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# Solving Fractional Hyperbolic Partial Differential Equations by the Generalized Differential Transform Method

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**Abstract:** In this paper, the Generalized Differential Transform Method (GDTM) is used to solve fractional hyperbolic partial differential equations. The idea is to modify the differential transform method used to solve fractional partial differential equation. The calculated results due to using the GDTM are given in tabulated form in order to give a good comparison with the exact solution.

**Key words:** Generalized differential transform method . generalized taylor formula . caputo fractional derivative . fractional hyperbolic partial differential equations

## INTRODUCTION

Fractional order partial differential equations, as generalizations of classical integer order partial differential equations, are increasingly used to model problems in fluid flow, finance and other areas of application. Fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. Half-order derivatives and integrals prove to be more useful for the formulation of certain electrochemical problems than the classical models [22]. Fractional differentiation and integration operators are also used for extensions of the diffusion and wave equations [23]. A great deal of effort has been expended over the last 10 years or so in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations of physical interest. Numerical and analytical methods have included finite difference method [6, 7, 24], Adomian Decomposition Method (ADM) [1, 8-12, 17], Variational Iteration Method (VIM) [5, 13, 14, 18] and Homotopy Perturbation Method (HPM) [5, 15, 19]. The VIM and the ADM have been extensively used to solve fractional partial differential equations, because they provide immediate and visible symbolic terms of analytic solutions, as well as numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization.

The present authors have written a series of papers solving linear partial differential equations and nonlinear partial differential equations of fractional order [1, 8-15, 17-19]. Recently they developed a semi-numerical method for solving linear partial differential equations of fractional order [20]. This method is named as Generalized Differential Transform Method (GDTM) and is based on the two dimensional Differential Transform Method (DTM) [2, 4, 25] and the generalized Taylor's formula [21]. The present paper may be regarded as an extension of the later paper [20] on nonlinear partial differential with space- and time-fractional derivatives of the form

$$\frac{\partial^{\mu}u}{\partial t^{\mu}} = \frac{\partial^{\nu}u}{\partial x^{\nu}} + N_f(u(x,y)),\ m-1<\mu\le m,\quad n-1<\nu\le n\ ,\ n,m\in N \tag{1.1}$$

where μ and v are parameters describing the order of the fractional time-and space-derivatives in the Caputo sense, respectively and $N_f$ is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t. The function u(x,t) is assumed to be a causal function of time and space, i.e., vanishing for t<0 and x<0. The general response expression contains parameters describing the order of the fractional derivatives that can be varied to obtain various responses.

There are several definitions of a fractional derivative of order α>0 [3, 22]. The two most commonly used definitions are the Riemann-Liouville and Caputo. The Riemann-Liouville fractional integration of order μ is defined as

$$J_a^{\mu}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (x-t)^{\mu-1}f(t)\ dt, \mu>0,\ x>0 \tag{1.2}$$

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The next two equations define Riemann-Liouville and Caputo fractional derivatives of order μ, respectively,

$$D_a^{\mu} f(x) = \frac{d^m}{dx^m}(J_a^{m-\mu} f(x)) \tag{1.3}$$

$$D_{*a}^{\mu} f(x) = J_a^{m-\mu}\left(\frac{d^m}{dx^m} f(x)\right) \tag{1.4}$$

where $m-1<\mu\leq m$ and $m\in N$. For now, the Caputo fractional derivative will be denoted by $D_{*a}^{\mu}$ to maintain a clear distinction with the Riemann-Liouville fractional derivative.

The Caputo fractional derivative is considered here because it allows traditional initial and boundary conditions to be included in the formulation of the problem. In this paper, we consider the one-dimensional space-and time fractional nonlinear partial differential equation (2.1), where the unknown function u = u(x,t) is a assumed to be a causal function of space and time, respectively and the fractional derivatives are taken in Caputo sense as follows.

**Definition 1.1:** For m to be the smallest integer that exceeds μ, the Caputo time-fractional derivative operator of order μ>0 is

$$D_{*t}^{\mu} u(x,t) = \frac{\partial^{\mu} u(x,t)}{\partial t^{\mu}} = \begin{cases} \dfrac{1}{\Gamma(m-\mu)}\displaystyle\int_0^t (t-\tau)^{m-\mu-1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}\,d\tau & \text{for } m-1<\mu<m \\ \dfrac{\partial^m u(x,\tau)}{\partial t^m} & \text{for } \mu=m\in N \end{cases} \tag{1.5}$$

and the space-fractional derivative operator of order ν > 0 is defined as

$$D_{*x}^{\nu} u(x,t) = \frac{\partial^{\nu} u(x,t)}{\partial x^{\nu}} = \begin{cases} \dfrac{1}{\Gamma(m-\nu)}\displaystyle\int_0^x (x-\theta)^{m-\nu-1}\frac{\partial^m u(\theta t)}{\partial\theta^m}\,d\theta & \text{for } m-1<\nu<m \\ \dfrac{\partial^m u(x,t)}{\partial x^m} & \text{for } \nu=m\in N \end{cases} \tag{1.6}$$

For mathematical properties of fractional derivatives and integrals one can consult the mentioned references.

## GENERALIZED TAYLOR'S FORMULA

In this section we present the generalized Taylor's formula that involves Caputo fractional derivatives. This generalization is presented in [21]. Suppose that $(D_{*a}^{\alpha})^k f(x)\in C(a,b]$ for k = 0,1,…,n+1 where 0<α≤1, then the generalized taylor's formula is

$$f(x) = \sum_{i=0}^{n}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_{*a}^{\alpha})^i f)(a+) + \frac{((D_{*a}^{\alpha})^{n+1} f)(\xi)}{\Gamma((n+1)\alpha+1)}(x-a)^{(n+1)\alpha} \tag{2.1}$$

with α≤ξ≤x, for each x ? (a,b] and $D_{*a}^{\alpha}$ is the Caputo fractional derivative of order α, where

$$(D_{*a}^{\alpha})^k = D_{*a}^{\alpha} D_{*a}^{\alpha}\cdots D_{*a}^{\alpha}$$

In case of α = 1, the generalized Taylor's formula (2.1) reduces to the classical Talyor's formula.

**Theorem 2.1:** [21] Suppose that $(D_{*a}^{\alpha})^k f(x)\in C(a,b]$ for k = 0,1,…,N+1 where 0<α≤1. If x∈[a,b], then

$$f(x) \simeq \sum_{i=0}^{N}\frac{(x-a)^{i\alpha}}{\Gamma(i\alpha+1)}((D_{*a}^{\alpha})^i f)(a+) \tag{2.2}$$

Furthermore, there is a value ξ with α≤ξ≤x, so that the error term $R_N^{\alpha}(x)$ has the form

$$R_N^{\alpha}(x) = \frac{((D_{*a}^{\alpha})^{N+1} f)(\xi)}{\Gamma((N+1)\alpha+1)}(x-a)^{(N+1)\alpha} \tag{2.3}$$

The accuracy of $f(x)=x^{\lambda}g(x)$ where $\lambda>-1$ increases when we choose large N and decreases as the value of x moves away from the center a. Hence, we must choose N large enough so that the error does not exceed a specified bound. In the following theorem, we find precise conditions under which the exponents hold for arbitrary fractional operators. This result is very useful on our approach for solving differential equations of fractional order.

**Theorem 2.2:** [16] Suppose that $f(x) = x^{\lambda}g(x)$ where $\lambda > -1$ and g(x) has the generalized power series expansion

$$g(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n\alpha}$$

with radius of convergence $\Re > 0$, where $0<\alpha\le 1$. Then

$$D_{*a}^{\gamma} D_{*a}^{\beta} f(x) = D_{*a}^{\gamma+\beta} f(x) \qquad (2.4)$$

for all $x \in (0,R)$ if one of the following conditions is satisfied:

(a) $\beta < \lambda+1$ and $\gamma$ arbitrary,
(b) $\beta \ge \lambda+1$, $\gamma$ arbitrary and $a_k = 0$ for $k = 0,1,\ldots,m-1$, where $m-1<\beta\le m$.

The proof of Theorem 2.1 is given in [20] and the proof of Theorem 2.2 is given in [21].

## THE GENERALIZED DIFFERENTIAL TRANSFORM METHOD

The Differential Transform Method (DTM) was first applied in the engineering domain by [25]. In general, the DTM is applied to the solution of electric circuit problems. The DTM is a numerical method based on the Taylor series expansion which constructs an analytical solution in the form of a polynomial. The traditional high order Taylor series method requires symbolic computation. However, the DTM obtains a polynomial series solution by means of an iterative procedure. The method is well addressed in [16-18].

In this section we shall derive the generalized two-dimensional DTM that we have developed for the numerical solution of linear partial differential equations with space-and time-fractional derivatives [20]. The proposed method is based on the combination of the classical two-dimensional DTM [17, 18] and generalized Taylor's formula [21]. Consider a function of two variables u(x,y) and suppose that it can be represented as a product of two single-variable functions, i.e., u(x,y) = $f$(x)g(y). Based on the properties of the generalized two-dimensional differential transform [2, 4], the function u(x,y) can be represented as

$$\begin{aligned} u(x,y) &= \sum_{k=0}^{\infty} F_{\alpha}(k)(x-x_0)^{\alpha k} G_{\beta}(h)(y-y_0)^{\beta h} \\ &= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U_{\alpha,\beta}(k,h)(x-x_0)^{\alpha k}(y-y_0)^{\beta h} \end{aligned} \qquad (3.1)$$

where $0<\alpha$, $\beta\le 1$, $U_{\alpha,\beta}(k,h) = F_{\alpha}(k)G_{\beta}(h)$ is called the spectrum of u(x,y). The generalized two-dimensional differential transform of the function u(x,y) is given by

$$U_{\alpha,\beta}(k,h) = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D_{*x_0}^{\alpha})^k (D_{*y_0}^{\beta})^h u(x,y)]_{(x_0,y_0)} \qquad (3.2)$$

where

$$(D_{*x_0}^{\alpha})^k = D_{*x_0}^{\alpha} D_{*x_0}^{\alpha} \cdots D_{*x_0}^{\alpha}$$

k-times. In this paper, the lowercase u(x,y) represents the original function while the uppercase $U_{\alpha,\beta}(k,h)$ stands for the transformed function. In case of $\alpha = 1$, $\beta = 1$ the generalized two-dimensional differential transform (3.1) reduces to the classical two-dimensional differential transform. Based on Eq(3.1) and Eq(3.2), we have the following results.

**Theorem 3.1:** Suppose that $U_{\alpha,\beta}(k,h)$, $V_{\alpha,\beta}(k,h)$ and $W_{\alpha,\beta}(k,h)$ are the differential transformations of the functions u(x,y), v(x,y) and w(x,y), respectively, then

(a) if $u(x,y) = v(x,y) \pm w(x,y)$, then $U_{\alpha,\beta}(k,h) = V_{\alpha,\beta}(k,h) \pm W_{\alpha,\beta}(k,h)$
(b) if $u(x,y) = a\, v(x,y)$, $a\in R$ then $U_{\alpha,\beta}(k,h) = aV_{\alpha,\beta}(k,h)$
(c) if $u(x,y) = v(x,y)w(x,y)$, then $U_{\alpha,\beta}(k,h) = \sum_{r=0}^{k}\sum_{s=0}^{h} V_{\alpha,\beta}(r,h-s)W_{\alpha,\beta}(k-r,h)$
(d) if $u(x,y) = (x-x_0)^{n\alpha}(y-y_0)^{m\beta}$, then $U_{\alpha,\beta}(k,h) = \delta(k-n)\delta(h-m)$

**Theorem 3.2:** Let $u(x,y) = D_{*x_0}^{\alpha} v(x,y)$, $\quad 0<\alpha\le 1$ Then

$$U_{\alpha,\beta}(k,h) = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h) \qquad (3.3)$$

**Proof:** By (3.2) we have

$$\begin{aligned} U_{\alpha,\beta}(k,h) &= \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D_{*x_0}^{\alpha})^k (D_{*y_0}^{\beta})^h D_{*x_0}^{\alpha} v(x,y)]_{(x_0,y_0)} = \frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D_{*x_0}^{\alpha})^{k+1}(D_{*y_0}^{\beta})^h v(x,y)]_{(x_0,y_0)}, \\ &= \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\alpha(k+1)+1)}[(D_{*x_0}^{\alpha})^{k+1}(D_{*y_0}^{\beta})^h v(x,y)]_{(x_0,y_0)} = \frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} V_{\alpha,\beta}(k+1,h) \end{aligned}$$

**Theorem 3.3:** Let u(x,y) = $f$(x)g(y) and $f(x) = x^{\lambda}h(x)$, where $\lambda>-1$,h(x) has the generalized Taylor series expansion

$$h(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^{\alpha k}$$

and

(a) $\beta<\gamma+1$, $\alpha$ arbitrary,

(b) $\beta\geq\lambda+1$, $\alpha$ arbitrary and $a_k = 0$ for k = 0,1,…,m-1, where $m-1<\beta\leq m$.

Then the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[D^{\alpha k}_{*x_0}(D^{\beta}_{*y_0})^h u(x,y)]_{(x_0,y_0)} \tag{3.4}$$

**Proof:** This follows immediately from the fact that $D^{\gamma_1}_{*x_0}D^{\gamma_2}_{*x_0}f(x)=D^{\gamma_1+\gamma_2}_{*x_0}f(x)$, under the conditions given in Theorem 2.2.

**Theorem 3.4:** Let $u(x,y)=D^{\gamma}_{*x_0}v(x,y)$, $m-1<\gamma\leq m$ and v(x,y) = $f$(x)g(y) where the function $f$(x) satisfies the conditions given in Theorem 2.2. Then

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h) \tag{3.5}$$

**Proof:** Using Theorem 3.3, we have

$$\begin{aligned}U_{\alpha,\beta}(k,h)&=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^{\alpha k}_{*x_0})(D^{\beta}_{*y_0})^h D^{\gamma}_{*x_0}v(x,y)]_{(x_0,y_0)}=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^{\alpha k+\gamma}_{*x_0})(D^{\beta}_{*y_0})^h v(x,y)]_{(x_0,y_0)}\\&=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)\Gamma(\beta h+1)\Gamma(\alpha k+\gamma+1)}[(D^{\alpha k+\gamma}_{*x_0})(D^{\beta}_{*y_0})^h v(x,y)]_{(x_0,y_0)}=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h)\end{aligned}$$

Now, if the function u(x,y) = $f$(x)g(y), $f$(x) and g(y) satisfy the conditions given in Theorem 2.2, then the generalized differential transform (3.2) becomes

$$U_{\alpha,\beta}(k,h)=\frac{1}{\Gamma(\alpha k+1)\Gamma(\beta h+1)}[(D^{\alpha k}_{*x_0})(D^{\beta h}_{*y_0})u(x,y)]_{(x_0,y_0)} \tag{3.6}$$

Therefore, in this case, if $u(x,y)=D^{\gamma}_{*x_0}D^{\mu}_{*y_0}v(x,y)$, where $m-1<\gamma\leq m$, $n-1<\mu\leq n$ and the functions $f$(x) and g(y) satisfy the conditions given in Theorem 2.2, then we have the following result:

$$U_{\alpha,\beta}(k,h)=\frac{\Gamma(\alpha k+\gamma+1)}{\Gamma(\alpha k+1)}\frac{\Gamma(\beta h+\mu+1)}{\Gamma(\alpha k+1)}V_{\alpha,\beta}(k+\gamma/\alpha,h+\mu/\beta) \tag{3.7}$$

## ANALYSIS OF THE METHOD

In this section we shall use the analysis presented in the previous section to construct our numerical method for solving the following nonlinear partial differential equation with space and time-fractional derivatives

$$\frac{\partial^{\mu}u}{\partial t^{\mu}}=\frac{\partial^{\nu}u}{\partial x^{\nu}}+N_f(u(x,t),\ m-1<\mu\leq m, n-1<\nu\leq n,\ n,m\in N \tag{4.1}$$

where $N_f$ is a nonlinear operator which might include other fractional derivatives with respect to the variables x and t.

First of all, if $0<\mu\leq 1$ and $0<v\leq 1$, then we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions $u(x,t) = v(x)w(t)$, where the function $w(t)$ satisfies the conditions given in Theorem 2.2. In this case, selecting $\alpha = \mu$, $\beta = v$ and applying Theorem 3.2 on both sides of Eq. (4.1), it transforms to

$$\frac{\Gamma(\alpha(h+1)+1)}{\Gamma(\alpha h+1)}U_{\alpha,\beta}(k,h+1)=\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}U_{\alpha,\beta}(k+1,h)+F_{\alpha,\beta}(k,h) \tag{4.2}$$

where $F_{\alpha,\beta}(k,h)$ is the generalized differential transformation of $N_f(u(x,y))$.

Secondly, if $m-1<\mu=m_1/m_2\leq m$ and $0<v\leq 1$, then we suppose that the solution the nonlinear equation (4.1) can be written as a product of single-valued functions $u(x,t) = v(x)w(t)$, where the function $w(t)$ satisfies the conditions given in Theorem 2.2. In this case, selecting $\alpha = 1/m_2$, $\beta = v$ and applying Theorem 3.2 on both sides of Eq. (4.1), it transforms to

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)}U_{\alpha,\beta}(k,h+m_1)=\frac{\Gamma(\beta(k+1)+1)}{\Gamma(\beta k+1)}U_{\alpha,\beta}(k+1,h)+F_{\alpha,\beta}(k,h) \tag{4.3}$$

Finally, if $m-1<\mu=m_1/m_2\leq m$ and $n-1<v=n_1/n_2\leq n$, then we suppose that the solution of the nonlinear equation (4.1) can be written as a product of single-valued functions $u(x,t) = v(x)w(t)$, where the functions $v(x)$ and $w(t)$ satisfy the conditions given in Theorem 2.2. In this case, selecting $\alpha = 1/m_2$, $\beta = 1/n_2$ and applying Theorem 3.2 on both sides of Eq. (4.1), it transforms to:

$$\frac{\Gamma(\alpha(h+1)+m_1)}{\Gamma(\alpha h+1)}U_{\alpha,\beta}(k,h+m_1)=\frac{\Gamma(\beta(k+1)+n_1)}{\Gamma(\beta k+1)}U_{\alpha,\beta}(k+n_1,h)+F_{\alpha,\beta}(k,h) \tag{4.4}$$

In all the above cases, the solution of the nonlinear space and time-fractional equation (4.1), using Definition (3.1), can be written as:

$$u(x,y)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{\alpha,\beta}(k,h)x^{\alpha k}y^{\beta h} \tag{4.5}$$

## FRACTIONAL HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH BOUNDARY VALUE PROBLEMS

We will study the simplest form of hyperbolic PDE of the form:

$$\frac{\partial^2 u(x,t)}{\partial t^2}=\frac{\partial^q u(x,t)}{\partial x^q},\ L\leq x\leq R,\ 0\leq t\leq T, 1\leq q\leq 2 \tag{5.1}$$

together with the initial and zero Dirichlet boundary conditions:

$$\begin{gathered}
u(x,0)=f(x) \quad \text{for} \quad L\leq x\leq R\ ,\ 0\leq t\leq T \\
\frac{\partial u(x,0)}{\partial t}=g(x) \quad \text{for} \quad L\leq x\leq R \\
u(L,t)=0 \quad \text{for} \quad 0\leq t\leq T\ , \\
u(R,t)=0 \quad \text{for} \quad 0\leq t\leq T
\end{gathered} \tag{5.2}$$

As a numerical example consider the fractional order hyperbolic partial differential equation:

$$\frac{\partial^2 u}{\partial t^2}=\frac{1}{\Gamma(0.5)}x^{1/2}\frac{d^{1.5}u}{dx^{1.5}}-4x^2+2x^3-2.546\,x^2t^2+2.546\,xt^2\quad, 0\le x\le 2,\ 0\le t\le 1 \tag{6.1}$$

together with initial and zero Dirichlet boundary conditions:

$$u(x,0)=0\ ,\ u_t(x,0)=0\ \text{ for }\ 0\le x\le 2,\quad u(0,t)=0\ , u(1,t)=-t^2; u(2,t)=0\ \text{ for }\ 0\le t\le 1 \tag{6.2}$$

Suppose that the solution u(x,t) can be represented as a product of single-valued functions, u(x,t) = v(x)w(t) where the function v(x) satisfies the conditions given in Theorem 2.2 Selecting $\alpha = 1$, $\beta = 0.5$ and applying Eq. (4.3), the recurrence relation for the hyperbolic partial differential equation (6.1) is given by

$$\sum_{r=0}^{k}\sum_{s=0}^{h}F(r)\delta(h-s)\frac{\Gamma(\frac{k-r}{2}+\frac{5}{2})}{\Gamma(\frac{k-r}{2}+1)}U_{1,1/2}(k-r+3,s)=\Gamma(0.5)\begin{bmatrix}(h+1)(h+2)U_{1,1/2}(k,h+2)+2\delta(h)(2\delta(k-4)-\delta(k-6))\\ +2.546\delta(h-2)(\delta(k-4)-\delta(k-2))\end{bmatrix} \tag{6.3}$$

where

$$F(r)=\frac{1}{r!}\left(\frac{d^{r}(x^{\frac{1}{2}})}{dx^{r}}\right)\Bigg|_{x=1} \tag{6.4}$$

The generalized two-dimensional differential transform of the initial and zero Dirichlet boundary conditions (6.2) is

$$U_{1,1/2}(k,0)=0\ ,\ U_{1,1/2}(k,2)=0\ ,\ U_{1,1/2}(0,h)=0,\ U_{1,1/2}(2,h)=0\ ,\ U_{1,1/2}(1,h)=\ -\delta(h-2) \tag{6.5}$$

Utilizing the recurrence relation (6.3) and the transformed initial conditions (6.5), we have

$$\sum_{r=0}^{k}\sum_{s=0}^{h}F(r)\delta(h-s)\frac{\Gamma(\frac{k-r}{2}+\frac{5}{2})}{\Gamma(\frac{k-r}{2}+1)}U_{1,1/2}(k-r+3,s)=\Gamma(0.5)\begin{bmatrix}(h+1)(h+2)U_{1,1/2}(k,h+2)+2\delta(h)(2\delta(k-4)-\delta(k-6))\\ +2.546\delta(h-2)(\delta(k-4)-\delta(k-2))\end{bmatrix} \tag{6.6}$$

Therefore, the approximate solution of the nonlinear fractional hyperbolic equation (6.1) with $\alpha=1,\beta=\frac{1}{2}$ can be derived as

$$u(x,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}U_{1,1/2}(k,h)\ x^{\frac{k}{2}}t^{h} \tag{6.7}$$

and using the inverse transformation rule in Eq. (3.2), u(x,t) is evaluated as

$$u(x,t)=x^2(x-2)t^2 \tag{6.8}$$

that is the exact solution of Eq. (6.1)

Using Matlab, Table 1 shows a comparison between the approximate solutions for Eq. (6.1) using the GDTM and the corresponding values out of the exact solution. From the numerical results in Table 1, it is clear that the approximate solution obtained using the GDTM is in good agreement with the exact solution for all values of x and t (Fig. 1).

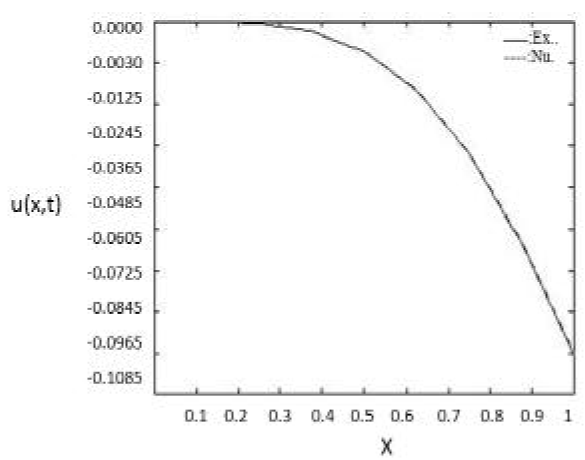


Fig. 1: A comparison between the approximate solutions for Eq. (6.1) using the GDTM and the corresponding values out of the exact solution

Table 1: Numerical values for Eq. ( 6.1) at t = 0.1,t = 0.2,t = 0.3 and t = 0.4

| | t = 0.1 | | t = 0.2 | |
|---|---|---|---|---|
| x | $u_{GDTM}$ | Exact | $u_{GDTM}$ | Exact |
| 0.0 | 0.00000000 | 0.0000 | 0.000000000 | 0.0000 |
| 0.1 | -0.000189998 | -0.00019 | -0.000759992 | -0.00076 |
| 0.2 | -0.000719992 | -0.00072 | -0.002879968 | -0.00288 |
| 0.3 | -0.001529983 | -0.00153 | -0.006119933 | -0.00612 |
| 0.4 | -0.002559972 | -0.00256 | -0.010239887 | -0.01024 |
| 0.5 | -0.003749959 | -0.00375 | -0.014999835 | -0.01500 |
| 0.6 | -0.005039945 | -0.00504 | -0.020159778 | -0.02016 |
| 0.7 | -0.00636993 | -0.00637 | -0.025479719 | -0.02548 |
| 0.8 | -0.007679915 | -0.00768 | -0.030719662 | -0.03072 |
| 0.9 | -0.008909902 | -0.00891 | -0.035639608 | -0.03564 |
| 1.0 | -0.009999890 | -0.01000 | -0.039999560 | -0.04000 |
| | t = 0.3 | | t = 0.4 | |
| x | $u_{GDTM}$ | Exact | $u_{GDTM}$ | Exact |
| 0.0 | 0.000000000 | 0.0000 | 0.000000000 | 0.0000 |
| 0.1 | -0.001709981 | -0.00171 | -0.003039967 | -0.00304 |
| 0.2 | -0.006479929 | -0.00648 | -0.011519873 | -0.01152 |
| 0.3 | -0.013769848 | -0.01377 | -0.024479730 | -0.02448 |
| 0.4 | -0.023039746 | -0.02304 | -0.040959549 | -0.04096 |
| 0.5 | -0.033749628 | -0.03375 | -0.059999339 | -0.06000 |
| 0.6 | -0.045359501 | -0.04536 | -0.080639112 | -0.08064 |
| 0.7 | -0.057329369 | -0.05733 | -0.101918878 | -0.10192 |
| 0.8 | -0.069119239 | -0.06912 | -0.122878647 | -0.12288 |
| 0.9 | -0.080189117 | -0.08019 | -0.142558430 | -0.14256 |
| 1.0 | -0.089999009 | -0.09000 | -0.159998238 | -0.16000 |

## CONCLUSIONS

In this paper a new generalization of the two dimensional differe-ntial transform method has been developed for fractional partial differential equations with boundary value problems. The new generalization is based on the generalized Taylor's formula and Caputo fractional derivative.

Comparison of the results obtained by using the GDTM with that obtained by other existing methods reveals that the present method for solving fractional partial differential equations is a very effective and convenient technique in finding numerical solutions for wide classes of problems and it increases the accuracy of the solutions.

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