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Codes in Anti-metric

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Abstract: In this paper, we introduce the concept of anti-metric and anti-weight on the vector space $\mathbf{F}_q^n$, the space of all n-tuples over the finite field $\mathbf{F}_q$ and study the properties of anti-codes which are subspaces of $\mathbf{F}_q^n$ equipped with the anti-metric.

Key words: Anti-metric, Linear Codes

INTRODUCTION

Error control coding basically deals with codes (linear or non linear) endowed with a metric function. The error correction and error detection capabilities of the codes are determined by the minimum distance of the code. In this paper, we introduce the concept of anti-weight and anti-metric on the vector space $\mathbf{F}_q^n$, the space of all n-tuples over the finite field $\mathbf{F}_q$ and study the properties of anti-codes which are subspaces of $\mathbf{F}_q^n$ equipped with the anti-metric. The concept of anti-weight and anti-distance will find applications for communication channels/systems where the noise in the system causes faults/errors near the end of the codeword i.e. system gets stuck up at some position and errors occur after that position. The author is motivated by the problem while trying to store an 8 digit telephone number and each time the retrieved telephone number was shifted one place to the right starting from the t^{th} position ($t \geq 5$), dropping the last digit and introducing a random digit at the t^{th} position due to fault in the system.

Error model. Suppose we are dealing with an n digit telephone number and the noise in the channel shifts the digits one place to the right beginning from some fixed t^{th} position and dropping the last digit and randomly introducing some digit at the t^{th} position. Such type of errors are called *cyclic errors of order* t. For example, cyclic error of order 3 applied on the vector $(12043) \in \mathbf{F}_5^5$ gives $(12b04)$ where $b \in \mathbf{F}_5$ is a randomly introduced digit at the 3rd position. The concept of anti-weight and anti-metric will be useful in the correction and detection of cyclic errors of order t.

2. ANTI-WEIGHT, ANTI-METRIC AND ANTI-CODES

In this section, we define anti-weight, anti-distance, anti-codes in the vector space $\mathbf{F}_q^n$ and then discuss basic results of anti-codes. We begin with the definition of anti-weight of a vector $v \in \mathbf{F}_q^n$.

Definition 1. The anti-weight $T_w(v)$ of a vector $v = (v_1, v_2, \cdots, v_n) \in \mathbf{F}_q^n$ is defined as

$$T_w(v) = \max_{i=1}^{n}\{i | v_i = 0\}.$$

Definition 2. The anti-distance $T_d : \mathbf{F}_q^n \times \mathbf{F}_q^n \to \{0, 1, 2 \cdots, n\}$ is defined as

$$T_d(x, y) = T_w(x - y) \quad \text{for all } x, y \in \mathbf{F}_q^n.$$

The anti-distance T_d satisfies the following properties:

(i) $T_d(x, y) \geq 0 \; \forall \; x, y \in \mathbf{F}_q^n$.

(ii) $T_d(x, y) = T(y, x) \; \forall \; x, y \in \mathbf{F}_q^n$.

(iii) $T_d(x, y) =$ maximum $= n$ iff $x = y$ and $T_d(x, y) =$ minimum $= 0$ iff $x_1 \neq y_1 \; \forall \; x = (x_1, \cdots x_n)$ and $y = (y_1, \cdots y_n) \in \mathbf{F}_q^n$.

(iv) $T_d(x, y) \geq T_d(x, z) + T(z, y) - n \; \forall \; x, y, z \in \mathbf{F}_q^n$

Remark 3. The anti-weight T_w is related to the RT-weight ρ [1] by the following relation:

$$T_w(x) = n - \rho(x_R)\ \forall\, x \in \mathbf{F}_q^n$$

where x_R is the reverse vector of x i.e. if $x = (x_1, \cdots, x_n)$ then $x_R = (x_n, x_{n-1}, \cdots x_1)$ and $\rho(x_R)$ is the RT-weight of x_R.

From now onwards, we will use the same symbol T for both T_w and T_d and call it as T-anti-weight- and T-anti-metric respectively.

Definition 4 (Anti-spheres.) Anti-spheres of radius r in $\mathbf{F}_q^n$ centered at $x \in \mathbf{F}_q^n$ are defined as

$$\begin{aligned} S_r^A(x) &= \{y \in \mathbf{F}_q^n | T(x,y) \geq r\} \\ &= \{y \in \mathbf{F}_q^n | r \leq T(x,y) \leq \\ &\quad \leq n\}. \end{aligned}$$

Definition 5 (Anti-codes). A T-anti-code or simply an anti-code V is a k-dimensional subspace of $\mathbf{F}_q^n$ equipped with the T-anti-metric.

Definition 6. The maximum anti-weight and maximum anti-distance of an anti-code V are defined as

$$T_w^{\max}(V) = \max_{\substack{x \in V \\ x \neq 0}} (T_w(x))$$

and

$$T_d^{\max}(V) = \max_{\substack{x,y \in V \\ x \neq y}} (T_d(x,y))$$

Theorem 7. *The maximum anti-weight and maximum anti-distance of an anti-code V coincide.*

Proof. Suppose the maximum anti-weight $(V) = w$.

Let $u, v \in V$ and $u \neq v$. Then

$$\begin{aligned} T(u,v) = T(u-v) &\leq w \\ \Rightarrow \max_{\substack{u,v \in V \\ u \neq v}} (T(u,v) &\leq w. \end{aligned} \tag{1}$$

Again, maximum anti-wight $(V) = w$ implies there exists a nonzero codeword say x such that $T(x) = w$. Also $\overline{0} = (0, 0, \cdots, 0) \in V$. Now

$$\begin{aligned} T(\overline{0}, x) &= T(x - \overline{0}) = T(x) = w \\ &\Rightarrow \max_{\substack{u,v \in V \\ u \neq v}} (T(u,v) \geq w. \end{aligned} \tag{2}$$

(1) and (2) prove the result. □

Theorem 8. *Let $t \geq \left(\dfrac{n+1}{2}\right)$ be a positive integer. If the maximum anti-distance of an $[n,k]$ anti-code V is at most $2t - n - 1$, then the anti-code V corrects all errors of anti-weight t or more.*

Proof. It suffices to show that the anti-spheres of radius t centered at codewords are all disjoint. If not, then there exists $x, y \in V, x \neq y$ and $z \in \mathbf{F}_q^n$ such that

$$z \in S_t^A(x) \cap S_t^A(y).$$

This gives

$$\begin{aligned} & T(x,z) \geq t \quad \text{and} \quad T(y,z) \geq t \\ \Rightarrow\ & T(x,z) + T(y,z) - n \geq 2t - n \\ \Rightarrow\ & T(x,y) \geq 2t - n \\ \Rightarrow\ & \max_{\substack{u,v \in V \\ u \neq v}} T(u,v) \geq 2t - n. \end{aligned} \tag{3}$$

(3) gives a contradiction. □

Remark 9(i). Theorem 8 also states that *If the maximum anti-distance of an $[n,k]$ anti-code V is at most T_{max}, then the anti-code V can correct all errors of anti-weight t or more where*

$$t \geq \frac{T_{max} + n + 1}{2}.$$

(ii)The condition in Theorem 8 or in Remark 9(i) is only a sufficient condition. It is not a necessary condition as seen from Example 14 in the next section.

3. STANDARD ANTI-ARRAY AND HAMMING ANTI-SPHERE PACKING BOUND

The standard anti-array for an $[n,k]$ anti-code V is the same as the standard array used in normal coding with coset leaders being replaced by anti-coset leaders where the anti-coset leaders are vectors of maximum anti-weights in their respective cosets and farthest neighbor decoding principle will be used for decoding purpose i.e. we find the code vector whose anti-distance from the received vector is maximum.

To obtain the Hamming anti-sphere packing bound, we enumerate the number of all vectors of anti-weight t or more in $\mathbf{F}_q^n$ and this number is given by

$$\begin{aligned} |S_t^A(0)| = V_{t,q} &= 1 + \sum_{i=0}^{n-t-1} (q-1)q^i \\ &= q^{n-t}. \end{aligned}$$

Theorem 10 (The Hamming anti-sphere packing Bound). *To correct all errors of anti-weight t or more, an $[n,k]$ anti-code V must have at least $(n-t)$ parity check digits or equivalently at most t information digits.*

Proof. The proof follows from the fact that the number of correctable errors of anti-weights t or more is q^{n-t} and the number of available cosets is q^{n-k} and hence

$$q^{n-k} \geq q^{n-t}.$$

□

Definition 11 (t-perfect anti-code.) An $[n, k]$ anti-code satisfying

$$q^{n-k} = q^{n-t}$$

or equivalently

$$k = t$$

is called a t-perfect anti-code.

Example 12. Let $q = 2, n = 4, t = 2, k = 2$. Let

$$V = \{(0000), (1100), (0100), (1000)\}.$$

Then $T_{max}(V) = 1$. The generator and parity check matrices for V are

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}_{2\times 4},$$

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{2\times 4}.$$

Standard anti-array for anti-code V is given by

0000	1100	0101	1000
0001	1101	0101	1001
0010	1110	0110	1010
0011	1111	0111	1011.

Note that errors vectors of anti-weight $t = 2$ or more belong to distinct cosets of the standard anti-array and hence are correctable and moreover the anti-code V corrects no other error. This is an example of a 2-perfect anti-code of length 4.

4. DECODING ALGORITHM

In this section, we describe two decoding algorithms for the correction of all errors of anti-weight t or more (including all cyclic errors of order t) using standard anti-arrays.

ALGORITHM 1.

Step 1. Let $v = (a_1, a_2, \cdots, a_n) \in \mathbf{F}_q^n$ be the received vector.

Step 2. Find syndrome of v viz. synd(v).

Step 3. Find the anti-coset leader e in the syndrome table(or standard anti-array table) such that synd(v)=synd(e).

Step 4. Subtract the anti-coset leader e from the received vector v to get the transmitted vector u i.e. $u = v - e$.

ALGORITHM 2.

Step 1. Let $v = (a_1, a_2, \cdots, a_n) \in \mathbf{F}_q^n$ be the received vector.

Step 2. Locate the position of v in the standard anti-array. Let it be row i with anti-coset leader e.

Step 3. Subtract e from the received vector v to get the transmitted vector u i.e. $u = v - e$.

Example 13. Consider the anti-code V in Example 12 Suppose $v = (1110)$ is the received vector. We compute synd$(v) = (10) =$ synd(e) where $e = (0010)$. $T(e) = 2$ implies that errors begin from the third position. Subtract e from v to get the transmitted vector $u = (1100)$.

Example 14. Let $q = 2, n = 4, k = 2$. Let V be an $[4, 2]$ binary anti-code given by

$$V = \{(0000), (1100), (0101), (1001)\}.$$

Then $T_{max}(V) = 1$.

In view of Remark 9(i), the anti-code V corrects all errors of anti-weight

$$\frac{T_{max} + n + 1}{2} = \frac{1 + 4 + 1}{2} = 3 \text{ or more.}$$

But this code also corrects all errors of anti-weight 2 as seen from the following standard anti-array.

0000	1100	0101	1001
0001	1101	0101	1000
0010	1110	0111	1011
0011	1111	0110	1010.

Thus the condition in Theorem 8 or in Remark 9(i) is only a sufficient condition.

The generator and parity check matrix for V are given by

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}_{2\times 4},$$

$$H = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}.$$

Let the transmitted vector be $u = (0101)$ and a cyclic error of order 2 occurs where the second bit is shifted to the third position and 3^{rd} bit is shifted to the 4^{th} place and a randomly chosen bit say 1 is introduced at the second position. So the received vector is $v = (0110)$. We locate the position of the received vector v in the standard anti-array. It lies in fourth row with anti-coset leader $e = (0011)$ where $T(e) = 2$. Subtract e from v to get the transmitted vector $u = (0101)$.

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