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A Generalization of Classical Connectivity Parameters for Graphs

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Abstract: In a weighted graph model, the reduction of flow between some pairs of nodes is more relevant and more frequent than the total disruption of the flow or the disconnection of the entire network. The concept of cuts are generalized to tackle with this type of problems. The classical edge and vertex connectivity parameters are generalized. Also a generalization of Whitney's theorem is presented.

Key words: weighted graph, partial cutnode, strength reducing set.

INTRODUCTION

Graph theory is applied in various fields like clustering analysis, operations reseedgeh, database theory, network analysis, information theory, etc. Connectivity concepts play a key role in applications related to graphs and weighted graphs. Several authors including Bondy and Fan [1-3], Broersma, Zhang and Li [13] introduced many connectivity concepts in weighted graphs following the works of Dirac [5] and Grĺotschel [6]. Also the authors have introduced the concepts of partial cutnodes, partial bridges and partial blocks in weighted graphs and characterized partial cutnodes and partial bridges recently [8-11]. Akram et al. introduced the concepts of bipolar fuzzy graphs and interval-valued fuzzy line graphs [15-19].

In this article, we generalize some of the connectivity concepts in graph theory. The best-fit model for any kind of network is a weighted graph. Also in many real world systems like information networks or electric circuits, the reduction of flow between pairs of nodes is more relevant and may frequently occur than the total disconnection of the entire network [7-8]. Motivated by this, we have introduced strength reducing sets of nodes and edges in weighted graphs [12]. This article generalizes the vertex and edge connectivity of a graph.

PRELIMINARIES

A *weighted graph* G is a graph in which every edge e is assigned a nonnegative number $w(e)$, called the *weight* of e. The set of all the neighbors of a vertex v in G is denoted by $N_G(v)$ or simply $N(v)$, and its cardinality by $d_G(v)$ or $d(v)$ [4]. The *weighted degree* of v is defined as $wd_G(v) = \sum_{x \in N(v)} w(vx)$. When no confusion occurs, we denote $wd_G(v)$ by $wd(v)$. The weight of a cycle is defined as the sum of the weights of its edges. An unweighted graph can be regarded as a weighted graph in which every edge e is assigned a weight $w(e) = 1$. Thus, in an unweighted graph, $wd(v) = d(v)$ for every vertex v, and the weight of a cycle is simply the length of the cycle. An *optimal cycle* is a cycle which has maximum weight[1].

In a weighted graph G, we can associate to each pair of nodes in G, a real number called strength of connectedness. It is evaluated using strengths of different paths joining the given pair of nodes. We have a set of new definitions which are given below.

Definition 1 [8] *Let G be a weighted graph. The strength of a path P (respectively, strength of a cycle C) of n edges e_i, for $1 \leq i \leq n$, denoted by $s(P)$ (respectively , $s(C)$), is equal to $s(P) = min_{1 \leq i \leq n}\{w(e_i)\}$.*

In a graph without weights, all paths are assumed to have strength one [4]. But in weighted graphs the strengths of paths may be different for different paths between pairs of nodes. Hence we have the definition,

Definition 2 [8] *Let G be a weighted graph. The strength of connectedness of a pair of nodes $u, v \in V(G)$, denoted by $CONN_G(u,v)$ is defined as $CONN_G(u,v) = Max\{s(P) : P$ is a $u-v$ path in $G\}$. If u and v are in different components of G, then $CONN_G(u,v) = 0$.*

Example 3 (Figure.1) Consider the following weighted graph with four vertices.

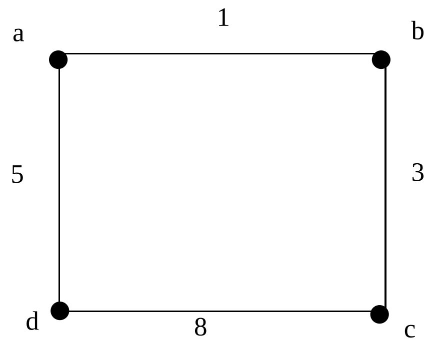

Figure.1: Strength of connectedness

Here, $CONN_G(a,b) = 3, CONN_G(a,c) = 5, CONN_G(a,d) = 5, CONN_G(b,c) = 3, CONN_G(b,d) = 3, CONN_G(c,d) = 8$.

Definition 4 [8] *A $u-v$ path in a weighted graph G is called a strongest $u-v$ path if $s(P) = CONN_G(u,v)$.*
Now we can generalize the concept of a cutnode as follows.
Definition 5 [8] *Let G be a weighted graph. A node w is called a partial cutnode (p-cutnode for short) of G if there exists a pair of nodes u, v in G, different from w, such that $CONN_{G-w}(u,v) < CONN_G(u,v)$.*
In example 3, the nodes c and d are p-cutnodes.
Analogous to that of p-cutnodes, we can define p-bridges. In a non-weighted graph, the removal of a bridge will disconnect the graph. But in a weighted graphs, the removal of a p-bridge will reduce the strength of connectedness between some pairs of nodes. Depending on these nodes, we can divide the p-bridges into three classes as given in the following definition.
Definition 6 [8] *Let G be a weighted graph. An edge $e = (u,v)$ is called a partial bridge(p-bridge for short) if $CONN_{G-e}(u,v) < CONN_G(u,v)$. A p-bridge is said to be a partial bond (p-bond for short) if $CONN_{G-e}(x,y) < CONN_G(x,y)$ with at least one of x or y different from both u and v and is said to be a partial cutbond (p-cutbond for short) if both x and y are different from u and v.*
In example 3, all edges except (a,b) are p-bridges. In particular $(b,c), (c,d)$ and (a,d) are partial bonds. Also edge (c,d) is a partial cutbond.
The concept of a strong edge is introduced in [8] by the authors as given below.
Definition 7 [8] *Let G be a weighted graph. Then an edge $e = (x,y) \in E$ is called α-strong if $CONN_{G-e}(x,y) < w(e)$, β-strong if $CONN_{G-e}(x,y) = w(e)$ and a δ-edge if $CONN_{G-e}(x,y) > w(e)$. A δ-edge e is called a δ^* - edge if e is not a weakest edge of G.*
Clearly an edge e is strong if it is either α-strong or β-strong. That is edge (x,y) is strong if its weight is at least equal to the strength of connectedness between x and y in G. If (x,y) is a strong edge, then x and y are said to be strong neighbors to each other.
Definition 8 [8] *A $u-v$ path P in G is called a strong $u-v$ path if all edges in P are strong. In particular if all edges of P are α-strong, then P is called an α-strong path and if all edges of P are β-strong, then P is called a β-strong path.*
Next the concepts of vertex cut and edge cut are generalized in [12] as follows.
Definition 9 [12] *A strength reducing set (srs) of nodes in a weighted graph G is a set of nodes $S \subseteq V(G)$ such that either $CONN_{G-S}(u,v) < CONN_G(u,v)$ for some pair of nodes $u, v \in V(G) - S$ or $G - S$ is trivial. If S contains a single node w, then w is a partial cutnode.*
Definition 10 [12] *A strength reducing set of edges in a weighted graph G is a set of edges $F \subseteq E(G)$ such that $CONN_{G-F}(u,v) < CONN_G(u,v)$ for some pair of nodes u, v in the graph obtained by deleting all edges in $G - F$ with at least one of u or v different from the end nodes of edges in F. If F contains a single edge e, then e is a partial bond.*
Example 11. (Figure.2) *Let $G(V,E)$ be a weighted graph with $V = \{a,b,c,d,f\}$ and $E = \{e_1 = (a,b), e_2 = (b,c), e_3 = (c,d), e_4 = (d,e), e_5 = (e,f), e_6 = (f,a), e_7 = (a,d)\}$ with $w(e_1) = w(e_3) = w(e_5) = 1, w(e_2) = w(e_4) = w(e_6) = 2, w(e_7) = 0.5$. $S = \{b,f\}$ is a strength reducing set of nodes since $CONN_{G-S}(a,d) = 0.5 < 1 = CONN_G(a,d)$. Also $F = \{(a,b),(e,f)\}$ is a strength reducing set of edges.*

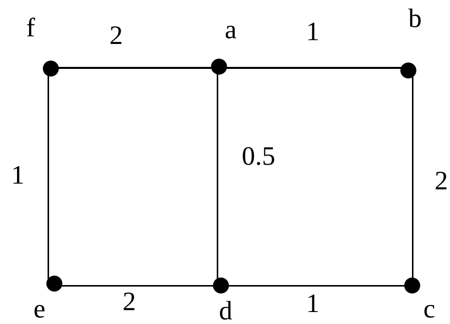

Figure.2: Strength reducing sets

The weighted degree of a weighted graph is discussed in [1]. We define a new type of degree in weighted graphs called strong degree as follows.
Definition 12 *Let G be a weighted graph. The* strong degree *of a node $v \in V(G)$ is defined as the sum*

of weights of all strong edges incident at v. *It is denoted by* $d_s(v)$. *The minimum strong degree of* G *is denoted by* $\delta_s(G)$ *and maximum strong degree* $\Delta_s(G)$.

Also if $N_s(v)$ denotes the set of all strong neighbors of v, then $d_s(v) = \sum_{u \in N_s(v)} w((u,v))$.

WEIGHTED CONNECTIVITY

In this section, we generalize the classical graph vertex and edge connectivity parameters κ and κ'. They will coincide with the classical parameters in an unweighted graph as all weights can be taken as one.

Definition 13 *Let* X *be a strength reducing set of nodes in a weighted graph. The strong weight of* X*, denoted by* $s(X)$ *is defined as* $s(X) = \sum_{x \in X} \mu(x,y)$*, where* $\mu(x,y)$ *is the minimum of the weights of strong edges incident on* x.

Definition 14 *The weighted node connectivity of a connected weighted graph* G *is defined as the minimum strong weight of strength reducing set of nodes in* G. *It is denoted by* $\kappa_w(G)$. *For a disconnected or trivial graph* G, $\kappa_w(G)$ *is defined to be* 0.

Example 15 *Let* $G(V,E)$ *be a weighted graph with* $V = \{u,v,w,x\}$ *and* $E = \{e_1 = (u,v), e_2 = (v,w), e_3 = (w,x), e_4 = (x,u)\}$ *with* $w(e_1) = 7, w(e_2) = 8, w(e_3) = 4, w(e_4) = 3$. *Clearly* $X_1 = \{v\}, X_2 = \{w\}, X_3 = \{v,w\}, X_4 = \{v,x\}$ *are strength reducing sets. In* G *all edges except* e_4 *are strong. Hence* $s(X_1) = 7, s(X_2) = 4, s(X_3) = 11$ *and* $s(X_4) = 11$. *Thus the weighted node connectivity of* G *is* $\kappa_w(G) = 4$.

As any δ -edge can be replaced by a path having more strength than its weight, any strength reducing set of edges will produce another strong strength reducing set with less number of edges while deleting all the δ-edge from the set. Thus we have,

Definition 16 *A strength reducing set of edges is said to be a strong strength reducing set of edges if it contains no* δ*-edges.*

Definition 17 The strong weight of a strength reducing set of edges F is defined as $s'(F) = \sum_{e_i \in E} \mu(e_i)$ where e_i is a strong edge in F.

Definition 18 The weighted edge connectivity $\kappa'_w(G)$ of a connected weighted graph G is defined as the minimum strong weight of strength reducing set of edges in G. $\kappa'_w(G)$ of a disconnected or trivial weighted graph is defined to be 0.

Example 19 *Let* $G(V,E)$ *be a weighted graph with* $V = \{u,v,w,x\}$ *and* $E = \{e_1 = (u,v), e_2 = (v,w), e_3 = (w,x), e_4 = (x,u), e_5 = (u,w)\}$ *with* $w(e_1) = 1, w(e_2) = 2, w(e_3) = 1, w(e_4) = 2, w(e_5) = 0.5$. *The reduction of a single edge will not reduce the strength of connectedness between any other pairs of nodes. Hence any strength reducing set of edge will contain at least two edges. Among them,* $F = \{(u,v),(w,x)\}$ *is the strength reducing set with a minimum strong weight of* 2. *Hence* $\kappa'_w(G) = 2$.

Theorem 20 *In a weighted tree* T, $\kappa_w(G) = \kappa'_w(G) =$ *minimum edge weight in* T

Proof. Let T be a weighted tree. Any internal node of T is a cutnode and hence is a partial cutnode. Thus any such node forms a strength reducing set and any strong edge in T must have at least one such node as an end node. Hence weighted node connectivity $\kappa_w(T)$ of T is the minimum weight of strong edges in T. Since each edge in T is strong, $\kappa_w(T)$ is the minimum weight of edges in T.

Also each edge in a tree is a bridge and hence is a partial bridge. Thus each edge is a strength reducing set. Since each edge of T is strong, it follows that the weighted edge connectivity of a weighted tree is the minimum weight of a strong edge in T. Since each edge of T is strong, it follows that the weighted edge connectivity of a weighed tree is the minimum weight of edges in T.

□

Next we present the weighted analogue of a famous result regarding node connectivity,edge connectivity and minimum degree of a graph due to Hassler Whitney

Theorem 21. In a connected weighted graph G, $\kappa_w(G) \le \kappa'_w(G) \le \delta_s(G)$.

Proof. First we shall prove the second inequality. Let G be a connected weighted graph . Let v be a node in G such that $d_s(v) = \delta_s(G)$. Let F be the set of strong edges incident at v. If these are the only edges incident at v, then $G - F$ is disconnected. If not, let (v,u) be an edge which is not strong, incident at v. Then u is a node different from the end nodes of edges in F. By definition of a strong edge,

$$w((u,v)) < CONN_G(u,v)$$

which implies that there exists a strongest $u - v$ path say P in G which should definitely pass through one of the strong edges at v. Thus the removal of F from G will reduce the strength of connectedness between v and u. Thus in both cases, F is a strength reducing set of edges. The strong weight of this srs is $\delta_s(G)$. Hence it follows that $\kappa'_w(G) \le \delta_s(G)$.

Next to prove $\kappa_w(G) \leq \kappa'_w(G)$, let F be an srs of edges with *strong weight* $\kappa'_w(G)$. We have the following cases.
Case 1: Every edge in F has one node in common, say, v.
In this case let $F = \{e_i = (v, v_i), i = 1, 2, .., n\}$.
Let $X = \{v_1, v_2, ..., v_n\}$. Then clearly X is an srs of nodes. Now,
$min_{u \in V(G)} w((v_i, u)) \leq w((v, v_i))$.
Therefore,
$\sum_i (min_{u \in V(G)} w((v_i, u)) \leq w((v_1, v)) + w((v_2, v)) + ... + w((v_n, v))$.
Thus $\kappa_w(G) \leq \kappa'_w(G)$.
Case 2: Not all edges in E have a node in common.
Let $F = \{e_i = (u_i, v_i), i = 1, 2, .., n\}$ for some n. Let $X_1 = \{u_1, u_2, ..., u_n\}$ and $X_2 = \{v_1, v_2, ..., v_n\}$. By assumption, $CONN_{G-F}(x, y) < CONN_G(x, y)$ for some pair of nodes $x, y \in V(G)$ with at least one of x or y is different from both u_i and v_i for $i = 1, 2, ..., n$.
Subcase 1: x and y are not members of $X_1 \bigcup X_2$
In this case, take $X = X_1$ or $X = X_2$. Then clearly X is an srs of nodes, since its deletion from G reduces the strength of connectedness between x and y and,
$\kappa_w(G) \leq$ *strong weight* of $X \leq$ *strong weight* of $F = \kappa'_w(G)$.
Subcase 2: Either x or y is in $X_1 \bigcup X_2$
Without loss of generality suppose that x is in $X_1 \bigcup X_2$. Let $x \in X_1$. Then take $X = X_2$. Clearly X is an srs of nodes for; the deletion of X from G will reduce the strength of connectedness between x and y. Thus,
$\kappa_w(G) \leq$ *strong weight* of $X \leq$ *strong weight* of $F = \kappa'_w(G)$.
Thus in all cases, $\kappa_w(G) \leq \kappa'_w(G) \leq \delta_s(G)$.

$\square$

CONCLUSIONS

The basic connectivity concepts in graph theory are generalized in this article. Connectivity concepts are important in the modelling of many real world situations. The classical parameters are dealing with the disconnection of the graph. In practical applications the reduction in the flow is more frequent than the disconnection. The authors made an attempt to generalize the connectivity concepts in weighted graphs. Also one of the major theorem in Graph theory due to Whitney is generalized.

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