

Harmonic p-Valent Functions with Respect to k-Symmetric Points

Faisal Al-Kasasbeh

Department of Mathematics and Statics, Faculty of Science,
 Mu'tah University, 61710 P.O. Pox: 7, Karak, Jordan

Abstract: In this paper new subclasses of p-valent harmonic analytic functions, within a generalization of integral operator with respect to k-symmetric points are introduced and explained. Coefficient bounds are estimated. Some properties of distortion, growth and extreme points are presented.

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INTRODUCTION

A continuous complex valued function $f = u+iv$ is said to be harmonic in a simply connected domain $D \in \mathbb{C}$ if both u and v are real harmonic in D . In any simply connected domain $D \in \mathbb{C}$ we can write $f = h + \bar{g}$, where h and g are analytic in D . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D [1]. Denote by \mathcal{SH} the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. Ahuja and Jahangiri [2] defined the class $\mathcal{H}_p(n)$ ($p, n \in \mathbb{N}$) consisting of all p-valent harmonic functions $f = h + \bar{g}$ that are sense preserving in \mathcal{U} and h and g are of the form

$$h(z) = z^p + \sum_{n=2}^{\infty} a_{n+p-1} z^{n+p-1}, g(z) = \sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1}, |b_p| < 1 \quad (1.1)$$

Observe that $\mathcal{H}_p(n)$ reduces to \mathcal{S}_p , the class of normalized univalent analytic functions, if the co-analytic part of f is zero.

A function f is said to be starlike of order α in \mathcal{U} denoted by $\mathcal{SH}^*(\alpha)$ [3] if

$$\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) = \Im \left\{ \frac{\frac{\partial}{\partial \theta} f(re^{i\theta})}{f(re^{i\theta})} \right\} = \Re \left\{ \frac{zh'(z) - z\overline{g'(z)}}{h(z) + \overline{g(z)}} \right\} \geq \alpha, |z| = r < 1$$

Corresponding Author: Faisal Al-Kasasbeh, Department of Mathematics and Statics, Faculty of Science, Mu'tah University, 61710 P.O. Pox: 7, Karak, Jordan

Also, a function f is p-valent harmonic starlike of order α in \mathcal{U} , denoted by $\mathcal{SH}^*(p, \alpha)$, if for $|z| = r < 1$.

$$\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) = \Im \left\{ \frac{\frac{\partial}{\partial \theta} f(re^{i\theta})}{f(re^{i\theta})} \right\} = \Re \left\{ \frac{zh'(z) - z\overline{g'(z)}}{h(z) + \overline{g(z)}} \right\} \geq p\alpha \quad (1.2)$$

Extending the above definitions to include the p-valent harmonic functions for $p \geq 1$ and $0 \leq \alpha < p^{-1}$, let $\mathcal{SH}_s^*(p, \alpha)$ denote the class of complex-valued, sense preserving, p-valent harmonic functions f of the form (1.1), which satisfy the condition

$$\Im \left\{ \frac{\frac{\partial}{\partial \theta} f(re^{i\theta})}{f_k(re^{i\theta})} \right\} \geq p\alpha$$

where $k \geq 2$ is a fixed positive integer, $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < 2\pi$, $\alpha \in [0, p^{-1})$ and $f_k(z) = h_k(z) + \overline{g_k(z)}$ satisfying:

$$h_k(z) = \frac{1}{k} \sum_{v=0}^{k-1} \epsilon^{-v} h(\epsilon^v z)$$

and

$$g_k(z) = \frac{1}{k} \sum_{v=0}^{k-1} \epsilon^{-v} g(\epsilon^v z), \text{ for } \epsilon = \exp\left(\frac{2\pi i}{k}\right) \quad (1.3)$$

Let j be an integer. Then the following identities follow directly:

$$f_k(\epsilon^j z) = \epsilon^j f_k(z)$$

$$f_k'(\epsilon^j z) = \epsilon^j f_k'(z)$$

$$f_k''(\epsilon^j z) = \epsilon^j f_k''(z)$$

The integral operator I^m is defined in [4] by:

- (i) $I^0 f(z) = f(z)$
- (ii) $I^1 f(z) = I f(z) = \int_0^z f(t) t^{-1} dt$
- (iii) $I^m f(z) = I(I^{m-1} f(z)), m \in \mathbb{N}$ and $f \in A$

and for a function $f = h + \bar{g}$ the integral operator I^m of f is defined as

$$I^m f(z) = I^m h(z) + (-1)^m \overline{I^m g(z)}$$

Also, several results have been discussed in [5-17] by many authers.

Definition 1.1: Let $f(z) \in \mathcal{SH}^*(p, \alpha)$, for $0 < \alpha < p^{-1}$ and $z = r e^{i\theta} \in U$. Then $f(z) \in \mathcal{ISH}_s^k(m, p, \alpha)$ if

$$\Im \left\{ \frac{I_\theta^m f(r e^{i\theta})}{I_\theta^{m+1} f_k(r e^{i\theta})} \right\} = \Re \left\{ \frac{I^m f(z)}{I^{m+1} f_k(z)} \right\} > \alpha p \quad (1.4)$$

where

$$I_\theta^m f(r e^{i\theta}) = \int_0^{r e^{i\theta}} f(r e^{i\theta}) (r e^{i\theta})^{-1} d(r e^{i\theta})$$

(integrate with respect to θ)
and

$$\begin{aligned} I^{m+1} f_k(z) &= I^{m+1} h_k(z) + (-1)^{m+1} \overline{I^{m+1} g_k(z)} \\ &= \frac{1}{k} \sum_{v=0}^{k-1} \epsilon^{-v} (I^{m+1} h(\epsilon^v z) + (-1)^{m+1} \overline{I^{m+1} g_k(\epsilon^v z)}) \end{aligned}$$

It is clear that

$$I^m f_k(\epsilon^j z) = \epsilon^j I^m f_k(z)$$

and

$$h_k(z) = z^p + \sum_{n=2}^{k-1} a_{n+p-1} \psi_{n+p-1} z^{n+p-1}$$

$$g_k(z) = \sum_{n=1}^{k-1} b_{n+p-1} \psi_{n+p-1} z^{n+p-1}$$

where

$$\psi_{n+p-1} = \frac{1}{k} \sum_{v=0}^{k-1} \epsilon^{-(n+p)v} = \begin{cases} 1, & n+p=jk, j \in \mathbb{N} \\ 0, & n+p=jk+1, j \in \mathbb{N} \end{cases}$$

Here we state this Lemma due to [5].

Lemma 1.2: [5] Let $f = h + \bar{g}$ if

$$\sum_{n=1}^{\infty} \left(\frac{n-\alpha}{1-\alpha} |a_n| + \frac{n+\alpha}{1-\alpha} |b_n| \right) \leq 2$$

for $|a_1|=1, 0 \leq \alpha < p^{-1}$. Then f is sense-preserving, harmonic univalent and starlike of order α in U .

Next corollary is an immediate result for the p -valent Harmonic functions that are given in (1.1).

Corollary 1.3: Let $f = h + \bar{g}$ be given in (1.1) if

$$\sum_{n=1}^{\infty} \left(\frac{n+p-1-\alpha}{1-\alpha} |a_{n+p-1}| + \frac{n+p+\alpha-1}{1-\alpha} |b_{n+p-1}| \right) \leq 2$$

for $|a_p|=1, 0 \leq \alpha < p^{-1}$. Then f is sense-preserving, harmonic univalent and starlike of order α in U .

THE MAIN RESULTS

A meaningful conclusion for the class $\mathcal{ISH}_s^k(m, p, \alpha)$ is introduced as follows.

Theorem 2.1: A function $f(z) \in \mathcal{SH}_s^*(p, \alpha)$ where $f = h + \bar{g}$ given by (1.1), then $f_k = h_k + \bar{g}_k$ given by (1.3) in $\mathcal{ISH}_s^k(m, p, \alpha)$.

Proof: Since $f(z) \in \mathcal{ISH}_s^k(m, p, \alpha)$, then replace $z = r e^{i\theta}$ by $\epsilon^v r e^{i\theta}$, ($v = 0, 1, \dots, k-1$), where $\epsilon^k = 1$ in (1.3) which is true such that:

$$\begin{aligned} \Im \left\{ \frac{I_\theta^m f(z)}{I_\theta^{m+1} f_k(z)} \right\} &= \Im \left\{ \frac{I_\theta^m f(\epsilon^v r e^{i\theta})}{I_\theta^{m+1} f_k(\epsilon^v r e^{i\theta})} \right\} \\ &= \Re \left\{ \frac{I^m f(\epsilon^v z)}{I^{m+1} f_k(\epsilon^v z)} \right\} > \alpha p, \text{ for } v = 0, 1, \dots, k-1 \end{aligned} \quad (2.1)$$

According to the definition of f^k , $\epsilon^k = 1$ and $I^m f_k(\epsilon^j z) = \epsilon^j I^m f_k(z)$, let $v = 0, 1, 2, \dots, k-1$ in (2.1) and sum all of them. Then we have

$$\begin{aligned} \Im \left\{ \frac{1}{k} \sum_{v=0}^{k-1} \frac{I_\theta^m f(\epsilon^v r e^{i\theta})}{I_\theta^{m+1} \epsilon^v f_k(r e^{i\theta})} \right\} &= \Im \left\{ \frac{I_\theta^m f_k(r e^{i\theta})}{I_\theta^{m+1} f_k(r e^{i\theta})} \right\} \\ &= \Re \left\{ \frac{I^m f_k(z)}{I^{m+1} f_k(z)} \right\} > \alpha p \end{aligned}$$

Thus $f_k \in \mathcal{ISH}_s^k(m, p, \alpha)$.

The sufficient coefficient bound for the p -valent harmonic functions in the subclass $\mathcal{ISH}_s^k(m,p,\alpha)$ is deduced and presented.

Theorem 2.2: Let a function $f = h + \bar{g}$ be given in (1.1). If

$$\sum_{n=1}^{\infty} \left(\frac{n+p-\alpha-1}{(n+p-1)^{m+1}} |a_{n+p-1}| + \left(\frac{n+p+\alpha-1}{(n+p-1)^{m+1}} \right) |b_{n+p-1}| \right) \leq \frac{2(1-p\alpha)}{p^m} \quad (2.2)$$

where $|a_p| = 1, m \in \mathbb{N}$ and $0 \leq \alpha < p^{-1}$. Then $f \in \mathcal{ISH}_s^k(m,p,\alpha)$ and it is sense-preserving in \mathcal{U} .

Proof: From the definition of $I^m f(z)$ and (1.2), since it obvious for $\alpha = 0$, we try to show that

$$\Re \left\{ \frac{I^m f(z) - \alpha p I^{m+1} f_k(z)}{I^{m+1} f_k(z)} \right\} > 0$$

For $0 < \alpha < p^{-1}$ it follows that

$$\begin{aligned} \Re \left\{ \frac{I^m f(z) - \alpha p I^{m+1} f_k(z)}{I^{m+1} f_k(z)} \right\} &= \Re \left\{ \frac{(p)^{-m} z^p (1-p^2\alpha) + \sum_{n=2}^{\infty} (n+p-1)^{-(m+1)} ((n+p-1)-\alpha) a_{n+p-1} z^{n+p-1}}{(p)^{-m} z^p + (-1)^{m+1} \bar{b}_p \bar{z}^p + \sum_{n=2}^{\infty} (n+p-1)^{-(m+1)} [a_{n+p-1} z^{n+p-1} + (-1)^{m+1} \bar{b}_{n+p-1} \bar{z}^{n+p-1}]} \right. \\ &\quad \left. + \frac{(-1)^m \sum_{n=1}^{\infty} (n+p-1)^{-(m+1)} ((n+p-1)+\alpha) \bar{b}_{n+p-1} \bar{z}^{n+p-1}}{(p)^{-m} z^p + (-1)^{m+1} \bar{b}_p \bar{z}^p + \sum_{n=2}^{\infty} (n+p-1)^{-(m+1)} [a_{n+p-1} z^{n+p-1} + (-1)^{m+1} \bar{b}_{n+p-1} \bar{z}^{n+p-1}]} \right\} \\ &= \Re \left\{ \frac{(1-p\alpha) + \sum_{n=2}^{\infty} p^m (n+p-1)^{-(m+1)} ((n+p-1)-\alpha) a_{n+p-1} z^{n-1}}{1 + (-1)^{m+1} p^m \bar{b}_p \bar{z}^p + \sum_{n=2}^{\infty} p^m (n+p-1)^{-(m+1)} [a_{n+p-1} z^{n-1} + (-1)^{m+1} \bar{b}_{n+p-1} \bar{z}^{n+p-1} z^{-p}]} \right. \\ &\quad \left. + \frac{(-1)^m \sum_{n=1}^{\infty} (n+p-1)^{-(m+1)} ((n+p-1)+\alpha) \bar{b}_{n+p-1} \bar{z}^{n+p-1} z^{-p}}{1 + (-1)^{m+1} p^m \bar{b}_p \bar{z}^p + \sum_{n=2}^{\infty} p^m (n+p-1)^{-(m+1)} [a_{n+p-1} z^{n-1} + (-1)^{m+1} \bar{b}_{n+p-1} \bar{z}^{n+p-1} z^{-p}]} \right\} = \Re \left\{ \frac{1-p\alpha + \phi(re^{i\theta})}{1 + \xi(re^{i\theta})} \right\} \end{aligned}$$

where

$$\phi(re^{i\theta}) = \sum_{n=2}^{\infty} (n+p-1)^{-(m+1)} p^m ((n+p-1)-\alpha) a_{n+p-1} r^{(n-1)} e^{(n-1)i\theta} + (-1)^m \sum_{n=1}^{\infty} (n+p-1)^{-(m+1)} p^m ((n+p-1)+\alpha) \bar{b}_{n+p-1} r^{(n-1)} e^{-(n-1)i\theta}$$

and

$$\xi(re^{i\theta}) = \sum_{n=2}^{\infty} (n+p-1)^{-(m+1)} p^m a_{n+p-1} r^{(n-1)} e^{(n-1)i\theta} + (-1)^{m+1} \sum_{n=1}^{\infty} (n+p-1)^{-(m+1)} p^m \bar{b}_{n+p-1} r^{(n-1)} e^{-(n-1)i\theta}$$

Consider

$$\frac{1-p\alpha + \phi(z)}{1 + \xi(z)} = (1-\alpha) \frac{1 + \omega(z)}{1 - \omega(z)}$$

Then

$$|\omega(z)| = \left| \frac{\phi(z) - (1-p\alpha) \xi(z)}{\phi(z) + (1-p\alpha) \xi(z) + 2(1-p\alpha)} \right| < 1$$

Thus the harmonic functions

$$f(z) = z^p + \sum_{n=2}^{\infty} (n+p-1)^{(m+1)} p^m \frac{(1-p\alpha)}{n+p(1-\alpha)-1} x_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} (n+p-1)^{(m+1)} p^m \frac{(1-p\alpha)}{n+p(1+\alpha)-1} \bar{y}_{n+p-1} \bar{z}^{n+p-1}$$

where $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} |x_{n+p-1}| + \sum_{n=1}^{\infty} |y_{n+p-1}| = 1$$

which show that the coefficient bound given by (2.2) is sharp.

Also, the sufficient coefficient condition for a function $\mathcal{SH}^*(p, \alpha)$ to be in $\mathcal{ISH}_s^k(m, p, \alpha)$ is derived.

Theorem 2.3: Let $f = h + \bar{g}$ be given by (1.1) and $f_k = h_k + \bar{g}_k$ be given by (1.3) satisfying

$$\sum_{n=1}^{\infty} \left(\frac{n+p(1+\alpha\psi_{n+p-1})-1}{(n+p-1)^{m+1}} \right) |a_{n+p-1}| + \left(\frac{n+p(1-\alpha\psi_{n+p-1})-1}{(n+p-1)^{m+1}} \right) |b_{n+p-1}| \leq \frac{(p+1)(1-p\alpha)}{p^m} \quad (2.3)$$

where $m \in \mathbb{N}_0 = \{0, 1, \dots\}$ and $0 \leq \alpha < p^{-1}$, then f is harmonic, orientation-preserving in \mathcal{U} and $f \in \mathcal{ISH}_s^k(m, p, \alpha)$.

Proof: To show that f is orientation-preserving in \mathcal{U} , it is enough to show that $|h'(z)| \geq g(z)|$ in \mathcal{U} . Thus

$$\begin{aligned} |h'(z)| &= \left| p z^{p-1} + \sum_{n=1}^{\infty} (n+p-1) a_{n+p-1} z^{n+p-2} \right| \geq p |z^{p-1}| - \sum_{n=1}^{\infty} (n+p-1) |a_{n+p-1}| |z^{n+p-2}| \\ &= p r^{p-1} - \sum_{n=1}^{\infty} (n+p-1) |a_{n+p-1}| r^{n+p-2} \geq p - \sum_{n=1}^{\infty} (n+p-1) |a_{n+p-1}| \\ &\geq \sum_{n=1}^{\infty} \left(\frac{(p^m)(n+p(1-\alpha\psi_{n+p-1})-1)}{(p+1)(n+p-1)^{m+1}(1-p\alpha)} \right) |b_{n+p-1}| \\ &\geq \sum_{n=1}^{\infty} (p+n-1) |b_{n+p-1}| \geq \sum_{n=1}^{\infty} (n+p-1) |b_{n+p-1}| r^{n+p-2} \\ &\geq \sum_{n=1}^{\infty} (n+p-1) |b_{n+p-1}| |z^{n+p-2}| = |g'(z)| \end{aligned}$$

To show that $f \in \mathcal{ISH}_s^k(m, p, \alpha)$, it suffices to prove that

$$\Re \left\{ \frac{I^m f(z) - \alpha p I^{m+1} f_k(z)}{I^{m+1} f_k(z)} \right\} > 0 \text{ or } \Re \left\{ \frac{I^m h(z) - (-1)^m \overline{I^m g(z)} - \alpha p (I^{m+1} h_k(z) - (-1)^{m+1} \overline{I^{m+1} g_k(z)})}{I^{m+1} h_k(z) + (-1)^{m+1} \overline{I^{m+1} g_k(z)}} \right\} > 0$$

Let

$$F(z) = I^m h(z) - (-1)^m \overline{I^m g(z)}$$

and

$$G(z) = I^{m+1} h_k(z) + (-1)^{m+1} \overline{I^{m+1} g_k(z)}$$

Then

$$|F(z) + (1-p\alpha)G(z)| - |F(z) - (1+p\alpha)G(z)| \geq 0$$

and

$$\begin{aligned} |F(z) + (1-p\alpha)G(z)| &= |I^m h(z) - (-1)^m \overline{I^m g(z)} - (1-p\alpha)(I^{m+1} h_k(z) + (-1)^{m+1} \overline{I^{m+1} g_k(z)})| \\ &= |(2-p\alpha)z^p - (-1)^m \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1 - (1-p\alpha)\psi_{n+p-1}) a_{n+p-1} z^{n+p-1} \\ &\quad + \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1 + (1-p\alpha)\psi_{n+p-1}) \bar{b}_{n+p-1} \bar{z}^{n+p-1}| \\ &\geq (2-p\alpha)z^p - \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1 + (1-p\alpha)\psi_{n+p-1}) a_{n+p-1} |z|^{n+p-1} \\ &\quad + \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1 + (1-p\alpha)\psi_{n+p-1}) |\bar{b}_{n+p-1}| |z|^{n+p-1} \end{aligned}$$

and

$$\begin{aligned}
 |F(z) - (1+p\alpha)G(z)| &= |I^m h(z) - (-1)^m \overline{I^m g(z)} - (1+p\alpha)(I^{m+1}h_k(z) + (-1)^{m+1}\overline{I^{m+1}g_k(z)})| \\
 &= |-(p\alpha)z^p - (-1)^m \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p+(1+p\alpha)\psi_{n+p-1}) a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1-(1+p\alpha)\psi_{n+p-1}) \bar{b}_{n+p-1} \bar{z}^{n+p-1}| \\
 &= |(p\alpha)z^p + (-1)^m \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1+(1+p\alpha)\psi_{n+p-1}) a_{n+p-1} z^{n+p-1} + \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1-(1-p\alpha)\psi_{n+p-1}) b_{n+p-1} z^{n+p-1}| \\
 &\leq (p\alpha) + \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1+(1+p\alpha)\psi_{n+p-1}) |a|_{n+p-1} |z|^{n+p-1} - \sum_{n=1}^{\infty} (p+n-1)^{-m} (n+p-1-(1-p\alpha)\psi_{n+p-1}) |b|_{n+p-1} |z|^{n+p-1}
 \end{aligned}$$

Thus

$$|F(z) + (1-p\alpha)G(z)| - |F(z) - (1+p\alpha)G(z)| \geq 0$$

which completes this proof.

Theorem 2.4: If $f = h + \bar{g} \in \mathcal{ISH}_s^k(m, p, \alpha)$ for $|z| = r < 1$, then

$$r^p - \frac{(p+1)(1-p\alpha)}{rp^m} \leq |f(z)| \leq r^p + \frac{(p+1)(1-p\alpha)}{rp^m}$$

Proof: By taking the absolute value for a function $f = h + \bar{g} \in \mathcal{ISH}_s^k(m, p, \alpha)$, it follows that

$$\begin{aligned}
 |f(z)| &= |z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} + (-1)^m \overline{\sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1}}| \leq r^p + \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) r^{n+p-1} \\
 &\leq r^p + r^{-1} \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \leq r^p + r^{-1} \left(\frac{(p+1)(1-p\alpha)}{(1-p\alpha)} \right) \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \cdot \left(\frac{(n+p(1-\alpha\psi_{n+p-1})-1)}{(n+p-1)^{m+1}} \right) \left(\frac{(n+p-1)^{m+1}}{n+p(1-\alpha\psi_{n+p-1})-1} \right) \\
 &\leq r^p + r^{-1}(p+1)(1-p\alpha) \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \left(\frac{1}{1-p\alpha} \right) \cdot \left(\frac{(n+p(1-\alpha\psi_{n+p-1})-1)}{(n+p-1)^{m+1}} \right) \\
 &\leq r^p + r^{-1}(p+1) \sum_{n=1}^{\infty} \left(\frac{n+p(1+\alpha\psi_{n+p-1})-1}{(n+p-1)^{m+1}} \right) |a_{n+p-1}| + \left(\frac{n+p(1-\alpha\psi_{n+p-1})-1}{(n+p-1)^{m+1}} \right) |b_{n+p-1}| \leq r^p + \frac{(p+1)(1-p\alpha)}{rp^m}
 \end{aligned}$$

and

$$\begin{aligned}
 |f(z)| &= |z^p + \sum_{n=1}^{\infty} a_{n+p-1} z^{n+p-1} + (-1)^m \overline{\sum_{n=1}^{\infty} b_{n+p-1} z^{n+p-1}}| \geq r^p - \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) r^{n+p-1} \geq r^p - r^{-1} \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \\
 &\geq r^p - r^{-1} \left(\frac{(1-p\alpha)}{(1-p\alpha)} \right) (p+1) \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \cdot \left(\frac{(n+p(1-\alpha\psi_{n+p-1})-1)}{(n+p-1)^{m+1}} \right) \left(\frac{(n+p-1)^{m+1}}{n+p(1-\alpha\psi_{n+p-1})-1} \right) \\
 &\geq r^p - r^{-1}(p+1)(1-p\alpha) \sum_{n=1}^{\infty} (|a_{n+p-1}| + |b_{n+p-1}|) \left(\frac{(p+1)}{1-p\alpha} \right) \cdot \left(\frac{(n+p(1-\alpha\psi_{n+p-1})-1)}{(n+p-1)^{m+1}} \right) \\
 &\geq r^p - r^{-1}(p+1) \sum_{n=1}^{\infty} \left(\frac{n+p(1+\alpha\psi_{n+p-1})-1}{(n+p-1)^{m+1}} \right) |a_{n+p-1}| + \left(\frac{n+p(1-\alpha\psi_{n+p-1})-1}{(n+p-1)^{m+1}} \right) |b_{n+p-1}| \geq r^p - \frac{(p+1)(1-p\alpha)}{rp^m}
 \end{aligned}$$

Next corollary of the growth is an immediate result due to the above theorem of distortion.

Corollary 2.5: If $f = h + \bar{g} \in \mathcal{ISH}_s^k(m, p, \alpha)$ for $|z| = r < 1$, then

$$pr^{p-1} - \frac{(p+1)(1-p\alpha)}{r^2 p^m} \leq |f(z)| \leq pr^{p-1} + \frac{(p+1)(1-p\alpha)}{r^2 p^m}$$

In the end of this article, the extreme points for these functions $f \in \mathcal{ISH}_s^k(m, p, \alpha)$ are estimated.

Theorem 2.6: A function $f = h + \bar{g} \in \mathcal{ISH}_s^k(p, \alpha)$ if and only if f can be expressed in the form:

$$f(z) = \sum_{n=0}^{\infty} (\xi_n^p h_n(z) + \mu_n^p g_n(z)) \quad (2.4)$$

where $z \in \mathcal{U}$, $0 \leq \alpha < p^{-1}$ and

$$h_o(z) = z^p, \quad h_n(z) = z^p + \frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1+\alpha\psi_{n+p-1})-1)} z^{n+p-1}$$

$$g_o(z) = z^p, \quad g_n(z) = z^p + \frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1-\alpha\psi_{n+p-1})-1)} \overline{z^{n+p-1}}$$

for $n = 1, 2, 3, \dots$. Also,

$$\sum_{n=0}^{\infty} (\xi_n^p + \mu_n^p) = 1, \quad \text{for } \xi_n^p, \mu_n^p \geq 0$$

In particular, the extreme points of $\mathcal{ISH}_s^k(p, \alpha)$ are h_n and g_n .

Proof: Assume that

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} (\xi_n^p h_n(z) + \mu_n^p g_n(z)) = \xi_o h_o(z) + \mu_o^p g_o(z) + \sum_{n=0}^{\infty} \xi_n^p \left(z^p + \frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1+\alpha\psi_{n+p-1})-1)} z^{n+p-1} \right) \\ &\quad + \sum_{n=0}^{\infty} \mu_n^p \left(z^p + \frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1-\alpha\psi_{n+p-1})-1)} \overline{z^{n+p-1}} \right) = \sum_{n=0}^{\infty} (\xi_n^p + \mu_n^p) z^{n+p-1} \\ &\quad + \sum_{n=0}^{\infty} \left[\xi_n^p \left(\frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1+\alpha\psi_{n+p-1})-1)} \right) z^{n+p-1} + \mu_n^p \left(\frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1-\alpha\psi_{n+p-1})-1)} \right) \overline{z^{n+p-1}} \right] \end{aligned}$$

Since

$$\begin{aligned} &\sum_{n=0}^{\infty} (n+p-1)^{m+1} \frac{(n+p(1+\alpha\psi_{n+p-1})-1)}{(p+1)(1-p\alpha)} \left(\frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1+\alpha\psi_{n+p-1})-1)} \xi_n^p \right) \\ &+ (n+p-1)^{m+1} \frac{(n+p(1-\alpha\psi_{n+p-1})-1)}{(p+1)(1-p\alpha)} \left(\frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1-\alpha\psi_{n+p-1})-1)} \mu_n^p \right) = \sum_{n=1}^{\infty} (\xi_n^p + \mu_n^p) = 1 - \xi_o^p - \mu_o^p \leq 1 \end{aligned}$$

Conversely, let $f \in \mathcal{ISH}_s^k(m, p, \alpha)$. Then (2.3) holds.

Set

$$\xi_n^p = \left(\frac{(n+p(1+\alpha\psi_{n+p-1})-1)(n+p-1)^{m+1}}{(p+1)(1-p\alpha)} \right) |a_{n+p-1}|$$

and

$$\mu_n^p = \left(\frac{(n+p-1)^{m+1}(n+p(1-\alpha\psi_{n+p-1})-1)}{(p+1)(1-p\alpha)} \right) |b_{n+p-1}|$$

Then

$$\begin{aligned} f(z) &= z^p + \sum_{n=0}^{\infty} |a_{n+p-1}| z^{n+p-1} + (-1)^m \sum_{n=1}^{\infty} |b_{n+p-1}| \overline{z^{n+p-1}} = z^p + \sum_{n=0}^{\infty} \frac{(1+p)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1+\alpha\psi_{n+p-1})-1)} \xi_n^p z^{n+p-1} \\ &\quad + (-1)^m \sum_{n=1}^{\infty} \frac{(p+1)(1-p\alpha)}{(n+p-1)^{m+1}(n+p(1-\alpha\psi_{n+p-1})-1)} \mu_n^p \overline{z^{n+p-1}} = z^p + \sum_{n=1}^{\infty} (h_n - z^p) \xi_n^p + \sum_{n=1}^{\infty} (g_n - z^p) \mu_n^p \\ &= z^p (1 - \sum_{n=1}^{\infty} \xi_n^p - \sum_{n=1}^{\infty} \mu_n^p) + \sum_{n=1}^{\infty} (h_n \xi_n^p + g_n \mu_n^p) = \xi_o h_o + \mu_o^p g_o + \sum_{n=1}^{\infty} h_n \xi_n^p + \sum_{n=1}^{\infty} g_n \mu_n^p = \sum_{n=0}^{\infty} (h_n \xi_n^p + g_n \mu_n^p) \end{aligned}$$

Which is completed our proof.

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