# A Model of the Width of an Oval via Differential Equations 

Kamal A.S. Al-Banawi and Omar K. Jaradat<br>Department of Mathematics, Faculty of Science, Mutah University, P.O.Box 7, Al-Karak, Jordan


#### Abstract

In this paper we produce a model for the width, the perpendicular displacement between the two supporting parallel tangent lines, of closed convex curves (ovals) in $\mathrm{R}^{2}$. The modelling uses linear second order ordinary differential equations with constant coefficients. We use such a model to deduce some results regarding ovals of constant width in $\mathrm{R}^{2}$.


Key words: Ovals, Width, Focal points, Ordinary differential equations

## INTRODUCTION

Throughout this paper, we deal with ovals in $\mathrm{R}^{2}$. An oval is a smooth ( $\mathrm{C}^{\infty}$ ) convex closed curve C in $\mathrm{R}^{2}$ [10]. Convexity of curves can be seen in two different but equivalent manners. A smooth simple closed curve C in $\mathrm{R}^{2}$ is convex if C lies entirely at one side of it's tangent at any point chosen on it [5, 11]. Also C is convex in $\mathrm{R}^{2}$ if the curvature of C is strictly positive at each point on $C$ [1]. Convex curves were studied using the support function [6]. For recent work regarding the study of convex domains using the support function [2,3].

Two points on a curve are said to be opposite to each other provided that tangent lines at the points are parallel. Ovals are characterized by the fact that every point has an opposite point. Also parallel tangents in this case are called supporting lines of the oval. The width of an oval is defined as the perpendicular distance between the supporting lines. An oval is said to be of constant width if the width does not depend on the choice of the point on the oval. Ovals of constant width in $\mathrm{R}^{2}$ were studied in [7]. Mellish's work was explained in [9]. Geometric and analytic properties of ovals of constant width were studied in [2] and most recently in [4].

## GEOMETRY OF CURVES IN R ${ }^{\mathbf{2}}$

A smooth $\left(\mathrm{C}^{\infty}\right)$ curve C in $\mathrm{R}^{2}$ is a differentiable function $\vec{f}: I \rightarrow R^{2}$ where $I$ is an (open) interval. Thus, if $t$ is a parameter of $\vec{f}, t \in I$, then we write $\vec{f}$ as $\overrightarrow{\mathrm{f}}(\mathrm{t})=\left(\mathrm{f}_{1}(\mathrm{t}), \mathrm{f}(\mathrm{t})\right)$ where $f_{1}$ and $f_{1}$ are differentiable real valued functions defined on I. The curve C is
regular provided that $\vec{f}^{\prime}(t) \neq \overline{0}$ for all $t \in I$. Now we take $\mathrm{I}=[\mathrm{a}, \mathrm{b}]$ and define the arclength along C by

$$
\begin{equation*}
s(t)=\int_{a}^{t}\left\|\vec{f}^{\prime}(t)\right\| t \tag{1}
\end{equation*}
$$

where $\left\|\bar{f}^{\prime}(\mathrm{t})\right\|$ is the norm of $\overline{\mathrm{f}}^{\prime}(\mathrm{t})$, usually called the speed of $\overrightarrow{\mathrm{f}}$. If C is a regular curve, then $\left\|\overline{\mathrm{f}}^{\prime}(\mathrm{t})\right\|>0$ for all $t \in I$ and so $s$ is a strictly increasing function of $t$ which has an inverse. That is, Eq.(1) can be solved for s and then C has a reparametrization by arclength.

Now take $\vec{f}$ with arclength reparametrization and let $\overrightarrow{\mathrm{T}}(\mathrm{s})=\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{s})$ be the unit tangent of $\overrightarrow{\mathrm{f}}$ at s . Then the curvature of $\overrightarrow{\mathrm{f}}$ is the function $\kappa(\mathrm{s})=\left\|\overrightarrow{\mathrm{T}}^{\prime}(\mathrm{s})\right\|=\| \overrightarrow{\mathrm{f}}^{\prime \prime}(\mathrm{s}) \mid$. Also let $\overrightarrow{\mathrm{N}}(\mathrm{s})$ be the unit normal of $\overrightarrow{\mathrm{f}}$ at s . Then the equations

$$
\begin{gather*}
\overrightarrow{\mathrm{T}}^{\prime}(\mathrm{s})=\kappa(\mathrm{s}) \stackrel{\mathrm{N}}{ }(\mathrm{~s})  \tag{2}\\
\overrightarrow{\mathrm{N}}^{\prime}(\mathrm{s})=-\kappa(\mathrm{s}) \overrightarrow{\mathrm{T}}(\mathrm{~s}) \tag{3}
\end{gather*}
$$

are called Frenet formulas [8].
Let $\psi(s)$ be the slope angle of the tangent line at $\vec{f}(\mathrm{~s})$. Then the unit tangent of $\overrightarrow{\mathrm{f}}$ at s is defined by

$$
\overline{\mathrm{T}}(\mathrm{~s})=(\cos \psi(\mathrm{s}), \sin \psi(\mathrm{s}))
$$

Now

$$
\mathrm{k}(\mathrm{~s})=\left|\overrightarrow{\mathrm{T}}^{\prime}(\mathrm{s})\right|=\left\|\psi^{\prime}(\mathrm{s})(-\sin \psi(\mathrm{s}), \cos \psi(\mathrm{s}))\right\|
$$

That is

$$
\begin{equation*}
\mathrm{k}(\mathrm{~s})=\psi^{\prime}(\mathrm{s}) \tag{4}
\end{equation*}
$$

or

$$
\begin{equation*}
\psi(s)=\int_{0}^{s} k(s) d s \tag{5}
\end{equation*}
$$

Now if $\vec{f}(\mathrm{~s})=(\mathrm{x}(\mathrm{s}), \mathrm{y}(\mathrm{s}))$, then

$$
\begin{equation*}
x(s)=\int_{0}^{s} \cos \psi(s) d s \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{y}(\mathrm{~s})=\int_{0}^{\mathrm{s}} \sin \psi(\mathrm{~s}) \mathrm{ds} \tag{7}
\end{equation*}
$$

Example 1: Let $\mathrm{k}(\mathrm{s})=\frac{1}{\mathrm{~s}^{2}+1}$ with $\overrightarrow{\mathrm{f}}(0)=(0,0)$. Then by Eq.(5),

$$
\psi(s)=\int_{0}^{s} k(s) d s=\int_{0}^{s} \frac{1}{s^{2}+1} d s=\tan ^{-1} s .
$$

Now by Eq.(6), we have

$$
x(s)=\int \cos \left(\tan ^{-1} s\right) d s=\int \frac{1}{\sqrt{s^{2}+1}} d s=\sinh ^{-1} s+c_{1}
$$

But $x(0)=0$ and so $x(s)=\sinh ^{-1} s$. Also by Eq.(7), we have

$$
y(s)=\int \sin \left(\tan ^{-1} s\right) d s=\int \frac{s}{\sqrt{s^{2}+1}} d s=\sqrt{s^{2}+1}+c_{2}
$$

But $\mathrm{y}(0)=0$ and so $\mathrm{y}(\mathrm{s})=\sqrt{\mathrm{s}^{2}+1}-1$. So $\overrightarrow{\mathrm{f}}(\mathrm{s})=\left(\sinh ^{-1} \mathrm{~s}, \sqrt{\mathrm{~s}^{2}+1}-1\right)$

## THE FOCAL CURVE OF A CURVE IN $\mathbf{R}^{\mathbf{2}}$

Definition 1: [8] The focal curve of a regular curve $\bar{f}$ is the curve $\overline{\mathrm{g}}=\overrightarrow{\mathrm{f}}+\frac{1}{\kappa} \stackrel{\mathrm{~N}}{ }$.

Similarly, the focal point with base $\overrightarrow{\mathrm{f}}(\mathrm{t})$ is the point $\stackrel{\mathrm{g}}{\mathrm{g}}(\mathrm{t})=\stackrel{\mathrm{f}}{\mathrm{f}}(\mathrm{t})+\frac{1}{\kappa(\mathrm{t})} \stackrel{\mathrm{N}}{\mathrm{N}}(\mathrm{t})$.

Now let $\Lambda_{\stackrel{\rightharpoonup}{p}}(\mathrm{t})=\mid \overrightarrow{\mathrm{f}}(\mathrm{t})-\overline{\mathrm{p}} \|^{2}$ be the distance function whose domain is the curve C with parametrization $\overrightarrow{\mathrm{f}}$.

Theorem 1: The point $\stackrel{\rightharpoonup}{\mathrm{p}}$ is a focal point of C with base $\overrightarrow{\mathrm{f}}(\mathrm{t})$ iff $\Lambda_{\mathrm{p}}^{\prime}(\mathrm{t})=0$ and $\Lambda_{\mathrm{p}}^{\prime \prime}(\mathrm{t})=0$.

Proof: Recall that

$$
\Lambda_{\mathrm{p}}(\mathrm{t})=\mid \stackrel{\mathrm{f}}{\mathrm{f}}(\mathrm{t})-\overline{\mathrm{p}} \|^{2}=(\stackrel{\mathrm{f}}{\mathrm{f}}(\mathrm{t})-\stackrel{\rightharpoonup}{\mathrm{p}}) \cdot(\overrightarrow{\mathrm{f}}(\mathrm{t})-\overline{\mathrm{p}})
$$

Thus

$$
\Lambda_{\stackrel{\rightharpoonup}{\mathrm{p}}}^{\prime}(\mathrm{t})=2(\overrightarrow{\mathrm{f}}(\mathrm{t})-\stackrel{\rightharpoonup}{\mathrm{p}}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})
$$

and

$$
\Lambda_{\stackrel{\rightharpoonup}{\mathrm{p}}}^{\prime \prime}(\mathrm{t})=2(\overrightarrow{\mathrm{f}}(\mathrm{t})-\stackrel{\rightharpoonup}{\mathrm{p}}) \cdot \overrightarrow{\mathrm{f}}^{\prime \prime}(\mathrm{t})+2 \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})
$$

If $\vec{p}$ is a focal point of $C$ with base $\vec{f}(t)$, then $\overrightarrow{\mathrm{p}}=\overrightarrow{\mathrm{f}}(\mathrm{t})+\frac{1}{\kappa(\mathrm{t})} \overline{\mathrm{N}}(\mathrm{t})$. Thus,

$$
\Lambda_{\stackrel{\rightharpoonup}{\mathrm{p}}}^{\prime}(\mathrm{t})=2\left(\overrightarrow{\mathrm{f}}(\mathrm{t})-\overrightarrow{\mathrm{f}}(\mathrm{t})-\frac{1}{\kappa(\mathrm{t})} \overline{\mathrm{N}}(\mathrm{t})\right) \cdot \overrightarrow{\mathrm{f}}(\mathrm{t})=\frac{-2}{\kappa(\mathrm{t})} \overrightarrow{\mathrm{N}}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}(\mathrm{t})=0
$$

Also

$$
\begin{aligned}
\Lambda_{\bar{p}}^{\prime \prime}(\mathrm{t}) & =2\left(\overrightarrow{\mathrm{f}}(\mathrm{t})-\overrightarrow{\mathrm{f}}(\mathrm{t})-\frac{1}{\kappa(\mathrm{t})} \overline{\mathrm{N}}(\mathrm{t})\right) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})+2 \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \\
& =\frac{2}{\kappa(\mathrm{t})} \overrightarrow{\mathrm{N}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})+2 \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \\
& =-2 \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})+2 \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}^{\prime}}(\mathrm{t})=0
\end{aligned}
$$

Conversely, the equation $\Lambda_{\overline{\mathrm{p}}}^{\prime}(\mathrm{t})=0$ implies that $\overrightarrow{\mathrm{f}}(\mathrm{t})-\overrightarrow{\mathrm{p}}=\mathrm{m} \stackrel{\rightharpoonup}{\mathrm{N}}$. Now put $\overrightarrow{\mathrm{p}}=\overrightarrow{\mathrm{f}}(\mathrm{t})-\mathrm{m} \stackrel{\rightharpoonup}{\mathrm{N}}$ in the equation $\Lambda_{\bar{p}}^{\prime \prime}(\mathrm{t})=0$ to get $\mathrm{m} \overrightarrow{\mathrm{N}} \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})+\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \overrightarrow{\mathrm{f}}(\mathrm{t})=0$, which is equivalent to

$$
-\mathrm{m} \overrightarrow{\mathrm{~N}}^{\prime} \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})+\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})=0
$$

or

$$
\mathrm{mk}(\mathrm{t}) \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})+\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \cdot \overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})=0
$$

Thus, $m=-\frac{1}{\kappa(t)}$. So $\overrightarrow{\mathrm{p}}=\overrightarrow{\mathrm{f}}(\mathrm{t})+\frac{1}{\kappa(\mathrm{t})} \overrightarrow{\mathrm{N}}(\mathrm{t})$, i.e. $\overrightarrow{\mathrm{p}}$ is a focal point of $C$ with base $\vec{f}(t)$.

Example 2: Let $\overrightarrow{\mathrm{f}}(\mathrm{t})=\left(\mathrm{t}, \mathrm{t}^{2}\right)$. Then $\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t})=(1,2 \mathrm{t})$ and $\overrightarrow{\mathrm{f}}^{\prime \prime}(\mathrm{t})=(0,2)$. So $\left\|\mathrm{f}^{\prime}(\mathrm{t})\right\|=\sqrt{1+4 \mathrm{t}^{2}}$ and $\left\|\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \times \overrightarrow{\mathrm{f}}^{\prime \prime}(\mathrm{t})\right\|=2 \mathrm{~S}$. As in [8], the curvature at $\vec{f}(t)$ is

$$
\mathrm{k}(\mathrm{t})=\frac{\left\|\overrightarrow{\mathrm{f}}^{\prime}(\mathrm{t}) \times \overrightarrow{\mathrm{f}}^{\prime \prime}(\mathrm{t})\right\|}{\|\overrightarrow{\mathrm{f}}(\mathrm{t})\|^{3}}=\frac{2}{\left(\sqrt{1+4 \mathrm{t}^{2}}\right)^{3}}
$$

By solving the equation

$$
(0,2)=\frac{4 \mathrm{t}}{\sqrt{1+4 \mathrm{t}^{2}}} \frac{(1,2 \mathrm{t})}{\sqrt{1+4 \mathrm{t}^{2}}}+\frac{2}{\left(\sqrt{\left.1+4 \mathrm{t}^{2}\right)^{3}}\right.}\left(1+4 \mathrm{t}^{2}\right) \stackrel{\mathrm{N}}{ }(\mathrm{t})
$$

for $\overrightarrow{\mathrm{N}}(\mathrm{t})$, we get $\stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t})=\frac{(-2 \mathrm{t}, 1)}{\sqrt{1+4 \mathrm{t}^{2}}}$. Thus,

$$
\overrightarrow{\mathrm{g}}(\mathrm{t})=\left(\mathrm{t}, \mathrm{t}^{2}\right)+\frac{\left(\sqrt{1+4 \mathrm{t}^{2}}\right)^{3}}{2} \frac{(-2 \mathrm{t}, 1)}{\sqrt{\left.1+4 \mathrm{t}^{2}\right)}}=\left(-4 \mathrm{t}^{3}, \frac{1}{2}+3 \mathrm{t}^{2}\right)
$$

Now $\Lambda_{\bar{p}}(t)=(t-a)^{2}+\left(t^{2}-b\right)^{2}$ with $\vec{p}=(a, b)$. So

$$
\Lambda_{\overline{\mathrm{p}}}^{\prime}(\mathrm{t})=(2-4 \mathrm{~b}) \mathrm{t}+4 \mathrm{t}^{3}-2 \mathrm{a}
$$

and

$$
\Lambda_{\mathrm{p}}^{\prime \prime}(\mathrm{t})=2-4 \mathrm{~b}+12 \mathrm{t}^{2}
$$

Equating the last two equations to zero, we get

$$
\mathrm{b}=1 / 2+3 \mathrm{t}^{2} \text { and } \mathrm{a}=-4 \mathrm{t}^{3}
$$

## THE WIDTH OF AN OVAL IN R $\mathbf{R}^{\mathbf{2}}$

Now assume that C is a smooth closed curve in $\mathrm{R}^{2}$ parametrized by the arclength s . That is, C is defined by $\overline{\mathrm{f}}(\mathrm{s})=\left(\mathrm{f}_{1}(\mathrm{~s}), \mathrm{f}_{2}(\mathrm{~s})\right)$ where the parameter s belongs to $[\mathrm{a}, \mathrm{b}]$ such that $\overrightarrow{\mathrm{f}}(\mathrm{a})=\overrightarrow{\mathrm{f}}(\mathrm{b})$. In fact we take C as a simple closed curve, i.e. C doesn't join up except at the end points. Then we restrict our point of research to convex simple closed curves in $R^{2}$. As in the introduction, convexity of curves in $R^{2}$ leads to ovals.

Assume that $\overrightarrow{\mathrm{f}}$ parameterizes C in an anticlockwise direction so that the bounded component of $\mathrm{R}^{2}$-C is on the left. Let $\overline{\mathrm{T}}$ and $\overline{\mathrm{N}}$ be respectively the unit tangent and the unit normal fields acting on C. Thus, at each point $\overrightarrow{\mathrm{f}}(\mathrm{s})$ on the oval there is a unique unit tangent vector $\overrightarrow{\mathrm{T}}(\mathrm{s})$ in the direction of the oval and a unique inward-pointing unit normal vector $\overline{\mathrm{N}}(\mathrm{s})$. In the case of an oval, the angle $\psi$ between $\overline{\mathrm{T}}$ and the positive xaxis is a strictly increasing function of $s$. Hence there is an orientation preserving diffeomorphism $\delta:[a, b] \rightarrow R$ that assigns to each $s \in R$ the unique $\stackrel{s}{*}^{*} \in \mathrm{R}$ such that $\overrightarrow{\mathrm{T}}(\mathrm{s})+\overrightarrow{\mathrm{T}}\left(\mathrm{s}^{*}\right)=0$ and then $\overrightarrow{\mathrm{N}}(\mathrm{s})+\overrightarrow{\mathrm{N}}\left(\mathrm{s}^{*}\right)=0$. It is natural to say that $\mathrm{p}^{*}=\overrightarrow{\mathrm{f}}\left(\mathrm{s}^{*}\right)$ is opposite to $\mathrm{p}=\overline{\mathrm{f}}(\mathrm{s})$.

Let $\mathrm{T}_{\text {aff }}$ and $\mathrm{N}_{\text {aff }}$ denote the affine lines in $\mathrm{R}^{2}$ tangent and normal to $C$ at $p$ and let $T_{\text {aff }}^{*}$ and $N_{\text {aff }}^{*}$ denote the corresponding lines at $\mathrm{p}^{*}$. Let $\mathrm{g}: \mathrm{R} \rightarrow \mathrm{R}$ be the function that assigns to each $s \in R$ the perpendicular displacement between $\mathrm{T}_{\text {aff }}$ and $\mathrm{T}_{\text {aff }}^{*}$ and $\mathrm{h}: \mathrm{R} \rightarrow \mathrm{R}$ the function that assigns to each $s \in R$ the perpendicular displacement between $\mathrm{N}_{\mathrm{aff}}$ and $\mathrm{N}_{\mathrm{aff}}^{*}$. Now consider the vector equation

$$
\begin{equation*}
\mathrm{p}^{*}=\mathrm{p}-\mathrm{h}(\mathrm{~s}) \stackrel{\rightharpoonup}{\mathrm{T}}(\mathrm{~s})+\mathrm{g}(\mathrm{~s}) \stackrel{\mathrm{N}}{\mathrm{~N}}(\mathrm{~s}) \tag{8}
\end{equation*}
$$

In functional notation

$$
\begin{equation*}
\overrightarrow{\mathrm{f}} \circ \delta=\overrightarrow{\mathrm{f}}-\mathrm{h} \overrightarrow{\mathrm{~T}}+\mathrm{g} \stackrel{\mathrm{~N}}{ } \tag{9}
\end{equation*}
$$

Differentiating both sides of Eq.(9) with respect to s leads to

$$
\begin{equation*}
\overline{\mathrm{f}}^{\prime}(\delta) \delta^{\prime}=\overline{\mathrm{T}}-\mathrm{h} \overline{\mathrm{~T}}^{\prime}-\mathrm{h} ' \mathrm{~T}+\mathrm{g}^{\prime} \mathrm{N}+\mathrm{g} \overline{\mathrm{~N}}^{\prime} \tag{10}
\end{equation*}
$$

Using Frenet formulas, Eq.(10) reduces to

$$
\begin{equation*}
-\delta^{\prime} \stackrel{\rightharpoonup}{\mathrm{T}}=\left(1-\mathrm{h}^{\prime}-\mathrm{kg}\right) \stackrel{\rightharpoonup}{\mathrm{T}}+\left(\mathrm{g}^{\prime}-\mathrm{kh}\right) \stackrel{\mathrm{N}}{ } \tag{11}
\end{equation*}
$$

Then

$$
\begin{equation*}
1+\delta^{\prime}=\mathrm{h}^{\prime}+\mathrm{kg} \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{g}^{\prime}-\mathrm{kh}=0 \tag{13}
\end{equation*}
$$

By Eq. (4), we have

$$
\begin{equation*}
\mathrm{ds}+\mathrm{ds}^{*}=\mathrm{dh}+\mathrm{gd} \psi \tag{14}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{dg}-\mathrm{hd} \psi=0 \tag{15}
\end{equation*}
$$

Let

$$
\mathrm{u}(\psi)=\frac{\mathrm{ds}}{\mathrm{~d} \psi}+\frac{\mathrm{ds}^{*}}{\mathrm{~d} \psi}
$$

Then

$$
\begin{equation*}
\frac{\mathrm{dh}}{\mathrm{~d} \psi}+\mathrm{g}=\mathrm{u}(\psi) \tag{16}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{h}=\frac{\mathrm{dg}}{\mathrm{~d} \psi} \tag{17}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\frac{\mathrm{d}^{2} \mathrm{~g}}{\mathrm{~d} \psi^{2}}+\mathrm{g}=\mathrm{u}(\psi) \tag{18}
\end{equation*}
$$

Eq.(18) is a linear second order ordinary differential equation with constant coefficients. Moreover, the equation is nonhomogeneous, which means that the general solution of it is affected by the behavior of $u(\psi)$. The general solution of Eq.(18) has the form

$$
\begin{equation*}
g(\psi)=c_{1} \cos \psi+c_{2} \sin \psi+Q(\psi) \tag{19}
\end{equation*}
$$

where Q is a particular solution. Now let $\mathrm{g}_{1}=\cos \psi$, $\mathrm{g}_{2}=\cos \psi$ and then the wronskian $\mathrm{W}\left(\mathrm{g}_{1}, \mathrm{~g}_{2}\right)=1$, is never zero. Using the method of variation of parameters [12], the particular solution is $\mathrm{Q}=\mathrm{u}_{1} \mathrm{~g}_{1}+\mathrm{u}_{2} \mathrm{~g}_{2}$ where $\mathrm{u}_{1}$ and $\mathrm{u}_{2}$ are smooth real valued functions satisfying

$$
\mathrm{u}_{1}^{\prime}=-\sin \psi \mathrm{u}(\psi) \text { and } \mathrm{u}_{2}^{\prime}=\cos \psi \mathrm{u}(\psi)
$$

Thus

$$
\mathrm{u}_{1}=-\int_{0}^{\psi} \sin \eta u(\eta) \mathrm{d} \eta \text { and } \mathrm{u}_{2}=\int_{0}^{\psi} \cos \eta u(\eta) \mathrm{d} \eta
$$

So

$$
\begin{equation*}
\mathrm{Q}(\psi)=-\cos \psi \int_{0}^{\psi} \sin \eta u(\eta) \mathrm{d} \eta+\sin \psi \int_{0}^{\psi} \cos \eta u(\eta) d \eta \tag{20}
\end{equation*}
$$

Thus, the general solution is

$$
\mathrm{g}(\psi)=\left(\mathrm{c}_{1}-\int_{0}^{\psi} \sin \eta \mathrm{u}(\eta) \mathrm{d} \eta\right) \cos \psi+\left(\mathrm{c}_{2}+\int_{0}^{\psi} \cos \eta \mathrm{u}(\eta) \mathrm{d} \eta\right) \sin \psi
$$

Also

$$
\frac{d g}{d \psi}=-\left(c_{1}-\int_{0}^{\psi} \sin \eta u(\eta) d \eta \sin \psi+\left(c_{2}+\int_{0}^{\psi} \cos \eta u(\eta) d \eta \cos \psi\right.\right.
$$

Observe that $\mathrm{g}(\pi)=\mathrm{g}(0)$. Also

$$
\frac{\mathrm{dg}}{\mathrm{~d} \psi}(\pi)=\mathrm{h}(\pi)=\mathrm{h}(0)=\frac{\mathrm{dg}}{\mathrm{~d} \psi}(0)
$$

Thus

$$
c_{1}=\frac{1}{2} \int_{0}^{\pi} \sin \eta u(\eta) d \eta \text { and } c_{2}=-\frac{1}{2} \int_{0}^{\pi} \cos \eta u(\eta) d \eta
$$

Thus, the width of a convex curve has the form

$$
\begin{align*}
\mathrm{g}(\psi)= & -\left(\int_{0}^{\psi} \sin \eta u(\eta) \mathrm{d} \eta-\frac{1}{2} \int_{0}^{\pi} \sin \eta u(\eta) \mathrm{d} \eta\right) \cos \psi \\
& +\left(\int_{0}^{\psi} \cos \eta u(\eta) \mathrm{d} \eta-\frac{1}{2} \int_{0}^{\pi} \cos \eta u(\eta) \mathrm{d} \eta\right) \sin \psi \tag{22}
\end{align*}
$$

Another way to look at $g(\psi)$ and $h(\psi)$ is to take

$$
\begin{equation*}
\mathrm{A}(\psi)=\int_{0}^{\psi} \sin \eta u(\eta) \mathrm{d} \eta-\frac{1}{2} \int_{0}^{\pi} \sin \eta u(\eta) \mathrm{d} \eta \tag{23}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{B}(\psi)=\int_{0}^{\psi} \cos \eta u(\eta) \mathrm{d} \eta-\frac{1}{2} \int_{0}^{\pi} \cos \eta u(\eta) \mathrm{d} \eta \tag{24}
\end{equation*}
$$

Then

$$
\begin{equation*}
g(\psi)=B(\psi) \sin \psi-A(\psi) \cos \psi \tag{25}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{h}(\psi)=\mathrm{B}(\psi) \cos \psi+\mathrm{A}(\psi) \sin \psi \tag{26}
\end{equation*}
$$

## OVALS OF CONSTANT WIDTH IN R $\mathbf{R}^{2}$

Here we introduce different proofs for the results in [7] regarding the concept of ovals of constant width in $R^{2}$, as well a new equivalent statement for such a concept. First of all, we state the formal definition of an oval of constant width in $\mathrm{R}^{2}$.

Definition 2: [7] An oval C in $R^{2}$ is of constant width if the perpendicular distance between tangent lines at opposite points p and $\mathrm{p}^{*}$ is independent of p , i.e. $\mathrm{g}=\mathrm{a}$ constant.

Theorem 2: An oval $C$ in $R^{2}$ is of constant width a iff the affine normal lines at opposite points coincide.

Proof: Observe that C is of constant width $\mathrm{g}=\mathrm{a}$ iff $\forall \mathrm{s} \in \mathrm{R}, \frac{\mathrm{dg}}{\mathrm{ds}}=0$
iff $\forall \mathrm{s} \in \mathrm{R}, \mathrm{h}=0$ (Eq. (13))
iff $\forall \mathrm{s} \in \mathrm{R}, \mathrm{N}_{\text {aff }}$ and $\mathrm{N}_{\text {aff }}^{*}$ coincide.
Theorem 3: Let $\rho(\mathrm{s})$ and $\rho\left(\mathrm{s}^{*}\right)$ be the corresponding radii of curvature at $p$ and $p^{*}$. Then the oval $C$ in $R^{2}$ is of constant width a iff $\forall s \in R, \rho(s)+\rho\left(s^{*}\right)=a$.

Proof: Recall that $\rho(\mathrm{s})=\frac{1}{\mathrm{k}(\mathrm{s})}$ and $\rho\left(\mathrm{s}^{*}\right)=\frac{1}{\mathrm{k}\left(\mathrm{s}_{\mathrm{s}}\right)}$. Then by Eq. (14), C is of constant width a
iff $\forall s \in R$, ds $+\mathrm{ds}^{*}=\mathrm{ad} \psi$
iff $\forall \mathrm{s} \in \mathrm{R}, \frac{1}{\frac{\mathrm{~d} \psi}{\mathrm{ds}}}+\frac{1}{\frac{\mathrm{~d} \psi}{\mathrm{ds}^{*}}}=\mathrm{a}$
iff $\forall \mathrm{s} \in \mathrm{R}, \frac{1}{\mathrm{k}(\mathrm{s})}+\frac{1}{\mathrm{k}\left({ }_{\mathrm{s}} \mathrm{*}\right)}=\mathrm{a}$ (Eq.(4))
iff $\forall \mathrm{s} \in \mathrm{R}, \rho(\mathrm{s})+\rho\left(\mathrm{s}^{*}\right)=\mathrm{a}$
Theorem 4: An oval $C$ in $R^{2}$ is of constant width a iff opposite points have the same focal point.

Proof: Recall that the focal point based at $s$ is $\overrightarrow{\mathrm{f}}(\mathrm{s})+\frac{1}{\mathrm{k}(\mathrm{s})} \stackrel{\mathrm{N}}{\mathrm{N}}(\mathrm{s})$ and the focal point based at $\mathrm{s}^{*}$ is $\overrightarrow{\mathrm{f}}\left(\mathrm{s}^{*}\right)+\frac{1}{\mathrm{k}\left(\mathrm{s}^{*}\right)} \overrightarrow{\mathrm{N}}\left(\mathrm{s}^{*}\right)$. By Theorem 3,C is of constant width a iff $\frac{1}{k\left(s^{*}\right)}+\frac{1}{k(s)}=\mathrm{a}$
iff $\mathrm{aN}(\mathrm{s})-\overrightarrow{\mathrm{N}}(\mathrm{s})\left[\frac{1}{\mathrm{k}\left(\mathrm{s}^{*}\right)}+\frac{1}{\mathrm{k}(\mathrm{s})}\right]=0$
iff $\overrightarrow{\mathrm{f}}\left(\mathrm{s}^{*}\right)-\overline{\mathrm{f}}(\mathrm{s})-\frac{1}{\mathrm{k}\left(\mathrm{s}^{*}\right)} \overline{\mathrm{N}}(\mathrm{s})-\frac{1}{\mathrm{k}(\mathrm{s})} \overline{\mathrm{N}}(\mathrm{s})=0$
iff $\overrightarrow{\mathrm{f}}\left(\mathrm{s}^{*}\right)-\overrightarrow{\mathrm{f}}(\mathrm{s})+\frac{1}{\mathrm{k}\left(\mathrm{s}^{*}\right)} \overrightarrow{\mathrm{N}}\left(\mathrm{s}^{*}\right)-\frac{1}{\mathrm{k}(\mathrm{s})} \overrightarrow{\mathrm{N}}(\mathrm{s})=0$
iff $\overrightarrow{\mathrm{f}}\left(\mathrm{s}^{*}\right)+\frac{1}{\mathrm{k}\left(\mathrm{s}^{*}\right)} \stackrel{\mathrm{N}}{ }\left(\mathrm{s}^{*}\right)-\left[\overrightarrow{\mathrm{f}}(\mathrm{s})+\frac{1}{\mathrm{k}(\mathrm{s})} \overrightarrow{\mathrm{N}}(\mathrm{s})\right]=0$
iff opposite points have the same focal point.
Now we introduce a new result regarding the function $u(\psi)$ in Eq.(18).

Theorem 5: An oval $C$ in $R^{2}$ is of constant width a iff $\forall \psi \in[0,2 \pi], \mathrm{u}(\psi)=\mathrm{a}$.

Proof: The oval C in $\mathrm{R}^{2}$ is of constant width a iff $\forall \psi \in[0,2 \pi], g(\psi)=\mathrm{a}$

$$
\text { iff } \forall \psi \in[0,2 \pi], \mathrm{h}(\psi)=0 \text { (Eq.(17)). }
$$

Solving Eq.(25) and Eq.(26) together, we arrive to the fact that C is of constant width a iff $\forall \psi \in[0,2 \pi]$,

$$
\mathrm{A}(\psi)=-\operatorname{acos} \psi \text { and } \mathrm{B}(\psi)=\operatorname{asin} \psi
$$

Now substitute $\forall(\psi)=-\operatorname{acos}(\psi)$ in Eq. (23), then differentiate both sides with respect to $\psi$ to get the fact that C is of constant width a iff $\forall \psi \in[0,2 \pi]$, $u(\psi)=a$.

Example 3: Consider the oval C in $\mathrm{R}^{2}$ defined by

$$
\begin{aligned}
\overrightarrow{\mathrm{f}}(\mathrm{t})= & (25 \cos \mathrm{t}+\operatorname{cost\operatorname {cos}5\mathrm {t}+5\operatorname {sin}\mathrm {s}\operatorname {sin}5\mathrm {t}} \\
& 25 \sin \mathrm{t}+\sin t \cos 5 \mathrm{t}-5 \cos t \sin 5 \mathrm{t}), \mathrm{t} \in[0,2 \pi]
\end{aligned}
$$

Since

$$
\overline{\mathrm{f}}^{\prime}(\mathrm{t})=(25-24 \cos 5 \mathrm{t})(-\sin \mathrm{t}, \cos \mathrm{t})
$$

we have

$$
\overrightarrow{\mathrm{T}}(\mathrm{t})=(-\sin \mathrm{t}, \cos \mathrm{t})
$$

Since

$$
\stackrel{\rightharpoonup}{\mathrm{T}}(\mathrm{t}+\pi)=(\sin \mathrm{t},-\cos \mathrm{t})=-\stackrel{\rightharpoonup}{\mathrm{T}}(\mathrm{t})
$$

we have

$$
\forall \mathrm{t} \in[0,2 \pi], \stackrel{\mathrm{T}}{\mathrm{~T}}(\mathrm{t})+\stackrel{\rightharpoonup}{\mathrm{T}}(\mathrm{t}+\pi)=0
$$

and so the points $\mathrm{p}=\overrightarrow{\mathrm{f}}(\mathrm{t})$ and $\mathrm{p}^{*}=\overrightarrow{\mathrm{f}}(\mathrm{t}+\pi)$ are opposite points.

If $\mathrm{d}\left(\mathrm{p}, \mathrm{p}^{*}\right)$ denotes the Euclidean distance between p and $\mathrm{p}^{*}$, then $\forall \mathrm{t} \in \mathrm{R}, \mathrm{d}\left(\mathrm{p}, \mathrm{p}^{*}\right)=50$ and so the curve is of constant width 50 .
A unit normal vector to $\vec{f}$ at $\vec{f}(t)$ is

$$
\overline{\mathrm{N}}(\mathrm{t})=(-\cos \mathrm{t},-\sin \mathrm{t})
$$

and so

$$
\stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t}+\pi)=(\cos \mathrm{t}, \sin \mathrm{t})=-\overline{\mathrm{N}}(\mathrm{t})
$$

If $N_{\text {eff }}$ is the affine normal line of $\overrightarrow{\mathrm{f}}$ at $\overrightarrow{\mathrm{f}}(\mathrm{t})$ and $\overline{\mathrm{q}} \in \mathrm{N}_{\mathrm{aff}}$, then

$$
\begin{aligned}
\overrightarrow{\mathrm{q}} & =\overrightarrow{\mathrm{f}}(\mathrm{t})+\lambda \stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t}), \quad \lambda \in \mathrm{R} \\
& =\overline{\mathrm{f}}(\mathrm{t}+\pi)+50 \stackrel{\mathrm{~N}}{ }(\mathrm{t}+\pi)-\lambda \stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t}+\pi) \\
& =\overrightarrow{\mathrm{f}}(\mathrm{t}+\pi)+(50-\lambda) \stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t}+\pi)
\end{aligned}
$$

Hence $\overrightarrow{\mathrm{q}} \in \mathrm{N}_{\mathrm{aff}}^{*}$, the affine normal line of $\overrightarrow{\mathrm{f}}$ at $\vec{f}(t+\pi)$. Similarly, if $\vec{q} \in N_{\text {aff }}^{*}$, then

$$
\begin{aligned}
\overrightarrow{\mathrm{q}} & =\overrightarrow{\mathrm{f}}(\mathrm{t}+\pi)+\lambda \stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t}+\pi), \quad \lambda \in \mathrm{R} \\
& =\stackrel{\overrightarrow{\mathrm{f}}}{ }(\mathrm{t})+50 \stackrel{\mathrm{~N}}{ }(\mathrm{t})-\lambda \stackrel{\rightharpoonup}{\mathrm{N}}(\mathrm{t}) \\
& =\overrightarrow{\mathrm{f}}(\mathrm{t})+(50-\lambda) \stackrel{\mathrm{N}}{ }(\mathrm{t})
\end{aligned}
$$

Hence $\overline{\mathrm{q}} \in \mathrm{N}_{\text {aff }}$. Thus, the affine normal lines at $\overrightarrow{\mathrm{f}}(\mathrm{t})$ and $\overrightarrow{\mathrm{f}}(\mathrm{t}+\pi)$ coincide (Theorem 2).
The curvature of $\vec{f}$ at $\vec{f}(t)$ is

$$
k(t)=\frac{\left\|\vec{f}^{\prime}(t) \times \vec{f}^{\prime \prime}(t)\right\|}{\|\vec{f}(t)\|^{3}}=\frac{1}{25-24 \cos 5 t}
$$

Thus, $\rho(\mathrm{t})+\rho(\mathrm{t}+\pi)=50$ (Theorem 3).
Now let $\overline{\mathrm{p}}=(\mathrm{a}, \mathrm{b})$. Then

$$
\begin{aligned}
\Lambda_{\overline{\mathrm{p}}}(\mathrm{t}) & =(25 \cos t+\cos t \cos 5 t+5 \sin t \sin 5 t-a)^{2} \\
& +(25 \sin t+\sin t \cos 5 t-5 \cos t \sin 5 t-b)^{2}
\end{aligned}
$$

So

$$
\Lambda_{\overline{\mathrm{p}}}^{\prime}(\mathrm{t})=2(25-24 \cos 5 \mathrm{t})(-5 \sin 5 \mathrm{t}+\mathrm{a} \sin \mathrm{t}-\mathrm{b} \cos \mathrm{t})
$$

$$
\Lambda_{\mathrm{p}}^{\prime \prime}(\mathrm{t})=2(25-24 \cos 5 \mathrm{t})(-25 \cos 5 \mathrm{t}+\mathrm{a} \cos \mathrm{t}+\mathrm{b} \sin \mathrm{t})
$$

$$
+240 \sin 5 t(-5 \sin 5 t+a \sin t-b \cos t)
$$

Equating the last two equations to zero, we get

$$
-5 \sin 5 t+\operatorname{asin} t-b \cos t=0
$$

and

$$
-25 \cos 5 t+a \cos t+b \sin t=0
$$

Solving the last two equations together, we get

$$
a=25 \cos t \cos 5 t+5 \sin t \sin 5 t
$$

and

$$
b=25 \sin t \cos 5 t-5 \cos t \sin 5 t
$$

Thus, the focal point of $\vec{f}$ with base $\vec{f}(t)$ is
(25costcos5t+5sintsin $5 t, 25 \sin t \cos 5 t-5 \cos t \sin 5 t)$
which is the focal point of the curve based at the opposite point $\overrightarrow{\mathrm{f}}(\mathrm{t}+\pi)$ (Theorem 4). By Eq.(18), we have $\forall \psi \in[0,2 \pi], \mathrm{u}(\psi)=50$ (Theorem 5).

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