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An Overview of Haar Wavelet Method for Solving Differential and Integral Equations

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Abstract: Investigation of various wavelet methods, for its capability of analyzing various dynamic phenomena through waves gained more and more attention in engineering research. Starting from 'offering good solution to differential equations' to capturing the nonlinearity in the data distribution, wavelets are used as appropriate tools that provide good mathematical model for scientific phenomena, which are usually modeled through linear or nonlinear differential equations. Review shows that the Haar wavelet method (HWM) is efficient and powerful in solving wide class of linear and nonlinear differential equations. The discrete wavelet transform has gained the reputation of being a very effective signal analysis tool for many practical applications. This review intends to provide the great utility of Haar wavelets to science and engineering problems which owes its origin to 1910. Besides future scope and directions involved in developing Haar wavelet algorithm for solving differential equations are addressed.

Key words: Haar wavelet method . differential equations . integral equations . operational matrices . nonlinear PDEs

INTRODUCTION

Wavelet analysis is a new branch of mathematics and widely applied in signal analysis, image processing and numerical analysis etc. The wavelet methods have proved to be very effective and efficient tool for solving problems of mathematical calculus. In recent years, these methods have attracted the interest of researchers of structural mechanics and many papers in this field are published. In most papers the Daubechies wavelets are applied. These wavelets are orthogonal, sufficiently smooth and have a compact support. Their shortcoming is that an explicit expression is lacking. This obstacle makes the differentiation and integration of these wavelets very complicated. For evaluation of such integrals the connection coefficients are introduced, but this complicates the course of the solution to a great extent. Among the wavelet families, which are defined by an analytical expression, special attention deserves the Haar wavelets. In 1910, Alfred Haar [1] introduced the notion of wavelets. His initial theory has been expanded recently into a wide variety of applications, but primarily it allows for the representation of various functions by a combination of step functions and wavelets over specified interval widths. The Haar wavelet transform is one of the earliest examples of what is known now as a compact, dyadic, orthonormal wavelet transform. Haar wavelets are made up of pairs of piecewise constant functions and are mathematically the simplest among all the wavelet families. A good feature of the Haar wavelets is the possibility to integrate them analytically arbitrary times. The Haar wavelets are very effective for treating singularities, since they can be interpreted as intermediate boundary conditions.

In the last two decades, the approximation of orthogonal functions has been playing an important role in the solution of problem such as parameter identification analysis and optimal control. The main characteristic of this technique is that it converts the differential equation used to describe problem to a set of algebraic equations. Chen and Hsiao [2] were the first to derive the approximation method via Walsh function. Subsequently, the set of orthogonal functions have been extensively applied to solve the parameter identification of linear lumped time invariant systems [3], bilinear systems [3] and multi-input multi-output systems [4]. The above mentioned orthogonal functions, however, are supported on the whole interval $a \le x \le b$. This kind of global support is evidently a drawback for certain analysis of work, particularly in systems involving abrupt variations or a local function vanishing outside a short interval of time or space [5]. Haar wavelets have been proved to be a useful mathematical tool for overcoming this disadvantage. The pioneering work in system analysis via Haar

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wavelets was led by Chen and Hsiao [5] who first derived a Haar operational matrix for the integrals of the Haar function vector and paved the way for the Haar analysis of the dynamical systems, later Hsiao [6] established the method to find solutions for time-varying systems by introducing Kronecker product of matrices for avoiding singularities [7] and the time-varying singular bilinear systems [8]. Kilicman and Al Zhour [9] had introduced the Kronecker operational matrices for fractional calculations and some applications. Some of the books include: C.K.Chui (1991), Daubechies (1992), Heller *et al*. (1992), Hernandez and Weiss (1996), Berrus (1997), Goswami (1999), Stephane Jaffard *et al*. (2001), Soesianto *et al*. (2002), Resnikoff *et al*. (2004) and Ge *et al*. (2007).

When the solution of a system contains components which change at significantly different rates to give changes in the independent variable, the system is said to be "stiff". Hsiao [10] dealt state analysis of linear time delayed systems via Haar wavelets and carried out comparative study of the same with finite difference method (FDM)

Wavelet methods have been applied for solving partial differential equations (PDEs) from beginning of the early 1990s. Haar wavelets have been applied extensively for signal processing in communications and physics research and have proved to be a wonderful mathematical tool. Hsiao and Wang [11] proposed a key idea to transform the time-varying function and its product with the states into a Haar product matrix. Since then, all related algorithms can be implemented easily. All severe mathematical constraints can be met perfectly. Chen and Hsiao [5, 12] established Haar wavelet method for solving lumped and distributed-parameter systems and also wavelet approach to optimizing dynamic systems. The RH transforms preserves all properties of the original Haar wavelet transform and can be efficiently implemented using digital pipeline architecture. Razzaghi and Ordokhani [13] had introduced the solution of differential equations via rationalized Haar functions. Hsiao and Wu [14] dealt the numerical solution of time-varying functional differential equations via Haar wavelets. Ohkita and Kobayayashi [15, 16] applied RH functions to solve ordinary differential equations and linear first and second order partial differential equations. Razzaghi and Ordokhani [17] introduced the RH functions to solve variational problems and differential equations. Maleknejad and Mirzaee [18] presented the rationalized Haar wavelet for solving linear integral equations. Shamsi *et al*. [19] used the Haar wavelet method for solving Pocklington's integral equation. Chen and Wu [20] applied a wavelet method for a class of fractional convection-diffusion equation with variable coefficients. Cattani [21, 22] showed Haar wavelet spline. In the Haar wavelet method for solving PDEs, the pioneering work by Cattani [21] is very important. Mehdi Rashidi Kouchi *et al*. [23] had solved the homogeneous and inhomogeneous harmonic differential equation using the Haar Wavelet method. Schwab and Stevenson [24] developed the adaptive wavelet algorithms for elliptic PDE's on product domains.

In the last two decades this problem has attracted great attention and numerous papers in this topic have been published. Lepik [25-38] introduced numerical solution of differential equations with higher order, integral equations and two dimensional partial differential equations using Haar wavelet method. Majak *et al*. [39] dealt weak formulation based Haar wavelet method for solving differential equations. Due to the simplicity the Haar wavelets are very effective for solving differential and integral equations [40-42]. Lepik [25-38] introduced the Haar wavelet method for solving differential, fractional differential, integral equations and integro-differential equations. The same author solved nonlinear integro-differential equations and fractional integral equations by the Haar wavelet method in which HW has been compared with other numerical methods [31, 38].

In recent years, the integral equations provide an efficient tool for modeling a numerous phenomena and processes and for solving boundary value problems for both ordinary and partial differential equations. The wavelet method was first applied to solving differential and integral equations in the 1990s. A survey of early results in this field can be found in [27]. Babolian *et al*. [43] presented the numerical solution of nonlinear fredholm integral equations of the second kind using Haar wavelets. Lately the number of respective papers has greatly increased and it is not possible to analyze them all here, but some are discussed in the following sections. Hesam-aldien Derili *et al*. [44] had developed the two-dimensional wavelets for integral equations. Adefemi Sunmonu [45] developed the Maple codes for Haar wavelets.

Hariharan *et al*. [46] introduced the Haar wavelet method for solving Fisher's equation. In the year 2010, the same author(s) [47-56] showed the superiority Haar wavelet method for solving FitzHugh-Nagumo equation, Cahn-Allen equation, finite length beam equation, Convection-Diffusion equation, some nonlinear parabolic equations, one-dimensional reaction-diffusion equations, some traveling wave equations, Klein-Gordon equation, over the other methods in application domain. Some Bratu-type equations and other partial differential equations in which the equations have been solved by Haar wavelet method and the numerical solutions have been compared with other methods like Aomain Decomposion Method (ADM), Restrictive Taylor' s series (RT) method, Homotopy perturbation method, Finite difference method and Upwind scheme. The same authors (s) have established Haar wavelet in estimating depth profile of soil temperature [47].

For applications of the Haar wavelet transform in logic design, efficient ways of calculating the Haar spectrum from reduced forms of Boolean functions are needed. Such methods were introduced for calculation of the Haar spectrum from disjoint cubes and different types of decision diagrams.

Optimal control theory is certainly the field of most extensive applications of Haar wavelets since its appearance, in view of its ability to model hereditary phenomena with long memory. This theory has also several applications, e.g. in structural dynamics, space flights, chemical engineering, economy. There is a significant interest in applications, which includes: process and manufacturing, aerospace and defence, marine and automotive systems, structural and mechanical design, robotics and manufacturing systems, chemical, petrochemical and industrial processes, electric power generation and distribution systems, energy systems and management, operations research and business, socio-economic models, biological and biomedical systems, environmental control, water treatment and ecology management, electrical and electronic systems and health care and support. It also covers a wide range of interdisciplinary and complex systems problems, where multi-agent software solutions, intelligent sensors and either dynamic or static optimization plays a major role. For more details on this, see thesis like state analysis and optimal control of linear time-varying systems via Haar wavelets. But we could refer to other applications like finance, stochastic processes and many branches of applied sciences and engineering, as proved by the increasing number of articles, congresses and treaties involving Haar wavelets (HW). In our opinion to cite review-survey papers on the applications of HW we have the risk to forget some because the list is long. Karimi *et al*. [57-61] illustrated the Haar wavelet-based approach for optimal control problems with linear systems in time domain. The same author(s) implemented a Haar wavelet-based robust optimal control for vibration reduction of vehicle engine-body system [58]. Hsiao and Wang [8] had solved the optimal control of linear time-varying systems via Haar wavelets.

In spite of great theoretical interest in applications of the discrete Haar transform in switching theory and logic design, exponential complexity of FHT in terms of both space and time was a restrictive factor for wider practical applications of the Haar transform. Due to its low computing requirements, the Haar transform has been mainly used for pattern recognition and image processing [62]. Hence, two dimensional signal and image processing is an area of efficient applications of Haar wavelet transforms due to their wavelet-like structure. In this field, it is usually reported that the simplest possible orthogonal wavelet system is generated from the Haar scaling function and wavelet. Moreover, wavelets are considered as a generalization of the Haar functions and transforms. Hence, HW is also well suited in communication technology for data coding, multiplexing and digital filtering. It is nowadays recognized the advantage of using Haar wavelet for solving differential equations which is demonstrated by the increasing number of papers and special issues in journals.

In recent decades the field of Haar wavelets for solving differential equations has attracted interest of researchers in several areas including mathematics, physics, chemistry, biology, engineering, statistics and even finance and social sciences.

But why Haar Wavelets are important?

For real time applications, hard-ware based fast Haar chips have been developed. Haar wavelet functions used in image processing [62], digital speech processing, voice controlled computing devices and robotics. The control system based on Haar spectrum has been used in military airplane. Indeed, at present, applications and/or activities related to Haar wavelet method for solving differential equations have appeared in at least the following fields. Schneider [63] used the matrix compression scheme.

APPLICATIONS OF HAAR WAVELET METHOD FOR SOLVING DIFFERENTIAL EQUATIONS

The Haar wavelet method exhibits several advantageous features.

(i) High accuracy is obtained already for a small number of grid points.
(ii) Possibility of implementation of standard algorithms. For calculation the integrals of the wavelet functions, universal subprograms can be put together. Another time consuming operation is the solving of high-order systems of linear equations and calculating high-order determinants; here the matrix programs of MATLAB are very effective.
(iii) The method is very convenient for solving boundary value problems since the boundary conditions are taken care of automatically.

Table 1: Applications of Haar wavelets for differential equations

S. No	Application	Field
1	Estimating depth profile of soil temperature, Modeling of Soil moisture	Civil Engineering
2	Advancement of calculus of variations and optimal control problems, lightning stroke problems, Lumped and distributed parameter systems	Electrical Engineering
3	Analytical and numerical tools and techniques	Mathematical Sciences
4	Image digital and signal processing	Computer Science and Engineering
5	Quantum field theory	Physics
6	Solving Ordinary, Partial, Integral and fractional order differential equations.	Mathematical Sciences
7	Bioengineering, Biomedical applications and Modeling of Biosensors	Biotechnology
8	Vibration problems, Heat and mass transfer problems, Fluid-flow problems	Mechanical Engineering
9	Reaction and Diffusion problems, Chemical kinetics problems	Chemical Engineering

(iv) Singularities can be treated as intermediate boundary conditions; this circumstance to a great extent simplifies the solution.
(v) The obtained solutions are mostly simpler compared with other known methods.

Yuanlu Li and Weiwei Zhao [64] dealt the Haar wavelet operational matrix of fractional order integration and its applications in solving the fractional order differential equations Bujurke *et al*. [65, 66] applied the single-term Haar wavelet series (STHWS) in the solution of nonlinear oscillator equations and stiff systems from nonlinear dynamics. Chang and Phang Piau [67] presented Haar wavelet matrices designation in numerical solution of ordinary differential equations. Chen *et al*. [68] established Haar wavelet method for solving a class of fractional convection-diffusion equation with variable coefficients. Li and Zhao [69] introduced the Haar wavelet operational matrix of fractional order integration and applications in solving the fractional order differential equations. Zhi Shi *et al*. [70] showed Haar wavelet method for solving wave equations and convection-diffusion equations. Li and Wang [71] solved a nonlinear fractional differential equations using Haar wavelet method. Goswami *et al*. [72] had solved first-kind integral equations using wavelets on a bounded interval. Gao and Lao [73] had established the discretization algorithm for fractional order integrals by Haar wavelets. Wanhai Geng *et al*. [74] used the wavelet method for nonlinear partial differential equations of fractional order. Xiao-Yan Lin *et al*. [75] introduced a Haar wavelet solution to Fredholm equations. Gu and. Jiang [76] had derived the Haar wavelets operational matrix of integration. Siraj-ul-Islam *et al*. [77-79] applied the Haar wavelet method for second order boundary value problems in which the performance of the Haar wavelets has been compared with other methods like Walsh wavelets, semi-orthogonal B-Spline wavelets, spline functions, Adomain Decomposition Method (ADM), Runge-Kutta (RK) method and nonlinear shooting method. Kazuhiro and Kazuhisa Abe *et al*. [80] presented the application of Haar wavelets to time-domain BEM for the transient scalar wave equation. Fazal-i-Haq *et al*. [81-84] had implemented a collocation method based on Haar wavelets for solving eight-order boundary value problems in which the accuracy and efficiency of the Haar wavelet method was established through comparison with the existing non-polynomial Spline based technique, modified decomposition method and homotopy perturbation method. Bayati *et al*. [85] established a modified wavelet algorithm to solve boundary value problems with an infinite number of boundary conditions. Hein and Feklistova [86] presented free vibrations of non-uniform and axially functionally graded beams using Haar wavelets. Celik [87] established the Haar wavelet method for solving generalized Burgers-Huxley equation. He proved that the Haar wavelet method is a very reliable, simple, small computation costs, flexible and convenient alternative method. Shi Zhi and Cao Yong-yan [88] addressed for solving 2D and 3D Poisson equations and biharmonic equations by the Haar wavelet method. Miaomiao Wang and Fengqun Zhao [89] had solved the two-dimensional Burgers' equation using Haar wavelet method. Kazuhiro Koro and Kazuhisa Abe [90] applied the Haar wavelets to time-domain BEM for the transient scalar wave equation. Kazuhisa Abe *et al*. [91] introduced the h-hierarchical Galerkin BEM using Haar wavelets. Beylkin *et al*. [92] established the wavelet-based boundary element method. More recently, hariharan *et al*. [93, 94] introduced the Haar wavelet method to film-pore diffusion model for methylene blue adsorption onto plant leaf powders and some wave-type equations.

HAAR WAVELET PRELIMINARIES

Haar wavelet was a system of square wave; the first curve was marked up as $h_0(t)$, the second curve marked up as $h_1(t)$ that is

$$h_0(t) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$h_1(t) = \begin{cases} 1, & 0 \le x < 1/2 \\ -1, & 1/2 \le x < 1 \\ 0, & \text{otherwise} \end{cases}$$

where $h_0(t)$ is scaling function, $h_1(t)$ is mother wavelet. In order to perform wavelet transform, Haar wavelet uses dilations and translations of function, i.e. the transform make the following function.

$$h_n(t) = h_1(2^j t - k), \ n = 2^j + k, \ j \ge 0, \ 0 \le k < 2^j$$

Chen and Hsiao [2] raised the ideology of operational matrix in 1975 and Kilichman and Al Zhour [95] investigated the generalized integral operational matrix, that is, the integral of matrix $\phi(t)$ can be approximated as follows:

$$\int_0^t \phi(t)\,dt \cong Q_\phi \phi(t) \tag{5}$$

where Q_ϕ is an operational matrix of one-time integral matrix $\phi(t)$, similarly, we can get operational matrix Q_ϕ^n of n-time integral of $\phi(t)$. Hsiao [6, 7, 10] proposed a uniform method to obtain the corresponding integral operational matrix of different basis. For example, the operational matrix of $\Phi(t)$ can be expressed by following:

$$Q_\Phi = \Phi Q_B \Phi^{-1} \tag{6}$$

Here Q_B is the operational matrix of the block pulse function.

$$Q_{B_m} = \frac{1}{2m}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{7}$$

where m is the dimension of matrix $\Phi(t)$ and usually $m = 2^\alpha$, α is positive integer.

If $\Phi(t)$ is a unitary matrix, then $Q_\Phi = \Phi Q_B \Phi^T$, Q_Φ is a matrix with characteristic of briefness and profound utility.

For $x \in [0,1]$, Haar wavelet function is defined as follows:

$$h_0(x) = \frac{1}{\sqrt{m}}$$

$$h_i(x) = \frac{1}{\sqrt{m}}\begin{cases} 2^{\frac{j}{2}}, & \frac{k-1}{2^j} \le x < \frac{k-(1/2)}{2^j} \\ -2^{\frac{j}{2}}, & \frac{k-(1/2)}{2^j} \le x < \frac{k}{2^j} \\ 0, & \text{otherwise} \end{cases}$$

Integer $m = 2^j$ ($j = 0,1,2,\ldots J$) indicates the level of the wavelet; $i = 0,1,2,\ldots m-1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m+k-1$; in the case of minimal values $m = 1$, $k = 0$ we have $i = 2$, the maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1$ in [0,1]. Let us define the collocation points $t_l = (l - 0.5)/2M, (l = 1,2....2M)$ and discretise the Haar function $h_i(x)$; in this way we get the coefficient matrix $H(i, l) = (h_i(x_l))$, which has the dimension $2M \times 2M$.

The operational matrix of integration P, which is a $2M$ square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t)\,dt$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^t h_i(t)\,dt$$

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$P_8 = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [2] showed that the following matrix equation for calculating the matrix P of order m holds

$$P_{(m)} = \frac{1}{2m}\begin{pmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & O \end{pmatrix}$$

where O is a null matrix of order $\frac{m}{2}\times\frac{m}{2}$

$$H_{m\times m} \triangleq \left[h_m(t_0) \;\; h_m(t_1) \;\; ------ h_m(t_{m-1}) \right]$$

Here $\frac{i}{m} \le t < i + \frac{1}{m}$ and

$$H_{mxm}^{-1} = \frac{1}{m} H_{mxm}^{T} \ \mathrm{diag}(r)$$

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations. First eight Haar functions and their integrals are presented in [46, 48].

Function approximation: Any square integrable function $y(x) \in L^2([0,1]\times[0,1])$ can be expanded by a Haar series of infinite terms

$$y(x,t) \approx \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(t) \qquad (8)$$

where the Haar coefficients c_{ij} are determined as

$$c_{i,j} = \int_0^1\int_0^1 y(x,t) h_i(x) h_j(t)\, dt\, dx, \quad (i, j = 0,1,2,...,m-1)$$

are coefficients, discrete y(x,t) by choosing the same step of x and t, we obtain

$$Y(x,t) = H^T(x) C H(t) \qquad (9)$$

where Y(x,t) is the discrete form of y(x,t) and

$$H = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,m-1} \end{bmatrix}$$

C is the coefficient matrix of Y and it can be obtained by formula:

$$C = \left(H^T\right)^{-1} Y H^{-1} \qquad (10)$$

H is an orthogonal matrix, then

$$C = H.Y.H^{-1} \qquad (11)$$

We even find

$$C = H Y H^T \qquad (12)$$

COMPARISON BETWEEN FOURIER TRANNSFORM (FT) AND WAVELET TRANSFORM(WT)

Wavelet transforms have advantages over traditional Fourier transforms for representing functions that have discontinuities and sharp peaks and for accurately deconstructing and reconstructing finite, non-periodic and/or nonstationary signals. Any numerical scheme for solving differential equations must adequately represent the derivatives and non linearities of the unknown function. In the case of wavelet bases, these approximations give rise to certain L_2 inner products of the basis functions, their derivatives and their translates, called the connection coefficients. In Fourier-based methods, since the products of the basis elements are also basis elements, the procedure does not face any difficulty. The numerical approximation of the connection coefficients which appear with the wavelet bases is unstable since the integrands are highly oscillatory [96]. Excellent discussions between the wavelet transform and the Fourier transform are presented in [96].

The most interesting dissimilarity between these two kinds of transforms is that individual wavelet functions are *localized in space*. Fourier sine and cosine functions are not. This localization feature, along with wavelets' localization of frequency, makes many

Table 2:Comparison of algorithmic complexity of the of the proposed method with FFT and WT [50]

Series	Numbers of additions	Numbers of multiplications
Haar Transform (HT)	2m-2	m
Walsh Transform (WT)	$m\log_2 m$	m
Fast Fourier Transform (FFT)	$m\log_2 m$	$M(\log_2 m+1)$

A FEW JOURNAL ARTICLES AND APPLICATIONS

Table 3: A few journal articles and applications

S.No	Title of the paper	Year	Author (s)	Application	Advantage
1	Haar wavelet method for solving lumped and distributed-parameter systems	1997	Chen, C.F., Hsiao,C.H	Haar Analysis of dynamic systems	Derived the Haar operational matrix for integrals of the Haar function vector
2	State analysis of linear time delayed systems via Haar wavelets	1997	Chen, C.H.,	Dynamic system	Product matrix and coefficient matrix were applied to solve time-delayed systems.
3	State analysis and optimal control of linear time-varying systems via Haar wavelets	1997	Hsiao., C.H	Wang, W.J	Dynamic systems to transform the time-varying function and its product with the states into a Haar product matrix
4	Wavelet approach to optimizing dynamic systems	1999	Chen, C.F., Hsiao,C.H	Haar Analysis of dynamic systems	A simple and complete procedure for optimizing a dynamic system is formulated
5	State analysis and parameters estimation of bilinear systems via Haar wavelets	2000	Chen, C.F., Hsiao,C.H.,	Wang, W.J	Dynamic systems Derived Haar product matrix and a coefficient matrix
6	Haar wavelet approach to nonlinear stiff systems	2001	Hsiao,C.H.,	Wang, W.J	Dynamic systems Haar wavelet method was compared with Runge-Kutta-Fehlberg approach.
7	An application of rationalized Haar functions for variational problems	2001	Razzagi, M., Ordokhani, Y	Rationalized Haar function was introduced	A direct method for solving variational problems using rationalized Haar method established
8	The Haar wavelet transform: its status and achievements	2003	Stankovic, R.S.	Falkowski,B.J	Applications of Haar wavelets in Engineering Various applications and generalized definitions of Haar wavelets were addressed
9	Haar wavelets based technique in evolution problems	2004	Cattani, C	Haar wavelet method was applied for solving application driven PDEs	Applications of HW for solving nonlinear PDE are addressed
10	Haar wavelet direct method for solving variational problems	2004	Hsiao, C.H	Extremization of a functional systems	The variational problems are solved by Haar wavelet method
11	Numerical solution of evolution equations by the Haar wavelet method	2007	Lepik, U	It is used in fluid dynamics teaching and in engineering as a simplified model for turbulence, boundary layer behavior, shock wave formation and mass transport.	Haar wavelet for solving some nonlinear PDEs like Burgers' equation, Sine-Gordon equation
12	Application of the Haar wavelet transform to solving integral and differential Equations	2007	Lepik, U	Engineering and Science	Haar wavelet method for solving integro-differential equations
13	Wavelets approach to time-varying functional differential equations	2008	Hsiao, C.H	The linear time-varying systems solved accurately by	The unknown wavelet coefficient matrix Haar wavelet method has found in the generalized Lyapunov equation
14	Haar wavelet method for solving Fisher's equation	2009	Hariharan, G., Kannan,K	Chemistry, biology and medicine	Haar wavelet method for solving differential equations with nonlinearity. HW has compared with ADM
15	Solving fractional integral equations by the Haar wavelet method	2009	Lepik, U	Modeling and control of many dynamical systems	HW method established for solving fractional integral equations
16	The numerical solution of second-order Boundary vaue problems by collocation with the Haar wavelets	2010	Siraj-ul-Islam, Imran Aziz, Bozidar Sarler	Mathematical modeling of deformation of beams and plate deflection theory	Haar wavelet method has compared with R-K, ADM, FDM and Nonlinear shooting methods and the convergence analysis of HW method was also addressed
17	Application of Haar wavelets to time-domain BEM for the transient scalar wave equation	2010	Kazuhiro Koro, Kazuhisa Abe	BE wave propagation analysis	The time variation of the unknown potential and flux is approximated
18	Solving fractional Riccati differential equations using Haar wavelet	2010	Yuan-lu Li, Li Hu	Modeling and control of many dynamical systems	Fractional Riccati equations are solved.
19	A Comparative Study of a Haar Wavelet Method and a Restrictive Taylor's Series Method for Solving Convection-diffusion Equations	2010	Hariharan, G., Kannan, K	Computational hydraulics and fluid dynamics to model convection-diffusion of quantities such as mass, heat, energy, vorticity	HW method established for solving a few Convection-Diffusion equations and the solutions are compared with Restrictive Taylor's series method and the conventional methods
20	Solving Finite Length Beam Equation by the Haar Wavelet Method	2010	Hariharan, G., Kannan, K	Elastic Mechanics	HW method established for solving finite length beam equation.
21	Haar wavelet method for solving some nonlinear parabolic equations	2010	Hariharan, G., Kannan, K	Mathematical modeling with gene propagation and	HW scheme derived for some nonlinear PDEs biological modeling and the solutions are compared with other solutions
22	Solving PDEs with the aid of two-dimensional Haar wavelets	2011	Lepik, U	Engineering applications	HW established for solving two-dimensional problems

functions and operators using wavelets “sparse" when transformed into the wavelet domain. This sparseness, in turn, results in a number of useful applications such as data compression, detecting features in images and removing noise from time series. One way to see the time-frequency resolution differences between the Fourier transform and the wavelet transform is to look at the basis function coverage of the time-frequency plane.

An advantage of wavelet transforms is that the windows vary. In order to isolate signal discontinuities, one would like to have some very short basis functions.

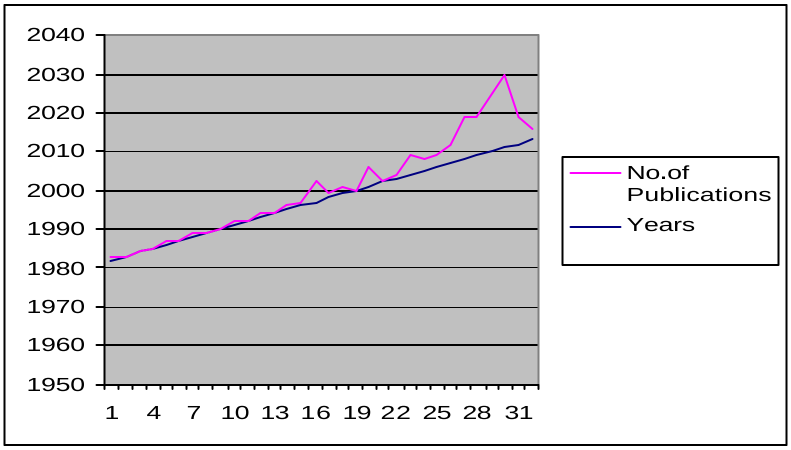

Fig. 1: Comparison between the years and no. of publications (1982-2013)

At the same time, in order to obtain detailed frequency analysis, one would like to have some very long basis functions. A way to achieve this is to have short high-frequency basis functions and long low-frequency ones. This happy medium is exactly what you get with wavelet transforms. One thing to remember is that wavelet transforms do not have a single set of basis functions like the Fourier transform, which utilizes just the sine and cosine functions. Instead, wavelet transforms have an infinite set of possible basis functions. Thus wavelet analysis provides immediate access to information that can be obscured by other time-frequency methods such as Fourier analysis. Table 1 shows the comparison of algorithmic complexity of the Haar Wavelet Method (HWT), Walsh Transform Method (WTM) and Fast Transform Method (FFM).

COMPARISON BETWEEN HAAR WAVELETS AND OTHER WAVELETS

A shortcoming of the Daubechies wavelets is that they do not have an explicit expression and therefore analytical differentiation or integration is not possible. Legendre multi-wavelets are a mutation of Haar's wavelet; they are piecewise linear and have short support, but they lack smoothness and are discontinuous. Moreover Legendre multi-wavelets are localized in time but not in frequency due to their discontinuity. Smooth multi-wavelets have the advantage of being mostly simultaneously localized in time and frequency, of course within the limit imposed by Heisenberg's uncertainty principle. Ghasemi *et al*. [97] obtained the solution of time-varying delay systems is obtained by using Chebyshev wavelets. They proved that the Chebyshev wavelets provide an exact solution for the cases when the exact solutions are polynomials. Cattani *et al*. [21, 22] established the wave propagation of Shannon wavelets and Harmonic wavelet analysis method for Fredholm equation of the second kind. Habibollah Saeedi *et al*. [98] and Siraj-ul-Islam *et al*. [78, 79] have established the convergence analysis of the Haar wavelet method. In order to analyze the convergence of Haar wavelet method, they have defined the error function and showed that the method is convergent for a special class of functions in the sense that the corresponding error tends to zero as m tends to infinity. Mehmet Sezer *et al*. [99] had solved the high-order linear differential equations by a Legendre matrix method based on hybrid Legendre and Taylor polynomials. Mohammadi and Hosseini [100] introduced a new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations. Mujeeb ur Rehman and Rahmat Ali Khan [101] presented a numerical method for solving boundary value problems for fractional differential equations. Razzaghi and Yousefi [102] introduced the Legendre wavelets method for the solution of nonlinear problems in the calculus of variations.

Preliminary note: Clearly, lists such as those assembled in this article, can never be complete and, besides, there must be selective decisions. We do apologize for all omissions. Moreover, we do not give any judgment on the references: we limit ourselves to cite (possibly/hopefully) most of them.

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FUTURE SCOPE AND DIRECTIONS

The main advantages of this method are its simplicity and less computation costs: it is due to the sparcity of the transform matrices and to the small number of significant wavelet coefficients. In comparison with existing numerical schemes used to solve the differential equations, the Haar wavelet method is an improvement over other methods in terms of accuracy. It is worth mentioning that Haar solution provides excellent results even for small values of m (m = 16).

For larger values of m (ie., m = 32, m = 64, m = 128, m = 256), we can obtain the results closer to the real values. The reason of use Haar wavelets is Haar wavelet method (HWM) are sparse matrix representation, fast transformation and possibility of implementation of fast and efficient algorithms. The method with far less degrees of freedom and with smaller CPU time provides better solutions than classical ones. The method is also very convenient for solving the boundary value problems, since the boundary conditions are taken care of automatically. Another benefit of our method is that the model equations including more mechanical, physical or biophysical effects, such as nonlinear convection, reaction, linear diffusion and dispersion can be solved easily.

The review of literature shows that there is still scope for applying Haar wavelet method for solving differential equations that will yield better quality solutions by addressing some of the issues as follows:

1. Eigen value problems of partial differential equations (PDEs).
2. Non-rectangular domains and nonlinear problems.
3. Fractional differential equations.
4. Application of differential equations arising in astrophysics.
5. Electrochemical modeling problems
6. Buckling and vibrations of elastic structures

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