)+\sum_{m-1}^{\infty}u_m(t) \tag{61}$$

where the terms $u_m(t)$ can be determined by the so-called high order deformation described below.

High-order deformation equation

Define the vector,

$$u_n-\{u_0(t),u_1(t),u_2(t)\ldots\ldots u_n(t) \tag{62}$$

Differentiating Eq. (63) m times with respect to embedding parameter q, the setting q=0 and dividing them by m!, we have the so-called m^{th} order deformation equation.

$$L[u_m(t) \quad \aleph_m u_{m-1}(t)] = hH(t)R_m(\overrightarrow{u_m}, t), \qquad (63)$$

where

$$\aleph_m = \begin{cases} 0, & m \leq 1 \\ 1, & otherwise \end{cases} \qquad (64)$$

and

$$R_m(\overrightarrow{u_m}, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\emptyset(t;q)]}{\partial q^{m-1}} \qquad (65)$$

For any given nonlinear operator N, the term $R_m(\overrightarrow{u_m}, t)$ can be easily expressed by Eq.(65). Thus, we can gain $u_1(t)$, $u_2(t)$ … … by means of solving the linear high-order deformation with one after the other order in order. The m^{th}-order approximation of u (t) is given by

$$u(t) = \sum_{k=0}^{m} u_k(t) \qquad (66)$$

ADM, VIM and HPM are special cases of HAM when we set h = -1 and H(r,t) = 1. We will get the same solutions for all the problems by above methods when we set h=-1 and H(r,t) = 1. When the base functions are introduced the H(r,t) = 1 is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence.

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