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A class of linear partition error control codes in γ-metric

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Abstract: Linear partition error control codes in the γ-metric is a natural generalization of error control codes endowed with the Rosenbloom-Tsfasman(RT) metric [4] to block coding and has applications in different area of combinatorial/discrete mathematics, e.g. in the theory of uniform distribution, experimental designs, cryptography etc. In this paper, we formulate the concept of linear partition codes in the γ-metric and derive results for the random block error detection and random bock error correction capabilities of these codes.

Key words: Linear codes, RT-metric, error-block code

INTRODUCTION

K. Feng and L.Xu and F.J.Hickernesll [2] initiated the concept of linear partition block code endowed with π-metric which is a natural generalization of the Hamming-metric codes. Also, we know that the Rosenbloom-Tsfasman metric (or RT-metric or ρ-metric) is stronger than the Hamming metric [1,5]. Motivated by the idea to have linear partition block code endowed with a metric generalizing the RT-metric, we formulate the concept of linear partition codes equipped with block ρ-metric and name this new metric as the γ-metric. We derive the basic results for linear partition codes in the γ-metric including various upper and lower bounds on their parameters and study their random block error detection and block error correction capabilities in Section 3 and Section 4 of this paper.

2. DEFINITIONS AND NOTATIONS

Let q, n be positive integers with $q = p^m$, a power of a prime number p. Let $\mathbf{F}_q$ be the finite field having q elements. A partition P of the positive integer n is defined as:

$$\begin{aligned} P \quad : \quad & n = n_1 + n_2 \cdots + n_s \quad \text{where} \\ & 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s, s \geq 1. \end{aligned}$$

The partition P is denoted as

$$P : n = [n_1][n_2] \cdots [n_s].$$

In the case, when

$$P : n = \underbrace{[n_1] \cdots [n_1]}_{r_1\text{- copies}} \underbrace{[n_2] \cdots [n_2]}_{r_2\text{- copies}} \cdots \underbrace{[n_t] \cdots [n_t]}_{r_t\text{- copies}},$$

we write

$$P : n = [n_1]^{r_1} [n_2]^{r_2} \cdots [n_t]^{r_t}$$

where

$$n_1 < n_2 < \cdots < n_t.$$

Further, given a partition $P : n = [n_1][n_2] \cdots [n_s]$ of a positive integer n, the linear space $\mathbf{F}_q^n$ over $\mathbf{F}_q$ can be viewed as the direct sum

$$\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s},$$

or equivalently

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s,$$

where $V = \mathbf{F}_q^n$ and $V_i = \mathbf{F}_q^{n_i}$ for all $i \leq i \leq s$.

Consequently, each vector $v \in \mathbf{F}_q^n$ can be uniquely written as a $v = (v_1, v_2, \cdots, v_s)$ where $v_i \in V_i = \mathbf{F}_q{}^{n_i}$ for all $1 \leq i \leq s$. Here v_i is called the i^{th} *block of block size* n_i of the vector v.

Definition 1. Let $v = (v_1, v_2, \cdots, v_s) \in \mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ be an s-block vector of length n over $\mathbf{F}_q$ corresponding to the partition $P : n = [n_1][n_2] \cdots [n_s]$ of n. We define the γ-weight of the block vector v as

$$w_\gamma^{(P)}(v) = \max_{i=1}^{s} \{i | v_i \neq 0\}.$$

The γ-distance $d_\gamma^{(P)}(u, v)$ between two s-block vectors of length n viz. $u = (u_1, u_2, \cdots, u_s)$ and

$v = (v_1, v_2, \cdots, v_s), u_i, v_i \in \mathbf{F}_q^{n_i} (1 \leq i \leq s)$ corresponding to the partition P is defined as

$$\begin{aligned} d_\gamma^{(P)}(u,v) &= w_\gamma^{(P)}(u-v) \\ &= \max_{i=1}^{s} \{i | u_i \neq v_i\} \end{aligned}$$

Then $d_\gamma^{(P)}(u,v)$ is a metric on $\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$.

Note. Once the partition P is specified, we will denote the γ-weight $w_\gamma^{(P)}$ by $w_\gamma(v)$ and γ-distance $d_\gamma^{(P)}$ by d_γ respectively.

Definition 2. A linear partition γ-code (or $lp\gamma$-code) V of length n corresponding to the partition $P : n = [n_1][n_2] \cdots [n_s], 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ is a $\mathbf{F}_q$-linear subspace of $\mathbf{F}_q^n = \mathbf{F}_q^{n_1} \oplus \mathbf{F}_q^{n_2} \oplus \cdots \oplus \mathbf{F}_q^{n_s}$ equipped with the γ-metric and is denoted as $[n, k, d_\gamma; P]$ code where $k = \dim_{\mathbf{F}_q}(V)$ and $d_\gamma = d\gamma(V) =$ minimum γ-distance of the code V.

Remark 3.

1. For $P : n = [1]^n$, the γ-metric (or γ-weight) reduces to the ρ-metric (or ρ-weight) respectively [4].
2. For a partition $P : n = [n_1][n_2] \cdots [n_s]$ of the positive integer n, the γ-distance (or γ-weight) is always greater than or equal to the π-distance (or π-weight) [2] respectively, i.e.

$$\begin{aligned} \pi\text{-metric} &\leq \gamma\text{-metric} \\ &\quad \text{and} \\ \pi\text{-weight} &\leq \gamma\text{-weight} \end{aligned}$$

Example 4. Let $n = q = 5$. Let $P : 5 = [1][2][2]$ be a partition of $n = 5$. Then F_5^5 can be viewed as $F_5^5 = F_5^1 \oplus F_5^2 \oplus F_5^2$ and $s = 3$. Let $v = (v_1, v_2, v_3) = (1\vdots10\vdots00)$. Then $w\gamma(v) = 2$. Similarly if $u = (u_1, u_2, u_3) = (1\vdots00\vdots00)$ and $x = (x_1, x_2, x_3) = (1\vdots10\vdots01)$, then $w\gamma(u) = 1$ and $w\gamma(x) = 3$ respectively. QED

Definition 5. The generator and parity check matrix of an $[n, k, d; P]$ $lp\gamma$-code over $\mathbf{F}_q$ where $P : n = [n_1][n_2] \cdots [n_s]$, $1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ are given as

$$G = [G_1, G_2, \cdots, G_s]$$

and

$$H = [H_1, H_2, \cdots, H_s],$$

where for all $1 \leq i \leq s, G_i = (G_1^{(i)}, G_2^{(i)}, \cdots, G_{n_i}^{(i)})$ is the i^{th} block of G of block size n_i consisting of n_i column vectors of length k each and $H_i = (H_1^{(i)}, H_2^{(i)}, \cdots, H_{n_i}^{(i)})$ is the i^{th} block of H of block size n_i consisting of n_i column vectors of length $(n-k)$ each.

Definition 6. A set of blocks $\{H_{i_1}, H_{i_2}, \cdots, H_{i_r}\} \subseteq \{H_1, H_2, \cdots, H_s\}$ of the parity check matrix H is said to be linearly independent if the union of all column vectors in the blocks $H_{i_1}, H_{i_2}, \cdots, H_{i_r}$ is a linearly independent set over $\mathbf{F}_q$. Otherwise, we say that the set of blocks $\{H_{i_1}, H_{i_2}, \cdots, H_{i_r}\}$ is linearly dependent. Equivalently, we can say that a set of blocks $\{H_{i_1}, H_{i_2}, \cdots, H_{i_r}\} \subseteq \{H_1, H_2, \cdots, H_s\}$ is linearly independent over $\mathbf{F}_q$ iff

$$\begin{aligned} &\alpha_{i_1}.H_{i_1} + \alpha_{i_2}.H_{i_2} + \cdots + \alpha_{i_r}.H_{i_r} = 0 \\ &\Rightarrow \alpha_{i_1} = \alpha_{i_2} \cdots = \alpha_{i_r} = 0, \end{aligned}$$

where for all $1 \leq j \leq r$, $\alpha_{i_j} = (\alpha_1^{(i_j)}, \alpha_2^{(i_j)}, \cdots, \alpha_{n_{i_j}}^{(i_j)}) \in \mathbf{F}_q^{n_{i_j}}$ and

$$\begin{aligned} \alpha_{i_j}.H_{i_j} &= \alpha_1^{(i_j)} H_1^{(i_j)} + \alpha_2^{(i_j)} H_2^{(i_j)} \\ &\quad + \cdots + \alpha_{n_{i_j}}^{(i_j)} H_{n_{i_j}}^{(i_j)}. \end{aligned}$$

3. SOME PROPERTIES OF $lp\gamma$-CODES

We begin by stating three results for $lp\gamma$-codes without proof as the proof is straightforward.

Theorem 7. *The minimum γ-weight and minimum γ-distance of an $lp\gamma$-code V coincide.* QED

Theorem 8.

(a) *An $lp\gamma$-code detects all block errors of γ-weight t or less iff the minimum γ-distance of the code is at least $t+1$.*

(b) *An $lp\gamma$-code V corrects all block errors of γ-weight t or less iff the minimum γ-distance of the code V is at least $2t+1$.* QED

Theorem 9. *Let V be an $[n, k; P]$ $lp\gamma$-code over $\mathbf{F}_q$ corresponding to the partition $P : n = [n_1][n_2] \cdots [n_s]$ of n. The minimum γ-distance of the code V is d iff first $(d-1)$ blocks of the parity check matrix H are linearly independent and first d blocks of H are linearly dependent over $\mathbf{F}_q$.* QED

Example 10. Let $n = 6$, $n - k = 5$ and $q = 3$. Let $P : n = 6 = [1][1][1][3]$ be a partition of $n = 6$. Then $s = 4$ and $n_1 = 1, n_2 = 1, n_3 = 1$ and $n_4 = 3$. Let $H = (H_1\vdots H_2\vdots H_3\vdots H_4)$ be the parity check matrix of a

$[6,1;P]$ $lp\gamma$-code V as given below

$$H=\begin{bmatrix} 1 & \vdots & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 1 & \vdots & 0 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 0 & \vdots & 1 & \vdots & 0 & 0 & 0 \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & 1 & 2 & 0 \\ 0 & \vdots & 0 & \vdots & 0 & \vdots & 0 & 2 & 1 \end{bmatrix}$$

Let

$$\alpha_1.H_1+\alpha_2.H_2+\alpha_3.H_3+ \\ +\alpha_4.H_4=0, \quad (1)$$

where $\alpha_1 = (\alpha_1^{(1)}) \in F_3^1, \alpha_2 = (\alpha_1^{(2)}) \in F_3^1, \alpha_3 = (\alpha_1^{(3)}) \in F_3^1$ and $\alpha_4 = (\alpha_1^{(4)}, \alpha_2^{(4)}, \alpha_3^{(4)}) \in F_3^3$.

Then $\alpha_1.H_1+\alpha_2.H_2+\alpha_3.H_3+\alpha_4.H_4=0$ implies

$$\alpha_1^{(1)}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+\alpha_1^{(2)}\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} \\ +\alpha_1^{(3)}\begin{bmatrix}0\\0\\1\\0\\0\end{bmatrix}+\alpha_1^{(4)}\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix} \\ +\alpha_2^{(4)}\begin{bmatrix}0\\0\\0\\2\\2\end{bmatrix}+\alpha_3^{(4)}\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix} \\ =\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}.$$

This gives

$$\alpha_1^{(1)}=0,\ \ \alpha_1^{(2)}=0,\ \ \alpha_1^{(3)}=0$$

and

$$\alpha_1^{(4)}=\alpha_2^{(4)}=\alpha_3^{(4)}.$$

Therefore, the only solutions of (1) are $\alpha_1=\alpha_2=\alpha_3=(0)$ and $\alpha_4=(a,a,a)$ where $a\in F_3$.

Thus, the first three blocks of H are linearly independent and the first four blocks of H are linearly dependent. Hence the $[6,1:P]$ $lp\gamma$-code V with H as the parity check matrix has minimum γ-distance equal to 4.

Theorem 11 [Singleton's Bound]. *If V is an $[n,k,d:P]$ $lp\gamma$-code over $\mathbf{F}_q$ corresponding to the partition $P: n=[n_1][n_2]\cdots[n_s], 1\le n_1\le n_2\le\cdots\le n_s$. Then*

$$n_1+n_2+\cdots+n_{d-1}\le n-k. \quad (2)$$

Proof. By Theorem 9, the columns of first $(d-1)$ blocks of the parity check matrix H of the code V are linearly independent. Since the number of rows in H is $n-k$, equation (2) follows. QED

Definition 12. An $[n,k,d:P]$ $lp\gamma$-code over $\mathbf{F}_q$ with $P: n=[n_1][n_2]\cdots[n_s]$ is said to be maximum γ-distance separable $(M\gamma DS)$ if equality holds in (2) i.e. if $(n-k)$ equals the sum of block sizes of first $(d-1)$ blocks.

Example 13. Let $q=5, n=2$. Let $P:2=[1][1]$ be a partition of $n=2$. The $[2,1;P]$ $lp\gamma$-code V with parity check matrix $H=(1\vdots0)$ over $\mathbf{F}_5$ is an $M\gamma DS$ code with maximum γ-distance equal to 2.

We now obtain Hamming sphere packing bound for $lp\gamma$-codes. For this, we first prove a lemma.

Lemma 14. *If $V_{t,q}^{(n_1,n_2,\cdots,n_s)}$ denote the number of all s-block vectors of length n over $\mathbf{F}_q$ of γ-weight t or less corresponding to the partition $P: n=[n_1][n_2]\cdots[n_s], 1\le n_1\le n_2\le\cdots\le n_s$, then*

$$V_{t,q}^{(n_1,n_2,\cdots,n_s)} \\ = 1+\sum_{r=1}^{t} q^{n_1+n_2+\cdots+n_{r-1}}\times \\ \times(q^{n_r-1}). \quad (3)$$

Proof. Let $u=(u_1,u_2,\cdots,u_s)\in\mathbf{F}_q^n=\mathbf{F}_q^{n_1}\oplus\mathbf{F}_q^{n_2}\oplus\cdots\oplus\mathbf{F}_q^{n_s}$. To make the γ-weight of u to be equal to $r(1\le r\le t)$, we have q^{n_j} choices for the j^{th} block $(1\le j\le r-1)$ and $(q^{n_r}-1)$ choices for the r^{th} block and only one choice viz. zero for the l^{th} block $(r+1\le l\le s)$. Therefore, the number of s-block vectors of length n of γ-weight r is given by

$$A_{r,q}^{(n_1,n_2,\cdots,n_s)} \\ = q^{n_1+n_2+\cdots+n_{r-1}}(q^{n_r-1}). \quad (4)$$

The result now follows by taking summation of (4) for $r=1$ to t and adding 1 to the resultant corresponding to the null vector. QED

Theorem 15 (Hamming Sphere Bound). *Let V be an $[n,k,d:P]$ $lp\gamma$-code over $\mathbf{F}_q$ corresponding to the partition $P: n=[n_1][n_2]\cdots[n_s], 1\le n_1\le n_2\le\cdots\le$*

n_s. Then

$$q^{n-k} \geq V_{[d-1]/2,q}^{(n_1,n_2,\cdots,n_s)}, \tag{5}$$

where $V_{[d-1]/2,q}^{(n_1,n_2,\cdots,n_s)}$ is given by (3) and $[x]$ denotes the largest integer less than or equal to x.

Proof. The proof follows from the fact that all the s-block vectors of length $n = \oplus_{i=1}^{s} n_i$ and γ-weight $[d-1]/2$ or less must belong to distinct cosets of the standard array and the number of available cosets is q^{n-k}. QED

4. GILBERT AND VARSHAMOV BOUNDS FOR $lp\gamma$-CODES

In this section, we obtain Gilbert bound, Varshmov bound and a bound for random block error correction in $lp\gamma$-codes. We derive Gilbert bound first.

Theorem 16 (Gilbert bound). *Let n, k, q be positive integers where $q = p^m (p$ prime$)$ and $1 \leq k \leq n$. Let $P : n = [n_1][n_2]\cdots[n_s], 1 \leq n_1 \leq n_2 \leq \cdots \leq n_s$ be a partition of n. Let d be a positive integer satisfying $1 \leq d \leq s$. Then there exists an $[n, k; P]$ $lp\gamma$-code over $\mathbf{F}_q$ with minimum γ-distance at least d provided*

$$n - k \geq log_q\left(V_{d-1,q}^{(n_1,n_2,\cdots,n_s)}\right), \tag{6}$$

where $V_{d-1,q}^{(n_1,n_2,\cdots,n_s)}$ is given by (3).

Proof. We shall show that if (6) holds then their exists an $(n-k)\times n$ matrix H over $\mathbf{F}_q$ such that no linear combination of $(d-1)$ or fewer blocks of H is zero. We define an algorithm for finding the blocks $H_1, H_2, \cdots, H_s$ of H where $H_i = (H_1^{(i)}, H_2^{(i)}, \cdots, H_{n_i}^{(i)})$ for all $1 \leq i \leq s$. From the set of all q^{n-k} column vectors of length $(n-k)$ over $\mathbf{F}_q$, we choose blocks of columns of the parity check matrix H as follows:

(1) The n_1 column vectors in the first block H_1 can be any vectors chosen from the set of q^{n-k} column vectors of length $n-k$ over $\mathbf{F}_q$ satisfying

$$\lambda_1.H_1 \neq 0,$$

where

$$0 \neq \lambda_1 = (\lambda_1^{(1)}, \lambda_2^{(1)}, \cdots, \lambda_{n_1}^{(1)}) \in \mathbf{F}_q^{n_1}.$$

(2) The second block $H_2 = (H_1^{(2)}, H_2^{(2)}, \cdots, H_{n_2}^{(2)})$ can be any set of n_2 column vectors of length $(n-k)$ satisfying

$$\lambda_1.H_1 + \lambda_2.H_2 \neq 0,$$

where for $1 \leq i \leq 2$,

$$\lambda_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{F}_q^{n_i},$$

and

$$w_\gamma(\lambda_1, \lambda_2) = \max_{i=1}^{2}\{i|\lambda_i \neq 0\} \leq d-1.$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

(j) The j^{th} block $H_j = (H_1^{(j)}, H_2^{(j)}, \cdots, H_{n_j}^{(j)})$ can be any set of n_j column vectors of length $(n-k)$ satisfying

$$\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_j.H_j \neq 0. \tag{7}$$

where for $1 \leq i \leq j$,

$$\lambda_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{F}_q^{n_i},$$

and

$$\begin{aligned} 1 &\leq w_\gamma(\lambda_1, \lambda_2, \cdots, \lambda_j) \\ &= \max_{i=1}^{j}\{i|\lambda_{i)} \neq 0\} \\ &\leq d-1. \end{aligned} \tag{8}$$

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{array}$$

(s) The s^{th} block $H_s = (H_1^{(s)}, H_2^{(s)}, \cdots, H_{n_s}^{(s)})$ can be any set of n_s column vectors satisfying

$$\lambda_1.H_1 + \lambda_2.H_2 + \cdots + \lambda_s.H_s \neq 0.$$

where

$$\lambda_i = (\lambda_1^{(i)}, \lambda_2^{(i)}, \cdots, \lambda_{n_i}^{(i)}) \in \mathbf{F}_q^{n_i} \quad \text{for all } 1 \leq i \leq s,$$

and

$$\begin{aligned} 1 &\leq w_\gamma(\lambda_1, \lambda_2, \cdots, \lambda_s) \\ &= \max_{i=1}^{s}\{i|\lambda_{i)} \neq 0\} \leq d-1. \end{aligned}$$

If we carry out this algorithm to completion, then, $H_1, H_2, \cdots, H_s$ are the blocks of size $n_1, n_2, \cdots, n_s$ respectively of an $(n-k)\times n$ $\left(\text{where } n = \sum_{i=1}^{s} n_i\right)$ block matrix H such that no linear combination of blocks of H of γ-weight $(d-1)$ or less is zero meaning thereby that this matrix is the parity check matrix of an $lp\gamma$-code with minimum γ-distance at least d. We show that the construction can indeed be completed. Let j be an integer such that $2 \leq j \leq s$ and assume that the blocks $H_1, H_2, \cdots, H_{j-1}$ have been chosen. Then the block H_j can be added to H provided (7) is satisfied. The number of distinct linear combinations in (7) satisfying (8) including the pattern of all zeros is given by

$$V_{d-1,q}^{(n_1,\cdots,n_j)}$$

where $V_{d-1,q}^{(n_1,\cdots,n_j)}$ is given by (3).

As long as the set of all linear combinations occuring in (7) satisfying (8) is less than or equal to the total number of $(n-k)$-tuples, the j^{th} block H_j can be added to H. Therfore, the block H_j can be added to H provided that

$$q^{n-k} \geq V_{d-1,q}^{(n_1,\cdots,n_j)}$$

or

$$n-k \geq log_q\left(V_{d-1,q}^{(n_1,\cdots,n_j)}\right).$$

Thus the fact that the blocks $H_1, H_2, \cdots, H_s$ can be chosen follows by induction on j and we get (6). QED

Corollary 17. *Let $P : n = [n_1][n_2]\cdots[n_3]$ be the partition of a positive integer n. Let t be a positive integer satisfying $2t+1 \leq s$. Then, a sufficient condition for the existence of an $[n,k;P]$ $lp\gamma$-code over $\mathbf{F}_q$ that corrects all random block errors of γ-weight t or less is given by*

$$n-k \geq log_q\left(V_{2t,q}^{(n_1,\cdots,n_s)}\right).$$

Proof. The proof follows from Theorem 16 and the fact that to correct all errors of γ weight t or less, the minimum γ-distance of an $lp\gamma$-code must be at least $2t+1$. QED

Example 18. Let $n=3, k=1, d=2$ and $q=5$. Let $P : 3 = [1][2]$ be a partition of $n=3$. We show that for these values of the parameters, equation (6) is satisfied. We note that here $n_1 = 1, n_2 = 2$. Equation (6) for these parameters becomes

$$5^{3-1} \geq V_{1,5}^{(1,3)},$$

or

$$25 \geq 5 \quad (\text{since } V_{1,5}^{(1,3)} = 5),$$

which is true.

Therefore, by Theorem 16, there exists a $[3,1;P]$ $lp\gamma$-code V over $\mathbf{F}_5$ with minimum γ-distance at least 2.

Consider the following 2×3 block parity check matrix H of a $[3,1;P]$ $lp\gamma$-code V over $\mathbf{F}_5$ constructed by the algorithm discussed in Theorem 16:

$$H = \begin{pmatrix} 1 & \vdots & 0 & 2 \\ 0 & \vdots & 1 & 3 \end{pmatrix}_{2\times 3}.$$

We claim that the $lp\gamma$-code which is the null space of the matrix H has minimum γ- distance at least 2.

The generator matrix of the $lp\gamma$-code corresponding to the parity check matrix H is given by

$$G = [-2 \vdots -3 \quad 1]_{1\times 3} = [3 \vdots 2 \quad 1]_{1\times 3}$$

The five codewords of the $lp\gamma$-code V with G as generator matrix and H as parity check matrix are given by:

$$\begin{aligned} v_0 &= (0\vdots 00); w_\gamma(v_0) = 0, \\ v_1 &= (3\vdots 21); w_\gamma(v_1) = 2, \\ v_2 &= (1\vdots 42); w_\gamma(v_2) = 2, \\ v_3 &= (4\vdots 13); w_\gamma(v_3) = 2, \\ v_4 &= (2\vdots 34); w_\gamma(v_4) = 2. \end{aligned}$$

Therfore, the minimum γ- weight of the $lp\gamma$-code V is equal to 2. Hence Theorem 16 is verified.

Theorem 19 (Varshamov Bound). *Let $B_q(n,d;P)$ denote the largest number of code vectors in an $[n,k;P]$ $lp\gamma$-code V over $\mathbf{F}_q$ with $P : n = [n_1][n_2]\cdots[n_s]$ having minimum γ-distance at least d. Then*

$$B_q(n,d;P) \geq q^{n-\lceil log_q(L)\rceil},$$

where $L = V_{d-1,q}^{(n_1,\cdots,n_s)}$ is given by (3) and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Proof. By Theorem 16, there exists an $[n,k;P]$ $lp\gamma$-code over $\mathbf{F}_q$ with minimum γ-distance at least d provided

$$\begin{aligned} q^{n-k} &\geq V_{d-1,q}^{(n_1,\cdots,n_s)} = L \\ \Rightarrow n-k &\geq log_q(L) \\ \Rightarrow k &\leq n - log_q(L). \end{aligned}$$

The largest integer k satisfying the above inequality is $n - \lceil log_q(L)\rceil$. Thus

$$B_q(n,d;P) \geq q^{n-\lceil log_q(L)\rceil}$$

where $L = V_{d-1,q}^{(n_1,\cdots,n_s)}$ is given by (3).   QED

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