World Applied Sciences Journal 22 (Special Issue of Applied Math): 45-61, 2013
ISSN 1818-4952

DOI: 10.5829/idosi.wasj.22.am.8.2013

Fuzzy Soft Sets and their Algebraic Structures

[1] *Munazza Naz and* [2] *Muhammad Shabir*

[1]Department of Mathematics, Fatima Jinnah Women University, The Mall, Rawalpindi, Pakistan
[2]Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan

Abstract: In this paper, we have studied the algebraic operations of fuzzy soft sets to establish their basic properties. We have discussed different algebraic structures of fuzzy soft sets under the restricted and extended operations of union and intersection in a comprehensive manner. Logical equivalences have also been made in order to give a complete overview of these structures.

2000 mathematics subject classification: [2000] Primary 05C38 . 15A15 . secondary 05A15 . 15A18

Key words: Soft set . fuzzy sets . union . intersection . extended operations . restricted operations

INTRODUCTION

While modeling most of the real world problems, vagueness or fuzziness of data creates difficulties and we do not get the required results. This inaccuracy and imprecision cause problems in the successful implementation of a mathematical model. To handle these uncertainties, different theories and approaches have been adapted so far. L.A. Zadeh [1] introduced his revolutionary concept of a fuzzy set in 1965. Fuzzy set theory is applied in almost all the branches of mathematics. After that, many theories like Theory of Rough Sets [2], Intuitionistic Fuzzy Sets [3] and Vague Set Theory [4] were introduced. All these theories have their own difficulties and limitations.

One problem was the inadequacy of the parameterization tool, which had been shown quite evidently by Molodtsov in his paper [5] in 1999. He proposed the new approach of soft set theory for dealing with the uncertainties of data. Molodtsov showed the applications of soft sets in various fields like stability and regularization, game theory, operations research and analysis. Maji and Roy presented a theoretical study [6] and defined several operations on soft sets. In 2001, they proposed the concept of "Fuzzy Soft Sets" [7] and later on applied the theories in decision making problem [8, 9]. Different algebraic structures and their applications have also been studied in soft and fuzzy soft context [10-16].

Irfan *et al.* pointed out some basic problems in the results related to the operations defined on fuzzy soft sets. The incorrectness of assertions comes before us while the operations are performed on more than two fuzzy soft sets. In their paper [17], some new operations are defined for fuzzy soft sets and modified results and laws are established. In this paper, we step forward in the same direction and define the associativity and distributivity of these operations. The paper is organized in five sections. First we have given preliminaries on the theories of fuzzy soft sets. We have used new and modified definitions and operations from [17] to discuss the properties of associativity and distributivity of these operations for fuzzy soft sets. Counter examples are provided to show that the converse of proper inclusions is not true in general. After accomplishing an account of algebraic properties of fuzzy soft sets, we study the overall algebraic structures of collections of fuzzy soft sets. Basically there are two types of collections of soft sets; one consists of those fuzzy soft sets with a fixed set of parameters while the other contains fuzzy soft sets defined over the same universe with different set of parameters. Both collections have some common and some different algebraic properties and therefore the algebraic structures also differ. We see four

Corresponding Author: Munazza Naz, Department of Mathematics, Fatima Jinnah Women University, The Mall, Rawalpindi

commutative monoid and eight semirings of fuzzy soft sets with respect to the extended or restricted operations. The lattice structure of these collections is discussed and we find that the collection of all fuzzy soft sets is a bounded distributive lattice and the collection of fuzzy soft sets with a fixed set of parameters becomes a Kleene algebra. At the end we define pseudo-complement of a fuzzy soft set and with these pseudo-complements; this collection becomes a stone algebra.

FUZZY SOFT SETS

In this section we present the theory of fuzzy sets and fuzzy soft sets, taken from [2, 9, 17]. Let U be an initial universe set, E be a set of parameters and A,B be the subsets of E. Let FP(U) denotes the set of all fuzzy subsets of U.

Definition 1: A fuzzy subset of U is a function from U into the unit closed interval l [0,1]. The set of all fuzzy subsets of U is called the fuzzy power set of U and is denoted by FP(U).

Definition 2: Let μ, $u \in \mathbf{FP}(U)$. If $m(x) \leq u(x)$ $(m(x) \geq u(x))$ for all x∈ U. Then μ is said to be contained in υ and we write $m \subseteq u($ or $u \supseteq m)$.
Clearly, the inclusion relation ⊆ is a partial order on FP(U).

Definition 3: Let μ, υ∈ FP(U). Then μ∨υ and μ∧υ are fuzzy subsets of U, defined as follows:
For all x∈ U,

$$(m \wedge u)(x) = m(x) \wedge u(x), \quad (m \vee u)(x) = m(x) \vee u(x).$$

The fuzzy subsets μ∨υ and μ∧υ are called the union and intersection of fuzzy subsets μ and υ, respectively.

Definition 4: [9] A pair (F,A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to \mathbf{FP}(U)$.

Definition 5: [9] For two fuzzy soft sets (F,A) and (G,B) over a common universe U, we say that (F,A) is a fuzzy soft subset of (G,B) if

1) A⊆B and
2) For all e∈ A, F(e) is a fuzzy subset of G(e).

We write $(F, A) \tilde{\subset} (G,B)$. (F,A) is said to be a fuzzy soft super set of (G,B), if (G,B) is a fuzzy soft subset of (F,A). We denote it by $(F, A) \tilde{\supset} (G,B)$.

Definition 6: [9] Two fuzzy soft sets (F,A) and (G,B) over a common universe U are said to be fuzzy soft equal if (F,A) is a fuzzy soft subset of (G,B) and (G,B) is a fuzzy soft subset of (F,A).

Definition 7: [11] Let E = $\{e_1, e_2, \ldots, e_n\}$ be a set of parameters. The NOT set of E denoted by $\neg E$ is defined by $\neg E = \{\neg e_1, \neg e_2, ..., \neg e_n\}$ where, $\neg e_i = not$ e_i for all i.

Definition 8: [9] The complement of a fuzzy soft set (F,A) is denoted by $(F,A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where $F^c : \neg A \to \mathbf{FP}(U)$ is a mapping given by $F^c(a) = \mathbf{1} - F(\neg a)$, for all $a \in \neg A$.

Let us call F^c to be the fuzzy soft complement function of F. Clearly $(F^c)^c$ is the same as F and $((F,A)^c)^c = (F,A)$.

Definition 9: [9] If (F,A) and (G,B) are two fuzzy soft sets over the same universe U then "$(F,A)\,AND\,(G,B)$" is a fuzzy soft set denoted by $(F,A)\wedge(G,B)$ and is defined by $(F,A)\wedge(G,B)=(H,A\times B)$ where, $H((a,b))=F(a)\wedge G(b)$ for all $(a,b)\in A\times B$. Here $\wedge$ is the operation of intersection of two fuzzy sets.

Definition 10: [9] If (F,A) and (G,B) are two fuzzy soft sets over the same universe U then "$(F,A)\,OR\,(G,B)$" is a fuzzy soft set denoted by $(F,A)\vee(G,B)$ and is defined by $(F,A)\vee(G,B)=(O,A\times B)$ where, $O((a,b))=F(a)\vee G(b)$ for all $(a,b)\in A\times B$. Here $\vee$ is the operation of union of two fuzzy sets.

The following De Morgan's types of results are true.

Proposition 1: [9]

1) $((F,A)\wedge(G,B))^c=(F,A)^c\vee(G,B)^c$,
2) $((F,A)\vee(G,B))^c=(F,A)^c\wedge(G,B)^c$.

Definition 11: [9] Union of two fuzzy soft sets (F,A) and (G,B) over the common universe U is the fuzzy soft set (H,C), where C = A∪B and for all e∈C,

$$H(e)=\begin{cases} F(e) & if\ e\in A-B \\ G(e) & if\ e\in B-A \\ F(e)\vee G(e) & if\ e\in A\cap B \end{cases}$$

We write $(F,A)\tilde{\cup}(G,B)=(H,C)$

Definition 12: [2] The extended intersection of two fuzzy soft sets (F,A) and (G,B) over a common universe U, is the fuzzy soft set (H,C) where C = A∪B and for all e∈C,

$$H(e)=\begin{cases} F(e) & if\ e\in A-B \\ G(e) & if\ e\in B-A \\ F(e)\wedge G(e) & if\ e\in A\cap B \end{cases}$$

We write $(F,A)\tilde{\cap}(G,B)=(H,C)$

Definition 13: [2] If (F,A) and (G,B) are two fuzzy soft sets over a common universe U such that $A\cap B\neq\emptyset$, then their restricted intersection is a fuzzy soft set $(H,A\cap B)$ denoted by $(F,A)\Cap(G,B)=(H,A\cap B)$ where H is a function from A∩B to FP(U) defined as $H(e)=F(e)\wedge G(e)$ for all $e\in A\cap B$.

Definition 14: [2] If (F,A) and (G,B) are two fuzzy soft sets over a common universe U such that $A \cap B \neq \varnothing$, then their restricted union is a fuzzy soft set $(H, A \cap B)$ denoted by $(F,A) \Cup (G,B) = (H, A \cap B)$ where H is a function from A∩B to FP(U) defined as $H(e) = F(e) \vee G(e)$ for all $e \in A \cap B$.

Definition 15: [2] Let U be an initial universe set, E be the set of parameters and A be a non-empty subset of E. Then

1) (F,A) is called a relative null fuzzy soft set (with respect to the parameter set A), denoted by Φ_A, if F(e) = ∅ for all e∈ A, where ∅ is the fuzzy subset of U mapping every element of U on 0.
2) (G,B) is called a relative whole fuzzy soft set (with respect to the parameter set A), denoted by U_E, if G(e) = 1 for all e∈ A, where 1 is the fuzzy subset of U mapping every element of U onto 1.

The relative whole fuzzy soft set U_E with respect to the set of parameters E is called the absolute fuzzy soft set over U.

Definition 16: [2] The relative complement of a fuzzy soft set (F,A) is denoted by (F,A)′ and is defined by (F,A)′ = (F′,A) where $F' : A \to FP(U)$ is a mapping given by $F'(a) = \mathbf{1} - F(a)$, for all α∈ A.

Theorem 1: [2] Let (F,A) and (G,B) be two fuzzy soft sets over a common universe U, such that $A \cap B \neq \varnothing$. Then

1) $\left((F,A) \Cap (G,B)\right)' = (F,A)' \Cup (G,B)'$
2) $\left((F,A) \Cup (G,B)\right)' = (F,A)' \Cap (G,B)'$.

Theorem 2: [2] Let (F,A) and (G,B) be two fuzzy soft sets over a common universe U. Then the following are true

1) $\left((F,A) \tilde{\cup} (G,B)\right)^c = (F,A)^c \tilde{\cap} (G,B)^c$
2) $\left((F,A) \tilde{\cap} (G,B)\right)^c = (F,A)^c \tilde{\cup} (G,B)^c$.

The Algebraic Structures of Fuzzy Soft Sets

For this section, U is an initial universe, E is a non-empty set of parameters and A, B, C are subsets of E. A fuzzy soft set (F,A) over U is said to be non-empty if $(F, A) \neq \Phi_A$. We denote the collections as follows:

1) $\mathbf{FSS}(U)^E$: The collection of all fuzzy soft sets defined over U
2) $\mathbf{FSS}(U)_A$: The collection of all fuzzy soft sets defined over U with a fixed set of parameters A.

We note that the collections are partially ordered by relation of soft inclusion " $\tilde{\subseteq}$ ". In this section we discuss associative and distributive laws with respect to different operations of union and intersection which are defined for fuzzy soft sets. Conventionally, we assume that $(F,A) \Cap (G,B) = (F,A) \Cup (G,B) = \mathrm{F}_{\varnothing}$ whenever $A \cap B = \varnothing$.

Proposition 2: "Associative Laws" Let $(F,A),(G,B)$ *and* (H,C) be any fuzzy soft sets over U. Then

1) $\left((F,A) \Cap (G,B)\right) \Cap (H,C) = (F,A) \Cap \left((G,B) \Cap (H,C)\right)$,
2) $\left((F,A) \Cup (G,B)\right) \Cup (H,C) = (F,A) \Cup \left((G,B) \Cup (H,C)\right)$,
3) $((F,A) \tilde{\cup} (G,B)) \tilde{\cup} (H,C) = (F,A) \tilde{\cup} \left((G,B) \tilde{\cup} (H,C)\right)$,
4) $((F,A) \tilde{\cap} (G,B)) \tilde{\cap} (H,C) = (F,A) \tilde{\cap} \left((G,B) \tilde{\cap} (H,C)\right)$.

Proof: Straightforward.

Remark 1: In general the operations $\vee$ and $\wedge$ are not associative because

$$(A \times B) \times C \neq A \times (B \times C)$$

but if we take

$$(A \times B) \times C = A \times B \times C = A \times (B \times C)$$

then

$$\left((F,A) \vee (G,B)\right) \vee (H,C) = (F,A) \vee \left((G,B) \vee (H,C)\right)$$

And

$$\left((F,A) \wedge (G,B)\right) \wedge (H,C) = (F,A) \wedge \left((G,B) \wedge (H,C)\right).$$

Proposition 3: "Distributive Laws" Let (F,A) and (G,B) and (H,C) be any fuzzy soft sets over U. Then

1) $(F,A) \Cap \left((G,B) \tilde{\cup} (H,C)\right) = \left((F,A) \Cap (G,B)\right) \tilde{\cup} ((F,A) \Cap (H,C))$,
2) $(F,A) \Cap \left((G,B) \Cup (H,C)\right) = \left((F,A) \Cap (G,B)\right) \Cup ((F,A) \Cap (H,C))$,
3) $(F,A) \Cap \left((G,B) \tilde{\cap} (H,C)\right) = \left((F,A) \Cap (G,B)\right) \tilde{\cap} ((F,A) \Cap (H,C))$,
4) $(F,A) \Cup \left((G,B) \tilde{\cup} (H,C)\right) = \left((F,A) \Cup (G,B)\right) \tilde{\cup} ((F,A) \Cup (H,C))$,
5) $(F,A) \Cup \left((G,B) \Cap (H,C)\right) = \left((F,A) \Cup (G,B)\right) \Cap ((F,A) \Cup (H,C))$,
6) $(F,A) \Cup \left((G,B) \tilde{\cap} (H,C)\right) = \left((F,A) \Cup (G,B)\right) \tilde{\cap} ((F,A) \Cup (H,C))$,
7) $(F,A) \tilde{\cup} \left((G,B) \tilde{\cap} (H,C)\right) \tilde{\supseteq} \left((F,A) \tilde{\cup} (G,B)\right) \tilde{\cap} ((F,A) \tilde{\cup} (H,C))$,
8) $(F,A) \tilde{\cup} \left((G,B) \Cup (H,C)\right) \tilde{\subseteq} \left((F,A) \tilde{\cup} (G,B)\right) \Cup ((F,A) \tilde{\cup} (H,C))$,
9) $(F,A) \tilde{\cup} \left((G,B) \Cap (H,C)\right) = \left((F,A) \tilde{\cup} (G,B)\right) \Cap ((F,A) \tilde{\cup} (H,C))$,
10) $(F,A) \tilde{\cap} \left((G,B) \tilde{\cup} (H,C)\right) \tilde{\subseteq} \left((F,A) \tilde{\cap} (G,B)\right) \tilde{\cup} ((F,A) \tilde{\cap} (H,C))$,
11) $(F,A) \tilde{\cap} \left((G,B) \Cup (H,C)\right) = \left((F,A) \tilde{\cap} (G,B)\right) \Cup ((F,A) \tilde{\cap} (H,C))$,

12) $(F,A)\tilde{\cap}\left((G,B)\Cap(H,C)\right)\tilde{\supseteq}\left((F,A)\tilde{\cap}(G,B)\right)\Cap((F,A)\tilde{\cap}(H,C))$.

Proof: We just prove (1), the other parts can be proved in a similar way.

1) For any $e\in A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$,

$$(G\tilde{\cup}H)(e)=\begin{cases}G(e) & if\ e\in B-C\\ H(e) & if\ e\in C-B\\ G(e)\vee H(e) & if\ e\in B\cap C\end{cases}$$

and

$$(F\Cap(G\tilde{\cup}H))(e)=\begin{cases}F(e)\wedge G(e) & if\ e\in A\cap(B-C)\\ F(e)\wedge H(e) & if\ e\in A\cap(C-B)\\ F(e)\wedge(G(e)\vee H(e)) & if\ e\in A\cap(B\cap C)\end{cases}$$

Considering the other side, we have

$$((F\Cap G)\tilde{\cup}(F\Cap H))(e)=\begin{cases}(F\Cap G)(e) & if\ e\in(A\cap B)-(A\cap C)\\ (F\Cap H)(e) & if\ e\in(A\cap C)-(A\cap B)\\ (F\Cap G)(e)\vee(F\Cap H)(e) & if\ e\in(A\cap B)\cap(A\cap C)\end{cases}$$

$$=\begin{cases}F(e)\wedge G(e) & if\ e\in A\cap(B-C)\\ F(e)\wedge H(e) & if\ e\in A\cap(C-B)\\ (F(e)\wedge G(e))\vee(F(e)\wedge H(e)) & if\ e\in A\cap(B\cap C)\end{cases}$$

$$=\begin{cases}F(e)\wedge G(e) & if\ e\in A\cap(B-C)\\ F(e)\wedge H(e) & if\ e\in A\cap(C-B)\\ F(e)\wedge(G(e)\vee H(e)) & if\ e\in A\cap(B\cap C)\end{cases}$$

$$=(F\Cap(G\tilde{\cup}H))(e)$$

Thus

$$(F,A)\Cap\left((G,B)\tilde{\cup}(H,C)\right)=\left((F,A)\Cap(G,B)\right)\tilde{\cup}((F,A)\Cap(H,C)).$$

Similarly right distributive laws are defined. Next example shows that generally equality does not hold in 7, 8, 10 and 12.

Example 1: Let U be the set of houses under consideration and E be the set of parameters,

U = $\{h_1, h_2, h_3, h_4, h_5\}$
E = $\{e_1, e_2, e_3, e_4, e_5\}$ = {beautiful, wooden, cheap, in good repair, furnished}
Suppose that A = $\{e_1, e_2, e_3\}$, B = $\{e_2, e_3, e_4\}$ and C = $\{e_3, e_4, e_5\}$
Let (F,A) and (G,B) and (H,C) be the fuzzy soft sets over U defined by the Table 1-3:

Table 1: (F,A)

F	e_1	e_2	e_3

h_1	0.1	0.7	0.3
h_2	0.2	0.9	0.7
h_3	0.3	0.2	0.5
h_4	0.7	0.4	0.2
h_5	0.4	0.1	0.6

Table 2: (G,B)

G	e_1	e_2	e_3
h_1	0.3	0.6	0.4
h_2	0.7	1.0	0.2
h_3	0.6	0.3	0.7
h_4	0.9	0.2	0.8
h_5	0.1	0.5	0.7

Table 3: (H,C)

H	e_3	e_4	e_5
h_1	0.7	0.5	0.7
h_2	0.8	0.5	0.8
h_3	0.5	0.3	0.2
h_4	0.4	0.1	0.3
h_5	0.4	0.4	0.9

Then

$F\,\tilde{\cup}\,(G \Cup H)$	e_1	e_2	e_3	e_4
h_1	0.1	0.7	0.7	0.5
h_2	0.2	0.9	1.0	0.3
h_3	0.3	0.2	0.5	0.7
h_4	0.7	0.4	0.4	0.8
h_5	0.4	0.1	0.6	0.7

and

$(F\,\tilde{\cup}\,G) \Cup (F\,\tilde{\cup}\,H)$	e_1	e_2	e_3	e_4
h_1	0.1	0.7	0.7	0.5
h_2	0.2	0.9	1.0	0.3
h_3	0.3	0.6	0.5	0.7
h_4	0.7	0.9	0.4	0.8
h_5	0.4	0.1	0.6	0.7

From these tables, we see that

$$(F\,\tilde{\cup}\,(G \Cup H))(e_2) \subset ((F\,\tilde{\cup}\,G) \Cup (F\,\tilde{\cup}\,H))(e_2).$$

Thus

$$((F,A)\tilde{\cup}(G,B)) \Cup \left(H,C\right) \not\simeq ((F,A)\,\tilde{\cup}\,(G,B)) \Cup ((F,A)\,\tilde{\cup}(\,H,C)).$$

Now

$F\tilde{\cap}(G\Cap H)$	e_1	e_2	e_3	e_4
h_1	0.1	0.7	0.3	0.4
h_2	0.2	0.9	0.7	0.2
h_3	0.3	0.2	0.3	0.2
h_4	0.7	0.4	0.2	0.1
h_5	0.4	0.1	0.4	0.4

and

$(F\tilde{\cap}G)\Cap(F\tilde{\cap}H)$	e_1	e_2	e_3	e_4
h_1	0.1	0.3	0.3	0.4
h_2	0.2	0.7	0.7	0.2
h_3	0.3	0.2	0.3	0.2
h_4	0.7	0.4	0.2	0.1
h_5	0.4	0.1	0.4	0.4

.

From the above tables we see that

$$(F\tilde{\cap}(G\Cap H))(e_2)\subset((F\tilde{\cap}G)\Cap(F\tilde{\cap}H))(e_2).$$

Thus

$$((F,A)\tilde{\cap}(G,B))\Cap(H,C)\not\simeq((F,A)\tilde{\cap}(G,B))\Cap((F,A)\tilde{\cap}(H,C)).$$

Similarly, it can be shown by computations that

$$(F,A)\tilde{\cup}((G,B)\tilde{\cap}(H,C))\neq((F,A)\tilde{\cup}(G,B))\tilde{\cap}((F,A)\tilde{\cup}(H,C))$$
$$(F,A)\tilde{\cap}((G,B)\tilde{\cup}(H,C))\neq((F,A)\tilde{\cap}(G,B))\tilde{\cup}((F,A)\tilde{\cap}(H,C))$$

In the next propositions, we give a necessary and sufficient condition that will ensure the equality.

Proposition 4: Let (F,A) and (G,B) and (H,C) are three fuzzy soft sets over U. Then

1) $(F,A)\tilde{\cup}((G,B)\tilde{\cap}(H,C))=((F,A)\tilde{\cup}(G,B))\tilde{\cap}((F,A)\tilde{\cup}(H,C))$ and $(F,A)\tilde{\cap}((G,B)\Cap(H,C))=((F,A)\tilde{\cap}(G,B))\Cap((F,A)\tilde{\cap}(H,C))$ if and only if $F(e)\subseteq G(e)$ for all $e\in(A\cap B)-C$ and $F(e)\subseteq H(e)$ for all $e\in(A\cap C)-B$.
2) $(F,A)\tilde{\cap}((G,B)\tilde{\cup}(H,C))=((F,A)\tilde{\cap}(G,B))\tilde{\cup}((F,A)\tilde{\cap}(H,C))$ and $(F,A)\tilde{\cup}((G,B)\Cup(H,C))=((F,A)\tilde{\cup}(G,B))\Cup((F,A)\tilde{\cup}(H,C))$ if and only if $F(e)\supseteq G(e)$ for

all $e\in(A\cap B)-C$ and $F(e)\supseteq H(e)$ for all $e\in(A\cap C)-B$.

Proof: For any $e\in A\cup(B\cup C)=(A\cap B)\cup(A\cap C)$

$$(F\tilde{\cup}(G\tilde{\cap}H))(e)=\begin{cases}F(e) & if\ e\in A-(B\cup C)\\ G(e) & if\ e\in B-(A\cup C)\\ H(e) & if\ e\in C-(A\cup B)\\ F(e)\vee G(e) & if\ e\in(A\cap B)-C\\ F(e)\vee H(e) & if\ e\in(A\cap C)-B\\ G(e)\wedge H(e) & if\ e\in(B\cap C)-A\\ F(e)\vee(G(e)\wedge H(e)) & if\ e\in A\cap(B\cap C)\end{cases}$$

$$=\begin{cases} F(e) & if\ e\in A-(B\cup C) \\ G(e) & if\ e\in B-(A\cup C) \\ H(e) & if\ e\in C-(A\cup B) \\ F(e) & if\ e\in (A\cap B)-C \\ F(e) & if\ e\in (A\cap C)-B \\ G(e)\wedge H(e) & if\ e\in (B\cap C)-A \\ F(e)\vee (G(e)\wedge H(e)) & if\ e\in A\cap (B\cap C) \end{cases}$$

$$=\begin{cases} F(e) & if\ e\in A-(B\cap C) \\ G(e) & if\ e\in B-(A\cup C) \\ H(e) & if\ e\in C-(A\cup B) \\ G(e)\wedge H(e) & if\ e\in (B\cap C)-A \\ F(e)\vee (G(e)\wedge H(e)) & if\ e\in A\cap (B\cap C) \end{cases}$$

Considering the other side, we have

$$((F\tilde{\cup} G)\tilde{\cap}(F\tilde{\cup} H))(e)=\begin{cases} F(e)\wedge F(e) & if\ e\in A-(B\cup C) \\ G(e) & if\ e\in B-(A\cup C) \\ H(e) & if\ e\in C-(A\cup B) \\ (F(e)\vee G(e))\wedge F(e) & if\ e\in (A\cap B)-C \\ F(e)\wedge (F(e)\vee H(e)) & if\ e\in (A\cap C)-B \\ G(e)\wedge H(e) & if\ e\in (B\cap C)-A \\ (F(e)\vee G(e))\wedge (F(e)\vee H(e)) & if\ e\in A\cap (B\cap C) \end{cases}$$

$$=\begin{cases} F(e) & if\ e\in A-(B\cup C) \\ G(e) & if\ e\in B-(A\cup C) \\ H(e) & if\ e\in C-(A\cup B) \\ F(e) & if\ e\in (A\cap B)-C \\ F(e) & if\ e\in (A\cap C)-B \\ G(e)\wedge H(e) & if\ e\in (B\cap C)-A \\ (F(e)\vee G(e))\wedge (F(e)\vee H(e)) & if\ e\in A\cap (B\cap C) \end{cases}$$

$$=(F\tilde{\cup}(G\tilde{\cap} H))(e)$$

Thus

$$(F,A)\tilde{\cup}((G,B)\tilde{\cap}(H,C))=((F,A)\tilde{\cup}(G,B))\tilde{\cap}((F,A)\tilde{\cup}(H,C))$$

Conversely if

$$(F,A)\tilde{\cup}((G,B)\tilde{\cap}(H,C))=((F,A)\tilde{\cup}(G,B))\tilde{\cap}((F,A)\tilde{\cup}(H,C))$$

then for any

$$e\in (A\cap B)-C, F(e)\vee G(e)=(F(e)\vee G(e))\wedge F(e)\Rightarrow$$

$$F(e)\vee G(e)\subseteq F(e)\subseteq F(e)\vee G(e)\Rightarrow F(e)=F(e)\vee G(e)\Rightarrow G(e)\subseteq F(e)$$

Similarly we can prove that $F(e)\subseteq H(e)$ for all $e\in(A\cap C)-B.$
Remaining parts can be proved in a similar way.

Corollary 1: Let (F,A) and (G,B) and (H,C) are three fuzzy soft sets over U. Then

1) $(F,A)\tilde{\cup}((G,B)\tilde{\cap}(H,C))=((F,A)\tilde{\cup}(G,B))\tilde{\cap}((F,A)\tilde{\cup}(H,C))$
2) $(F,A)\tilde{\cap}((G,B)\tilde{\cup}(H,C))=((F,A)\tilde{\cap}(G,B))\tilde{\cup}((F,A)\tilde{\cap}(H,C))$

if and only if F(e) = G(e) for all $e\in(A\cap B)-C$ and F(e) = H(e) for all $e\in(A\cap C)-B.$

Corollary 2: Let (F,A) and (G,B) and (H,C) are three fuzzy soft sets over U, such that $(A\cap B)-C=(A\cap C)-B=\varnothing$ (Empty Set) then
1) $(F,A)\tilde{\cup}((G,B)\tilde{\cap}(H,C))=((F,A)\tilde{\cup}(G,B))\tilde{\cap}((F,A)\tilde{\cup}(H,C))$,
2) $(F,A)\tilde{\cap}((G,B)\tilde{\cup}(H,C))=((F,A)\tilde{\cap}(G,B))\tilde{\cup}((F,A)\tilde{\cap}(H,C))$.

Corollary 3: Let (F,A) and (G,A) and (H,A) are three fuzzy soft sets over U. Then

$$(F,A)\boldsymbol{a}((G,A)\boldsymbol{b}(H,A))=((F,A)\boldsymbol{a}(G,A))\boldsymbol{b}((F,A)\boldsymbol{a}(H,A))$$

for distinct $\boldsymbol{a},\boldsymbol{b}\in\{\tilde{\cup},\Cup,\tilde{\cap},\Cap\}$.

Proposition 5: Let (F,A) be a fuzzy soft set over the universe set U. Then

1) "Idempotent Law" (F,A) α(F,A) = (F,A), for all $\boldsymbol{a}\in\{\tilde{\cup},\Cup,\tilde{\cap},\Cap\}$
2) $(F,A)\boldsymbol{a}\Phi_A=\Phi_A$, for all $\boldsymbol{a}\in\{\tilde{\cap},\Cap\}$
3) $(F,A)\boldsymbol{a}\Phi_A=(F,A)$, for all $\boldsymbol{a}\in\{\tilde{\cup},\Cup\}$
4) (F,A)αU_E = (F,A), for all $\boldsymbol{a}\in\{\tilde{\cap},\Cap\}$
5) (F,A)αU_E = U_E, for all $\boldsymbol{a}\in\{\tilde{\cup},\Cup\}$.

Proof: Straightforward.

Proposition 6: "Least upper and greatest lower bounds" Let (F,A) and (G,B) be two fuzzy soft sets over a common universe U. Then the following are true:

1) $(F,A)\tilde{\cup}(G,B)$ is the smallest fuzzy soft set over U which contains both (F,A) and (G,B),
2) $(F,A)\Cap(G,B)$ is the largest fuzzy soft set over U which is contained in both (F,A) and (G,B).

Proof

1) $(F,A)\tilde{\subseteq}(F,A)\tilde{\cup}(G,B)$ and $(G,B)\tilde{\subseteq}(F,A)\tilde{\cup}(G,B)$, because $A\subseteq A\cup B$, $B\subseteq A\cup B$ and $F(e)\subseteq F(e)\vee G(e)$, $G(e)\subseteq F(e)\vee G(e)$. Let (H,C) be any fuzzy soft set over U, such that $(F,A)\tilde{\subseteq}(H,C)$ and $(G,B)\tilde{\subseteq}(H,C)$. Then $A\cup B\subseteq C$ and $F(e)\subseteq H(e)$, for all e$\in$A, $G(e)\subseteq H(e)$ for all e$\in$B implies that $(F\tilde{\cup}G)(e)\subseteq H(e)$ for all $e\in A\cup B$. Thus $(F,A)\tilde{\cup}(G,B)\tilde{\subseteq}(H,C)$.

2) $\left(F,A\right)\Cap\left(G,B\right)\tilde{\subset}(F,A)$ and $\left(F,A\right)\Cap\left(G,B\right)\tilde{\subset}(G,B)$, because $A\cap B\subseteq A$, $A\cap B\subseteq B$ and $F(e)\wedge G(e)\subseteq F(e)$, $F(e)\wedge G(e)\subseteq G(e)$ for all $e\in A\cap B$. Let (H,C) be any fuzzy soft set over U, such that and $(H,C)\tilde{\subseteq}(G,B)$. Then $C\subseteq A\cap B$ and $H(e)\subseteq F(e)$, $H(e)\subseteq G(e)$ for all e$\in$C implies that $H(e)\subseteq F(e)\wedge G(e)=(F\Cap G)(e)$ for all e$\in$C. Thus $(H,C)\tilde{\subseteq}(F,A)\Cap(G,B)$.

Proposition 7: Let (F,A) and (G,B) be two fuzzy soft sets over a common universe U. Then,

$$(F,A)\Cap(G,A)=(F,A)\tilde{\cap}(G,A),\ (F,A)\Cup(G,A)=(F,A)\tilde{\cup}(G,A).$$

Proof: Straightforward.

Proposition 8: "Commutative Laws" Let (F,A) and (G,B) be two fuzzy soft sets over U. Then

$$(F,A)\,\boldsymbol{a}\,(G,B)=(G,B)\,\boldsymbol{a}\,(F,A),\ \text{for all}\ \ \boldsymbol{a}\in\{\tilde{\cup},\Cup,\tilde{\cap},\Cap\}$$

Proof: Straightforward.

Proposition 9: "Absorption Laws" If (F,A) and (G,B) are two fuzzy soft sets over U. Then

$$(F,A)\,\tilde{\cap}\,((F,A)\Cup(G,B))=(F,A)$$
$$(F,A)\Cup((F,A)\,\tilde{\cap}\,(G,B))=(F,A)$$
$$(F,A)\Cap((F,A)\,\tilde{\cup}\,(G,B))=(F,A)$$
$$(F,A)\,\tilde{\cup}\,((F,A)\Cap(G,B))=(F,A)$$

Proof: For any e$\in$A,

$$(F\,\tilde{\cap}\,(F\Cup G))(e)=\begin{cases} F(e) & \text{if } e\in A-(A\cap B)\\ (F\Cup G)(e) & \text{if } e\in (A\cap B)-A\\ F(e)\wedge(F\Cup G)(e) & \text{if } e\in A\cap(A\cap B)\end{cases}$$
$$=\begin{cases} F(e) & \text{if } e\in A-(A\cap B)\\ F(e)\wedge(F(e)\vee G(e)) & \text{if } e\in A\cap B\end{cases}$$
$$=\begin{cases} F(e) & \text{if } e\in A-(A\cap B)\\ F(e) & \text{if } e\in A\cap B\end{cases}$$
$$=F(e)$$

The remaining parts can also be proved similarly.

Definition 17: A Semigroup (S,*) is a non-empty set with an associative binary operation *. Usually we write xy instead of x*y. If there exists an element e$\in$S such that ex = xe = x for all x in S then we say that

S is a monoid with identity element e. An element $x \in S$ is called idempotent if xx = x. We say that S is idempotent, if every element of S is idempotent.

Definition 18: A Semiring $(R,+,\cdot)$ is an algebra consisting of a non-empty set R together with two binary operations called addition and multiplication (denoted in usual manner) such that the following conditions hold:

1) $(R,+,\cdot)$ is a commutative monoid with additive identity 0
2) $(R,\cdot)$is a monoid with additive identity 1
3) Multiplication is distributive over addition from either side,

$$a\cdot(b+c)=a\cdot b+a\cdot c \text{ and } (b+c)\cdot a=b\cdot a+c\cdot a \text{ for all } a,b,c\in R$$

4) 0.a = a.0 = 0 for all $a\in R$.

We summarize the above results by giving the following observations about the algebraic structures of fuzzy soft sets:

1) $(\mathbf{FSS}(U)^E, \Cup)$ is a commutative monoid with identity Φ_E, $(\mathbf{FSS}(U)_A, \Cup)$ is its subsemigroup with identity Φ_A,
2) $(\mathbf{FSS}(U)^E, \Cap)$ is a commutative monoid with identity U_E, $(\mathbf{FSS}(U)_A, \Cap))$ is its subsemigroup with identity U_A,
3) $(\mathbf{FSS}(U)^E, \tilde{\cap})$ is a commutative monoid with identity $\Phi_?$, $(\mathbf{FSS}(U)_A, \tilde{\cap})$ is its subsemigroup with identity U_A,
4) $(\mathbf{FSS}(U)^E, \tilde{\cap})$ is a commutative monoid with identity $\Phi_?$, $(\mathbf{FSS}(U)_A, \tilde{\cup})$ is its subsemigroup with identity Φ_A.

We see the following semiring structures:

1) $(\mathbf{FSS}(U)^E, \tilde{\cap}, \Cup)$ is a commutative, idempotent semiring with identity element Φ_E because $(F,A)\Cup\Phi_E=(F,A)$, for every (F,A) in $FSS(U)^E$,
2) $(\mathbf{FSS}(U)^E, \tilde{\cup}, \Cap)$ is a commutative, idempotent semiring with identity element U_E because $(F,A)\Cap U_E=(F,A)$, for every (F,A) in $FSS(U)^E$,
3) $(\mathbf{FSS}(U)^E, \Cap, \tilde{\cup})$ is a commutative, idempotent semiring with identity element $\Phi_?$ because $(F,A)\tilde{\cup}\Phi_f=(F,A)$, for every (F,A) in $FSS(U)^E$,
4) $(\mathbf{FSS}(U)^E, \Cap, \Cup)$ is a commutative, idempotent semiring with identity element Φ_E because $(F,A)\Cup\Phi_E=(F,A)$, for every (F,A) in $FSS(U)^E$,

5) $(\mathbf{FSS}(U)^E, \Cap, \tilde{\cap})$ is a commutative, idempotent semiring with identity element $\Phi_?$ because $(F,A)\,\tilde{\cap}\,\Phi_f = (F,A)$, for every (F,A) in $FSS(U)^E$,

6) $(\mathbf{FSS}(U)^E, \Cup, \tilde{\cup})$ is a commutative, idempotent semiring with identity element $\Phi_?$ because $(F,A)\,\tilde{\cup}\,\Phi_f = (F,A)$, for every (F,A) in $FSS(U)^E$,

7) $(\mathbf{FSS}(U)^E, \Cup, \tilde{\cap})$ is a commutative, idempotent semiring with identity element $\Phi_?$ because $(F,A)\,\tilde{\cap}\,\Phi_f = (F,A)$, for every (F,A) in $FSS(U)^E$,

8) $(\mathbf{FSS}(U)^E, \Cup, \Cap)$ is a commutative, idempotent semiring with identity element U_E because $(F,A) \Cap U_E = (F,A)$, for every (F,A) in $FSS(U)^E$.

Proposition 10: $(\mathbf{FSS}(U)^E, \tilde{\cap}, \Cup)$, $(\mathbf{FSS}(U)^E, \Cup, \tilde{\cap})$, $(\mathbf{FSS}(U)^E, \tilde{\cup}, \Cap)$ and $(\mathbf{FSS}(U)^E, \Cap, \tilde{\cup})$ are distributive lattices.

Proof: Straightforward.

Proposition 11: $(\mathbf{FSS}(U)^E, \Cap, \tilde{\cup})$ is a bounded distributive lattice, with least element $\Phi_?$ and greatest element U_E while $(\mathbf{FSS}(U)^E, \tilde{\cup}, \Cap, U_E, \Phi_f)$ is its dual.

Proof: Straightforward.

Proposition 12: $(\mathbf{FSS}(U)_A, \Cap, \tilde{\cup}) = (\mathbf{FSS}(U)_A, \tilde{\cap}, \Cup)$ is a bounded distributive lattice, with least element Φ_A and greatest element U_A.

Proposition 13: "Involution" Let (F,A) and (G,B) be two fuzzy soft sets over U. Then

1) $((F,A)')' = (F,A)$
2) $(F,A) \tilde{\subseteq} (G,A)$ implies $(G,A)' \tilde{\subseteq} ((F,A)'$

Proof

1) ((F,A)′)′ = ((F′,A))′ = ((F′)′,A) = ((F′)′,A)

Now,

$$((F')'(e))(x) = (\mathbf{1} - F'(e))(x) = 1 - (F'(e))(x) = 1 - (\mathbf{1} - F(e))(x) = 1 - 1 + (F(e))(x)$$
$$= 1 - 1 + (F(e))(x) = (F(e))(x)$$

for all $e \in A$, $x \in U$. Thus ((F,A)′)′ = (F,A)

2) If $(F,A) \tilde{\subseteq} (G,A)$ then

$$(F(e))(x) \leq (G(e))(x) \text{ and so } 1-(G(e))(x) \leq 1-(F(e))(x)$$

which gives $(G'(e))(x) \leq (F'(e))(x)$ for all $e \in A, x \in U$.

Hence $(G,A)' \tilde{\subseteq} ((F,A)'$.
Thus $'$ is an involution on $FSS(U)_A$.

Proposition 14: "De Morgan's Laws" Let (F,A) and (G,B) be two fuzzy soft sets over U. Then the following are true

1) $((F,A) \Cup (G,B))' = (F,A)' \Cap (G,B)'$,
2) $((F,A) \Cap (G,B))' = (F,A)' \Cup (G,B)'$,
3) $((F,A) \tilde{\cup} (G,B))' = (F,A)' \tilde{\cap} (G,B)'$,
4) $((F,A) \tilde{\cap} (G,B))' = (F,A)' \tilde{\cup} (G,B)'$,
5) $((F,A) \vee (G,B))' = (F,A)' \wedge (G,B)'$,
6) $((F,A) \wedge (G,B))' = (F,A)' \vee (G,B)'$.

Proof: Straightforward.

Definition 19: If De Morgan's laws hold for a bounded distributive lattice having an involution, then it is called a De Morgan algebra.

Proposition 15: $(\mathbf{FSS}(U)_A, \Cup, \Cap, ', \Phi_A, U_A)$ is a De Morgan algebra.
Proof: Straightforward.

Definition 20: A De Morgan's algebra $(L, \wedge, \vee, ', 0, 1)$ that satisfies $x \wedge x' \leq y \vee y'$ for all x, y∈ L, is called a Kleene algebra.

Proposition 16: $(\mathbf{FSS}(U)_A, \Cup, \Cap, ', \Phi_A, U_A)$ is a Kleene algebra.
Proof: We have already seen that $(\mathbf{FSS}(U)_A, \Cup, \Cap, ', \Phi_A, U_A)$ is a De Morgan algebra. Now, for any (F,A), $(G,A) \in \mathbf{FSS}(U)_A$, if we have

$$(F,A) \Cap (F,A)' \tilde{\supset} (G,A) \Cup (G,A)'$$

where $(F,A) \Cap (F',A) \neq (G,A) \Cup (G,A)'$.

Then there exists some e∈ A such that $(F \Cap F')(e) \supset (G \Cup G')(e)$ and so there exists some x∈ U such that

$$((F \Cap F')(e))(x) > ((G \Cup G')(e))(x) \text{ or } (F(e) \wedge F'(e))(x) > (G(e) \vee G'(e))(x)$$

or

$$(F(e))(x) \wedge (F'(e))(x) > (G(e))(x) \vee (G'(e))(x).$$

But $(F(e))(x) \wedge (F'(e))(x) \leq 0.5$ and $(G(e))(x) \vee (G'(e))(x) \geq 0.5$ which gives

$$(F(e))(x) \wedge (F'(e))(x) \leq (G(e))(x) \vee (G'(e))(x).$$

A contradiction, thus our supposition is wrong. Hence

$$(F,A) \Cap (F,A)' \tilde{\subset} (G,A) \Cup (G,A)'.$$

Therefore $(\mathbf{FSS}(U)_A, \Cup, \Cap, ', \Phi_A, U_A)$ is a Kleene algebra.

Definition 21: Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice and x∈L. Then an element x^* is called a pseudo-complement of x, if $x \wedge x^*$ and y?x^* whenever x∧y = 0. If every element has a pseudo-complement then L is pseudo-complemented. The equation $x^* \vee x^{**} = 1$ is called Stone's identity. A Stone algebra is a pseudo-complemented, distributive lattice satisfying Stone's identity.

Definition 22: For a fuzzy soft set (F,A) over U, we define a fuzzy soft set over U, which is denoted by $(F, A)^*$ and is given by $(F, A)^* = (F^*, A)$ where

$$(F^*(e))(x) = \begin{cases} 0 & \text{if } (F(e))(x) \neq 0 \\ 1 & \text{if } (F(e))(x) = 0 \end{cases}$$

for all x∈U, e∈A.

Theorem 3: Let (F,A) be a fuzzy soft set over U. Then the following are true:

1) $(F,A) \Cap (F,A)^* = \Phi_A$,
2) $(G,A) \tilde{\subseteq} (F,A)^*$ whenever $(F,A) \Cap (G,A) = \Phi_A$,
3) $(F,A)^* \Cup (F,A)^{**} = U_A$.

Thus $(\mathbf{FSS}(U)_A, \Cup, \Cap, ^*, \Phi_A, U_A)$ is a Stone algebra.

Proof:

1) Straightforward.
2) If $(F,A) \Cap (G,A) = \Phi_A$. Then for any x∈U, e∈A,
   if $(G(e))(x) = 0$ then $(G(e))(x) \leq (F^*(e))(x)$
   If $(G(e))(x) \neq 0$ then $(F(e))(x) \wedge (G(e))(x) = 0$
   Implies that $(F(e))(x) = 0$, so $(F^*(e))(x) = 1$
   and hence $(G(e))(x) \leq 1 = (F^*(e))(x)$

Thus, $(G(e))(x) \leq (F^*(e))(x)$ for all $x \in U$, $e \in A$. i.e. $(G,A) \tilde{\subseteq} (F,A)^*$.

3) For any x∈U, e∈A,

$$((F^{*} \Cup F^{**})(e))(x) = (F^{*}(e) \vee F^{**}(e))(x)$$
$$= \max\{(F^{*}(e))(x), (F^{**}(e))(x)\}$$
$$= \begin{cases} \max\{1,0\} & \text{if } (F(e))(x) \neq 0 \\ \max\{0,1\} & \text{if } (F(e))(x) = 0 \end{cases} = 1$$

Thus $(F, A)^{*} \Cup (F, A)^{**} = U_A$ and so, $(\mathbf{FSS}(U)_A, \Cup, \Cap, ^{*}, \Phi_A, U_A)$ is a Stone algebra.

CONCLUSION

We see that different algebraic structures of fuzzy soft sets have their own properties which are different from those of fuzzy sets. Further research may be done to explore the applications of these properties in the models developed by using fuzzy soft sets. Fuzzy sets and soft sets are studied theoretically in a combined way and therefore, we can use fuzzy soft sets to solve the problems involving particular sets of parameters and fuzziness in data.

REFERENCES

1. Zadeh, L.A., 1965. Fuzzy sets. Inform. and Control, 8: 338-353.
2. Pawlak, Z., 1982. Rough sets. Internat. J. Inform. Comput. Sci., 11: 341-356.
3. Atanassov, K., 1986. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20: 87-96.
4. Gau, W.L. and D.J. Buehrer, 1993. Vague sets. IEEE Trans. System, Man and Cybernet., 23 (2): 610-614.
5. Molodtsov, D., 1999. Soft set theory first results. Comput. Math. Appl., 37: 19-31.
6. Maji, P.K., R. Biswas and A.R. Roy, 2003. Soft set theory, Computers and Math. with Appl., 45: 555-562.
7. Maji, P.K., R. Biswas and A.R. Roy, 2001. Fuzzy soft sets. The J. Fuzzy Math., 9: 589-602.
8. Maji, P.K., A.R. Roy and R. Biswas, 2002. An application of soft sets in a decision making problem. Comput. Math. Appl., 44: 1077-1083.
9. Roy, A.R. and P.K. Maji, 2007. A fuzzy soft set theoretic approach to decision making problems. J. of Comp. and App. Math., 203: 412-418.
10. Ali, M.I., F. Feng, X.Y. Liu, W.K. Min and M. Shabir, 2008. On some new operations in soft set theory. Comput. Math. Appl., pp: 2621-2628.
11. Ali, M.I., M. Shabir and M. Naz, 2011. Algebraic structures of soft sets associated with new operations. Comput. Math. Appl., 61: 2647-2654.
12. Feng, F., Y.B. Jun and X.Z. Zhao, 2008. Soft semirings, Computers and Math. with Appl., 56: 2621-2628.
13. Feng, F., C. Li, B. Davvaz and M.I. Ali, 2010. Soft sets combined with fuzzy sets and rough sets: A tentative approach, Soft Computing, 14: 899-911.
14. Kong, Z., L. Gao, L. Wong and S. Li, 2008. The normal parameter reduction of soft sets and its algorithm. Comp. Appl. Math., 56: 3029-3037.
15. Pie, D. and D. Miao, 2005. From soft sets to information systems. Granular computing, IEEE Inter. Conf. 2: 617-621.
16. Shabir, M. and M. Naz, 2011. On soft topological spaces, Comp. and Math. with App., 61: 1786 - 1799.

17. Ali, M.I. and M. Shabir, 2010. Comments on De Morgan's law in fuzzy soft sets. The J. of Fuzzy Math., 18 (3): 679-686.