World Applied Sciences Journal 22 (Special Issue of Applied Math): 36-40, 2013
ISSN 1818-4952

DOI:10.5829/idosi.wasj.22.am.6.2013

Array codes in m-metric correcting independent and clustered errors simultaneously

Sapna Jain

Department of Mathematics, University of Delhi, Delhi 110 007, India

sapnajain@gmx.com

Abstract: Array codes or two-dimensional codes in m-metric are subsets/subspaces of the space $\text{Mat}_{m\times s}(F_q)$, the linear space of all $m \times s$-matrices with entries from a finite field F_q endowed with the m-metric [15]. In this paper, we obtain a lower bound over the number of parity checks of a two-dimensional m-code that corrects any two-dimensional array which has both clustered errors as well as errors in separate positions (or independent errors).

Key words: Bursts, Linear codes, Array Codes

INTRODUCTION

Burst error correcting m-metric array codes are developed [7] to protect the clustered errors over a particular subarray part of the transmitted array message. These types of errors occur in many practical situations e.g. due to lightening and thunder in deep space and satellite communication. One important and practical situation is that in which the array message is disturbed over a particular subarray part of the transmitted code array together with occasional disturbances, thus creating simultaneously burst as well as independent (or random) errors. Therefore, in actual communication, while it is important to consider correction of burst array errors, care must be taken to correct independent (or independent) array errors of up to a specified weight, no matter where they occur. In this paper, we consider the problem of burst array error correction together with the independent (or random) array error correction with weight constraint in m-metric array codes and obtain a lower bound (which is in fact a necessary condition) over the number of parity checks in m-codes for the correction of the same.

2. DEFINITIONS AND NOTATIONS

Let F_q be a finite field of q elements. Let $\text{Mat}_{m\times s}(F_q)$ denote the linear space of all $m\times s$ matrices with entries from F_q. An m-metric array code is a subset of $\text{Mat}_{m\times s}(F_q)$ and a linear m-metric array code is an F_q-linear subspace of $\text{Mat}_{m\times s}(F_q)$. Note that the space $\text{Mat}_{m\times s}(F_q)$ is identifiable with the space F_q^{ms}. Every matrix in $\text{Mat}_{m\times s}(F_q)$ can be represented as a $1 \times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in F_q^{ms} can be represented as an $m \times s$ matrix in $\text{Mat}_{m\times s}(F_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates.

The weight and metric defined by Rosenbloom and Tsfasman [15] on the space $\text{Mat}_{m\times s}(F_q)$ are as follows :

Let $X \in \text{Mat}_{m\times 1}(F_q)$ with

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_m \end{pmatrix},$$

then column weight (or weight) of X is given by

$$wt_c(X) = \begin{cases} m - \ \max \{ \ i \mid x_k = 0 \\ \text{for any } k \leq i\} \ \text{ if } X \neq 0 \\ \\ 0 \ \text{ if } \ X = 0. \end{cases}$$

This definition of wt_c can be extended to $m \times s$ matrices in the space $\text{Mat}_{m\times s}(F_q)$ as

$$wt_c(A) = \sum_{j=1}^{s} wt_c(A_j)$$

where $A = [A_1, A_2, \cdots, A_s] \in \text{Mat}_{m\times s}(F_q)$ and A_j denotes the j^{th} column of A. Then wt_c satisfies $0 \leq wt_c(A) \leq n(= ms)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ if we set $d(A, A') = wt_c(A - A') \ \forall \ A, A' \in \text{Mat}_{m\times s}(F_q)$. We call this metric as column-metric. Note that for $m = 1$, it is just the usual Hamming metric.

There is an alternative equivalent way of defining the weight of an $m \times s$ matrix using the weight of its rows [4]:

Let $Y \in \text{Mat}_{1\times s}(F_q)$ with $Y = (y_1, y_2, \cdots, y_s)$. Define row weight (or weight) of Y as

$$wt_\rho(Y) = \begin{cases} \max\{\, i \mid y_i \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } Y = 0. \end{cases}$$

Extending the definitions of wt_ρ to the class of $m \times s$ matrices as

$$wt_\rho(A) = \sum_{i=1}^{m} wt_\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in \text{Mat}_{m\times s}(F_q)$ and R_i denotes the i^{th} row of A. Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq n(= ms)\ \forall\ A \in \text{Mat}_{m\times s}(F_q)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ known as row-metric.

It turns out that row weight of a vector is equal to the column weight of transpose of the vector with its component reversed and hence the two metrics viz. row-metric and column-metric give rise to equivalent codes and both the metrics have been known as m-metric or RT-metric.

In this paper, we take distance and weight in the sense of row-metric. Throughout this paper, $[x]$ denotes the greatest integer less than or equal to x.

3. LOWER BOUND FOR m-CODES CORRECTING INDEPENDENT AND BUSRY ARRAY ERRORS

We begin with the definition of bursts in m-metric array codes [7].

Definition 1. A burst of order pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is an $m \times s$ matrix in which all the nonzero entries are confined to some $p \times r$ submatrix which has non-zero first and last rows as well as non-zero first and last columns.

Note. For $m = p = 1$, Definition 3.1 reduces to the definition of burst for classical codes [5].

Definition 2. A burst of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is a burst of order cd(or $c \times d$) where $1 \leq c \leq p \leq m$ and $1 \leq d \leq r \leq s$.

To obtain the desired bound, we need to find all $m \times s$ arrays of ρ-weight t or less and additional arrays of ρ-weight w_ρ or less which are bursts of order $p \times r$ or less. We obtain in the next two lemmas the number of these arrays separately.

Lemma 3. *If V_t denotes all $m \times s$ arrays in $Mat_{m\times s}(F_q)$ of ρ-weight t or less then*

$$V_t = \sum_{k_1,k_2,\cdots,k_s} \frac{m!}{\prod_{i=1}^{s} k_i!\left(m - \sum_{i=1}^{s} k_i\right)!} \times \left(\frac{q-1}{q}\right)^{\sum_{i=1}^{s} k_i} (q)^{\sum_{i=1}^{s} ik_i} \tag{1}$$

where $k_1, k_2, \cdots, k_s$ are nonegative integers such that

$$\begin{aligned} &\sum_{i=1}^{s} k_i \leq m, \\ &\sum_{i=1}^{s} ik_i \leq t. \end{aligned} \tag{2}$$

Proof. Let $A \in \text{Mat}_{m\times s}(F_q)$ be an $m \times s$ array over F_q having ρ-weight t or less. Out of m rows of A let $k_i(\geq 0)$ denote the number of rows having ρ-weight $i(1 \leq i \leq s)$. Also, the ρ-weight $i(1 \leq i \leq s)$ of a row can be obtained by filling the i^{th} entry in the row by a nonzero elements of F_q and the preceding $(i-1)$ entries by any of the q elements of F_q. Therefore, number of ways in which we obtain ρ-weight $i(1 \leq i \leq s)$ of a row-vector is $(q-1)q^{i-1}$.

Since k_i is the number of rows of array $A \in \text{Mat}_{m\times s}(F_q)$ having ρ-weight $i(1 \leq i \leq s)$, therefore, number of ways in which rows of array A can be selected is given by

$$\begin{aligned}
&= \frac{m!}{\prod_{i=1}^{s} k_i!\left(m - \sum_{i=1}^{s} k_i\right)!} \times ((q-1)q^0)^{k_1} \times \\
&\quad \times ((q-1)q^1)^{k_2} \times \cdots \times ((q-1)q^{s-1})^{k_s} \\
&= \frac{m!}{\prod_{i=1}^{s} k_i!\left(m - \sum_{i=1}^{s} k_i\right)!} \times \\
&\quad \times (q-1)^{\sum_{i=1}^{s} k_i} (q)^{\sum_{i=2}^{s} (i-1)k_i} \\
&= \frac{m!}{\prod_{i=1}^{s} k_i!\left(m - \sum_{i=1}^{s} k_i\right)!} \times \\
&\quad \times \left(\frac{q-1}{q}\right)^{\sum_{i=1}^{s} k_i} (q)^{\sum_{i=1}^{s} ik_i}.
\end{aligned} \tag{3}$$

Condition (2) follows immediately as the total number of rows of array A is m and sum of ρ-weights of the m-

rows of A is atmost t. Now summing (3) for all possible values of $k_i's$, we get (1). □

Lemma 4. *If $B^{p\times r}_{m\times s}(F_q, t+1, w_\rho)$ is the total number of arrays of order $m \times s$ in Mat$_{m\times s}(F_q)$ which are bursts of order $p \times r$ having ρ-weight between $t+1$ and w_ρ, then*

$$B^{p\times r}_{m\times s}(F_q, t+1, w_\rho) = \begin{cases} m \times \min(w_\rho - t, s-t) \times (q-1) \\ \qquad\qquad \text{if } p = r = 1, \\ m \times \{\min(w-r+1, s-r+1) - \\ -\max(t-r+2, 1)+1\} \times \\ \times (q-1)^2 q^{r-2} \qquad \text{if } p = 1, r \geq 2, \\ (m-p+1) \times \\ \times \displaystyle\sum_{j=1}^{\min([w_\rho/2], s)} \sum_{\substack{\eta=0: \\ t+1-2j \leq \eta j \leq w_\rho - 2j}}^{p-2} \\ \displaystyle\binom{p-2}{\eta}(q-1)^{\eta+2} \qquad \text{if } p \geq 2, r = 1, \\ (m-p+1) \displaystyle\sum_{j=1}^{\min(w-r+1, s-r+1)} \\ (L^p_j - 2L^{p-1}_j + L^{p-2}_j) \quad \text{if } p \geq 2, r \geq 2, \end{cases} \tag{4}$$

where

$$L^p_j = \sum_{k_j, k_{j+1}, \cdots, k_{j+r-1}} \times \frac{p!}{\prod_{l=0}^{r-1} k_{j+l}! \left(p - \sum_{l=0}^{r-1} k_{j+l}\right)!} \times \times \left(\frac{q-1}{q}\right)^{\sum_{l=0}^{r-1} k_{j+l}} \times \times q^{\sum_{l=0}^{r-1}(l+1)k_{j+l}}, \tag{5}$$

and $k_j, k_{j+1}, \cdots, k_{j+r-1}$ being nonnegative integers such that

$$\begin{aligned} &k_j > 0, \quad k_{j+1}, k_{j+2}, \cdots, k_{j+r-2} \geq 0, \\ &k_{j+r-1} > 0, \\ &k_j + k_{j+1} + k_{j+2} + \cdots + k_{j+r-1} \leq p, \\ &t+1 \leq \sum_{l=0}^{r-1} (j+l)k_{j+l} \leq w_\rho. \end{aligned} \tag{6}$$

Proof. Consider a burst $A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ A_m \end{bmatrix}$ where $A_i = (a_{i_1}, a_{i_2}, \cdots, a_{i_s})$, of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ having ρ-weight lying between $t+1$ and w_ρ. Let B be the $p\times r$ nonzero submatrix of A such that all the nonzero entries of A are confined to B with first and last rows as well as first and last columns nonzero. There are four cases depending upon the values of p and r.

Case 1. When $p = 1,\ r = 1$.

In this case, number of starting positions for the 1×1 nonzero submatrix B in $m \times s$ matrix A is $m \times \min(w_\rho - t, s-t)$ and these $m \times \min(w_\rho - t, s-t)$ positions can be filled by $(q-1)$ nonzero elements from F_q. Therefore, number of bursts of order 1×1 having ρ-weight lying between $t+1$ and w_ρ in Mat$_{m\times s}(F_q, t+1, w_\rho)$ is given by

$$\begin{aligned} & B^{1\times 1}_{m\times s}(F_q, t+1, w_\rho) \\ = \ & m \times \min(w_\rho - t, s-t) \times (q-1). \end{aligned}$$

Case 2. When $p = 1,\ r \geq 2$.

In this case, number of starting positions for the $1 \times r$ nonzero submatrix B in $m \times s$ matrix A is $m \times \{\min(w-r+1, s-r+1) - \max(t-r+2, 1)+1\}$ and entries in the $1 \times r$ submatrix B can be selected in $(q-1)^2 q^{r-2}$ ways as the first and last components of the single rowed submatrix B can be chosen in $(q-1)^2$ ways and intermediate $(r-2)$ components can be chosen in q^{r-2} ways. Therefore, number of bursts of order $1 \times r$ having ρ-weight between $t+1$ and w_ρ in Mat$_{m\times s}(F_q, t+1, w_\rho)$ is given by

$$\begin{aligned} & B^{1\times r}_{m\times s}(F_q, t+1, w_\rho) \\ = \ & m \times \{\min(w-r+1, s-r+1) - \\ & -\max(t-r+2, 1)+1\} \times \\ & \times (q-1)^2 q^{r-2}. \end{aligned}$$

Case 3. When $p \geq 2,\ r = 1$.

In this case, the $p \times 1$ nonzero column vector B can have (i, j) as its feasible starting positions in $m \times s$ matrix A where i can vary from 1 to $(m-p+1)$ and j can vary from 1 to $\min([w_\rho/2], s)$ subject to the condition that there exists η lying between 0 and $p-2$ satisfying $t+1-2j \leq \eta j \leq w_\rho - 2j$. If no such η exists then (i, j) cannot be taken as the feasible starting position of the $p \times 1$ nonzero column vector B. Now with (i, j) as the feasible starting position of $p \times 1$ nonzero column matrix B, entries in B can be filled in

$$\sum_{\substack{\eta=0: \\ t+1-2j\leq \eta j\leq w_\rho-2j}}^{p-2} (q-1)^2 \binom{p-2}{\eta} (q-1)^\eta$$

ways as first and last components of the column matrix B can be chosen in $(q-1)^2$ ways and intermediate $(p-2)$ components can be chosen in $\sum_{\eta=0}^{p-2} \binom{p-2}{\eta} (q-1)^\eta$ ways subject to constraint $t+1-2j \leq \eta j \leq w_\rho - 2j$ as $2j$ ρ-weight has already been taken from the first and last components. Therefore, number of bursts of order $p \times 1$ having ρ-weight lying between $t+1$ and w_ρ or less in $\mathrm{Mat}_{m\times s}(F_q, t+1, w_\rho)$ is given by

$$\begin{aligned} & B^{p\times 1}_{m\times s}(F_q, t+1, w_\rho) = \\ = \; & (m-p+1) \sum_{j=1}^{\min([w_\rho/2], s)} \\ & \sum_{\substack{\eta=0: \\ t+1-2j\leq nj\leq w_\rho-2j}}^{p-2} (q-1)^2 \binom{p-2}{\eta} \times \\ & \times (q-1)^\eta \\ = \; & (m-p+1) \sum_{j=1}^{\min([w_\rho/2], s)} \\ & \sum_{\substack{\eta=0: \\ t+1-2j\leq nj\leq w_\rho-2j}}^{p-2} \binom{p-2}{\eta} (q-1)^{\eta+2} \end{aligned}$$

Case 4. When $p \geq 2, \; r \geq 2$.

In this case, let the $p\times r$ nonzero submatrix B starts at the $(i,j)^{th}$ position in A. Out of p rows of B, let $k_j, k_{j+1}, k_{j+2}, \cdots, k_{j+r-1}$ be the number of rows of B having ρ-weight $j, j+1, \cdots, j+r-1$ respectively. The number of ways in which p rows of B can be selected is given by

$$L^p_j - 2L^{p-1}_j + L^{p-2}_j, \tag{7}$$

where L^p_j is given by (5) and $k_j, k_{j+1}, k_{j+2}, \cdots, k_{j+r-1}$ being nonnegative integers satisfying (6). Since in the starting position (i,j) of the submatrix B, i can vary from 1 to $(m-p+1)$ and j can vary from 1 to $\min(w-r+1, s-r+1)$, therefore, summing (7) over i and j, we get number of bursts of order pr (or $p\times r)(2 \leq p \leq m, 2 \leq r \leq s)$ having ρ-weight lying between $t+1$ and w_ρ and is given by

$$\begin{aligned} & B^{p\times r}_{m\times s}(F_q, t+1, w_\rho) = \\ = \; & (m-p+1) \times \\ & \sum_{j=1}^{\min(w-r+1, s-r+1)} (L^p_j - 2L^{p-1}_j + L^{p-2}_j), \end{aligned}$$

where L^p_j is given by (5) satisfying the constraints (6). □

Now, we obtain the desired bound.

Theorem 5. *An $[m\times s, k]$ linear m-metric code of order $m\times s$ that simultaneously corrects independent errors of ρ-weight t or less and bursts of order $p\times r$ or less with ρ-weight w_ρ or less $(W_\rho \geq t)$ should have at least*

$$log_q(V_t + E^{p\times r}_{m\times s}(F_q, t+1, w_\rho)) \tag{8}$$

parity checks where

$$\begin{aligned} & E^{p\times r}_{m\times s}(F_q, t+1, w_\rho) \\ = \; & \sum_{c=1}^{p}\sum_{d=1}^{r} B^{c\times d}_{m\times s}(F_q, t+1, w_\rho). \end{aligned} \tag{9}$$

Proof. The total number of correctable error patterns for an m-code correcting simultaneously independent errors of ρ-weight t or less and bursts of order $p\times r$ or less with ρ-weight w_ρ or less is given by

$$V_t + E^{p\times r}_{m\times s}(F_q, t+1, w_\rho),$$

where $E^{p\times r}_{m\times s}(F_q, t+1, w_\rho)$ is given by (4) and (9) and V_t is given by (1).

Also, the number of available cosets is q^{ms-k} and since a linear $(m\times s, k)$ code should have at least as many cosets as the number of correctable error patterns, we have

$$q^{ms-k} \geq V_t + E^{p\times r}_{m\times s}(F_q, t+1, w_\rho)$$

i.e.

$$ms - k \geq log_q(V_t + E^{p\times r}_{m\times s}(F_q, t+1, w_\rho)).$$

□

Particular Case. The weight constraint over the burst can be removed by taking w_ρ to be the maximum possible weight for a burst of order $p\times r$. This requires that we take $w_\rho = ps$. The result in that case reduces to the one given by the following corollary:

Corollary 6. *A linear m-code of order $m\times s$ that simultaneously corrects independent errors of ρ-weight t*

or less and any burst of order pr or less should have at least

$$log_q(V_t + E^{p\times r}_{m\times s}(F_q, t+1, ps)).$$

parity checks. □

On the other hand, we can derive results for burst correction only. This requires the dropping of the independent error correction constraint. Taking $t = 0$, the corresponding result which is obtained, can be given in the following corollary.

Corollary 7. *A linear m-code of order $m \times s$ that corrects all bursts of order $p \times r$ or less with ρ-weight w_ρ or less should have at least*

$$log_q(V_t + E^{p\times r}_{m\times s}(F_q, 1, w_\rho))$$

parity checks. □

This result has been obtained differently by the author in [7]. A bound for independent error correction can also be deduced from the result obtained in Theorem 5. This requires the dropping of the burst correction constraint. On taking $w_\rho = t$, we have $E^{p\times r}_{m\times s}(F_q, t+1, w_\rho) = 0$. The result so obtained can be given in the following corollary.

Corollary 8[15]. *A linear m-code of order $m \times s$ that corrects independent errors of ρ-weight t or less should have at least*

$$log_q(V_t)$$

parity checks. □

Acknowledgment. The author would like to thank her spouse Dr. Arihant Jain for his constant support and encouragement for pursuing research.

REFERENCES

1. Blaum, M., P.G. Farrell and H.C.A. van Tilborg, 1998. Array Codes, in Handbook of Coding Theory, (Ed.: V. Pless and Huffman), Vol. II, Elsevier, North-Holland, pp.1855-1909.
2. Campopiano, C.N., 1962. Bounds on Burst Error Correcting Codes, IRE. Trans., IT-8:257-259.
3. Dougherty S.T. and M.M. Skriganov, 2002. MacWilliams duality and the Rosenbloom-Tsfasman metric, Moscow Mathematical Journal, 2:83-99.
4. Dougherty, S.T. and M.M. Skriganov, 2002. Maximum Distance Separable Codes in the ρ-metric over independent Alphabets, Journal of Algebraic Combinatorics, 16:71-81.
5. Fire, P., 1959. A Class of Multiple-Error-Correcting Binary Codes for Non-Independent Errors, Sylvania Reports RSL-E-2, Sylvania Reconnaissance Systems, Mountain View, California.
6. Gabidulin, E.M. and V.V. Zanin, 1993. Matrix codes correcting array errors of size 2×2, International Symp. on Communication Theory and Applications, Ambleside, U.K., 11-16 June, 1993.
7. Jain, S., 2006. Bursts in m-Metric Array Codes, Linear Algebra and Its Applications, 418:130-141.
8. Jain, S., 2007. Campopiano-Type Bounds in Non-Hamming Array Coding, Linear Algebra and Its Applications, 420: 135-159.
9. Jain, S., 2008. An Algorithmic Approach to Achieve Minimum ρ-Distance at least d in Linear Array Codes, Kyushu Journal of Mathematics, 62:189-200.
10. Jain, S., 2008. CT Bursts- From Classical to Array Coding, Discrete Mathematics, 308:1489-1499.
11. Jain, S., On a Class of Cyclic Bursts in Array Codes, to appear in Ars Combinatoria.
12. Jain, S. and K.P. Shum, 2010. Correction of CT Burst Array Errors in the Generalized-Lee-RT Spaces, Acta Mathematica Sinica, English Series, 26:1475-1484.
13. Peterson, W.W. and E.J. Weldon, Jr., 1972. Error Correcting Codes, 2nd Edition, MIT Press, Cambridge, Massachusetts.
14. Reiger, S.H., 1960. Codes for the Correction of Clustered Errors, IRE-Trans., IT-6:16-21.
15. Rosenbloom,M.Yu. and M.A. Tsfasman, 1997. Codes for m-metric, Problems of Information Transmission, 33:45-52.