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On Strongest Paths, Delta Arcs and Blocks in Fuzzy Graphs

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Abstract: In this paper we study strongest paths in a fuzzy graph. A necessary and sufficient condition for an arc in a fuzzy graph to be a strongest path and a sufficient condition for a path in a fuzzy graph to be a strongest path are obtained. A characterization of $\delta-$arcs and the relationship between fuzzy cutnodes and $\delta-$arcs are also obtained. Also a characterization of blocks is obtained using strongest paths.

Key words: Fuzzy graph, Strongest path, Fuzzy cutnodes, $\delta-$arcs, Blocks

INTRODUCTION AND PRELIMINARIES

Fuzzy graphs were introduced by Rosenfeld [1] in 1975. Rosenfeld has obtained the fuzzy analogues of several basic graph-theoretic concepts like bridges, paths,cycles,trees and connectedness and established some of their properties. Recently, Akram *et al.* introduced the concepts of bipolar fuzzy graphs and interval-valued fuzzy line graphs [2, 11-15]. Further he has defined length, distance, eccentricity, radius and diameter of a bipolar fuzzy graph and has introduced the concept of self centered bipolar fuzzy graphs[3].The author has also introduced the concept of an antipodal intuitionistic fuzzy graph and self median intuitionistic fuzzy graph of the given intuitionistic fuzzy graph[4]. Fuzzy graph theory is now finding numerous applications in modern science and technology especially in the fields of information theory, neural networks, expert systems,cluster analysis,medical diagnosis,control theory etc.

A fuzzy graph(f-graph) [5] is a triplet $G : (V,\sigma,\mu)$ where V the vertex set, σ is a fuzzy subset of V and μ is a fuzzy relation on σ such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ $\forall u,v \in V$. We assume that V is finite and non empty, μ is reflexive and symmetric. In all the examples σ is chosen suitably. Also we denote the underlying crisp graph [6] by G^* :(σ^*, μ^*) where $\sigma^* = \{u \in V : \sigma(u) > 0 \}$ and $\mu^* = \{(u, v) \in V \text{ x } V : \mu(u, v) > 0\}$. Here we assume $\sigma^* = V$. A fuzzy graph $H : (V,\tau,\nu)$ is called a partial fuzzy subgraph of $G : (V,\sigma,\mu)$ if $\tau(u) \leq \sigma(u)$ $\forall$u $\in \tau^*$ and $\nu(u, v) \leq \mu(u, v)$ $\forall (u, v) \in \nu^*$. In particular we call $H : (V,\tau,\nu)$ a fuzzy subgraph of $G : (V,\sigma,\mu)$ if $\tau(u) = \sigma(u)$ $\forall$u $\in \tau^*$ and $\nu(u, v) = \mu(u, v)$ $\forall (u, v) \in \nu^*$ and if in addition $\tau^* = \sigma^*$, then H is called a spanning fuzzy subgraph of G. A weakest arc of $G : (V, \sigma,\mu)$ is an arc with least membership value. A path P of length n is a sequence of distinct nodes $u_0,u_1,.....u_n$ such that $\mu(u_{i-1}, u_i) > 0$, i = 1,2,3 ...n and the degree of membership of a weakest arc in the path is defined as its strength. If $u_0 = u_n$ and $n \geq 3$, then P is called a cycle and a cycle P is called a fuzzy cycle(f-cycle) if it contains more than one weakest arc. The strength of connectedness between two nodes u and v is defined as the maximum of the strengths of all paths between u and v and is denoted by $CONN_G(u, v)$. A $u - v$ path P is called a strongest $u - v$ path if its strength equals $\text{CONN}_G(u, v)$. A fuzzy graph $G : (V, \sigma,\mu)$ is connected if for every u,v in σ^*, $CONN_G(u, v) > 0$. Throughout this , we assume that G is connected. An arc of a fuzzy graph is called strong if its weight is at least as great as the strength of connectedness of its end nodes when it is deleted and a $u - v$ path is called a strong path if it contains only strong arcs [7]. In a fuzzy graph a strongest path need not be a strong path and a strong path need not be a strongest path [8].

An arc (u, v) is a fuzzy bridge(f-bridge) of G if deletion of (u, v) reduces the strength of connectedness between some pair of nodes [5]. Equivalently, (u, v) is a fuzzy bridge if and only if there exist x, y such that (u, v) is an arc on every strongest $x - y$ path. A node is a fuzzy cutnode (f-cutnode) of G if removal of it reduces the strength of connectedness between some other pair of nodes [5]. Equivalently, w is a fuzzy cutnode if and only if there exist u, v distinct from w such that w is on every strongest $u - v$ path. A connected fuzzy graph $G : (V,\sigma,\mu)$ is a block if G has no fuzzy cutnodes.

A connected fuzzy graph $G : (V,\sigma,\mu)$ is a fuzzy tree (f-tree) if it has a spanning fuzzy subgraph $F : (V,\sigma,\nu)$, which is a tree, where for all arcs (x, y) not in F there exists a path from x to y in F whose strength is more than $\mu(x, y)$. Thus for all arcs (x, y) which are not in F, $\mu(x, y) < CONN_F(x, y)$.

Depending on the $CONN_G(u, v)$ of an arc (u, v) in a fuzzy graph G, strong arcs are further classified as $\alpha-$strong & $\beta-$strong and the remaining arcs are termed as $\delta-$arcs [8] as follows. Note that $G - (u, v)$ denotes the fuzzy subgraph of G obtained by deleting the arc (u, v) from G.

Definition 1 *An arc (u, v) in G is called $\alpha-$strong if $\mu(u, v) > CONN_{G-(u,v)}(u, v)$.*

Definition 2 *An arc (u, v) in G is called $\beta-$strong if $\mu(u, v) = CONN_{G-(u,v)}(u, v)$.*

Definition 3 *An arc (u, v) in G is called a $\delta-$arc if $\mu(u, v) < CONN_{G-(u,v)}(u, v)$.*

Definition 4 *A $\delta-$arc (u, v) is called a δ^*- arc if $\mu(u, v) > \mu(x, y)$ where (x, y) is a weakest arc of G.*

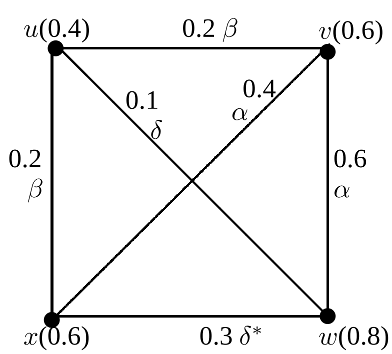

Figure 1. Fuzzy graph with different types of arcs.

Example 5 *In Figure 1, arcs (v, x) and (v, w) are $\alpha-$strong arcs, (u, v) and (u, x) are $\beta-$strong arcs, and (w, u) and (w, x) are $\delta-$arcs . Arc (w, x) is a δ^*-arc. The path $u - x - v - w$ is a strongest as well as a strong path. The path $u - x - w$ is a strongest path but not a strong path since the arc (x, w) is a δ^*-arc.*
In Figure 1, if $\mu(w, x)$ = 0.4 then arc (w, x) will become a $\beta-$strong arc and the path $v - x - w$ will become a strong path which is not strongest since $CONN_G(v, w)$ = 0.6 .

Remark 6 *Theorem 2.3.1 of [8] states that an arc (x, y) of a fuzzy graph $G : (V,\sigma,\mu)$ is an f-bridge if and only if it is $\alpha-$strong. Henceforth, an arc is $\alpha-$strong means it is an f-bridge and vice versa.*

STRONGEST PATH IN A FUZZY GRAPH

Remark 7 *If (u, v) is a fuzzy bridge then the arc (u, v) is the strongest $u - v$ path [9]. The following theorem generalizes this concept.*

Theorem 8 *An arc (u, v) in a fuzzy graph $G : (V,\sigma,\mu)$ is a strongest $u - v$ path if and only if (u, v) is either $\alpha-$strong or $\beta-$strong.*

Proof. Let $G : (V,\sigma,\mu)$ be a fuzzy graph. Let (u, v) be an arc in G. Consider the path $P : u - v$ in G. Then by definition of strength of a path,
Strength of $P = \mu(u, v)$ ...(1).
Suppose P is a strongest path, then Strength of $P = CONN_G(u, v)$. From (1), $\mu(u, v) = CONN_G(u, v)$ ...(2).
Strength of $P \geq$ Strength of all other $u - v$ paths. In particular, Strength of $P \geq CONN_{G-(u,v)}(u, v)$.Thus $CONN_G(u, v) \geq CONN_{G-(u,v)}(u, v)$...(3).
Now from (2) & (3), $\mu(u, v) \geq CONN_{G-(u,v)}(u, v) \Rightarrow$ Arc (u, v) is either $\alpha-$strong or $\beta-$strong.
Conversely assume that arc (u, v) is either $\alpha-$strong or $\beta-$strong. Then $\mu(u, v) \geq CONN_{G-(u,v)}(u, v) \Rightarrow CONN_G(u, v) = \mu(u, v)$. i.e ; $CONN_G(u, v) =$ Strength of P.
$\therefore P$ is a strongest path in G. □

Theorem 9 *Let $G : (V,\sigma,\mu)$ be a fuzzy graph and let (u, v) be any arc in a $u_0 - u_n$ path P such that strength of $P = \mu(u,v)$. Then P is a strongest $u_0 - u_n$ path if (u, v) is a strong arc and it is the unique weakest arc of P.*

Proof. Let $G : (V,\sigma,\mu)$ be a fuzzy graph. Let $P : u_0 - u_1 - u_2 - u_3 - ... - u_n$ be a $u_0 - u_n$ path in G where Strength of $P = \mu(u_{i-1}, u_i)$ for some $i = 1, 2, 3...n$. Let (u_{i-1}, u_i) be a strong arc and let it be the unique weakest arc in P. To prove P is a strongest $u_0 - u_n$ path. If possible suppose P is not a strongest $u_0 - u_n$ path. Let $P_1 : u_0 - v_1 - v_2 - v_3 - ... - v_{n-1} - u_n$ be a strongest $u_0 - u_n$ path in G, where all of $u_i, i = 1, 2, 3...n - 1$ and $v_j, j = 1, 2, 3...n - 1$ need not be distinct. Since Strength of P_1 is greater than Strength of P, we have strength of each arc of $P_1 > \mu(u_{i-1}, u_i)$. Also note that arc (u_{i-1}, u_i) is not a common arc for P and P_1. Then $P \cup P_1$ will contain at least one cycle and let C be one such cycle in which the only weakest arc is (u_{i-1}, u_i). Consider a $u_{i-1} - u_i$ path P' in C not containing the arc (u_{i-1}, u_i) . Clearly $\mu(u_{i-1}, u_i) <$ Strength of P'

and Strength of $P' \leq CONN_{G-(u_{i-1},u_i)}(u_{i-1}, u_i)$. $\therefore \mu(u_{i-1}, u_i) < CONN_{G-(u_{i-1},u_i)}(u_{i-1}, u_i)$, which implies (u_{i-1}, u_i) is a $\delta - arc$, which contradicts that (u_{i-1}, u_i) is a strong arc . Hence P must be a strongest $u_0 - u_n$ path in G.

□

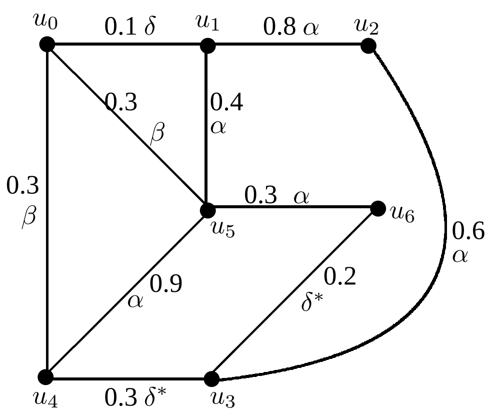

Figure 2. Strongest path containing δ^*- arc.

Remark 10 *The condition in the above theorem is not necessary. ie. If P is a strongest $u_0 - u_n$ path in G and (u, v) is an arc in P such that strength of $P = \mu(u,v)$, need not imply that (u, v) is the unique weakest arc in the path P and all arcs in P having strength equal to $\mu(u,v)$ need not be strong. In Figure 2, path $P : u_0 - u_4 - u_3 - u_2$ is a strongest path where strength of $P = 0.3$. The weakest arcs in the path are (u_0, u_4) and (u_4, u_3) where (u_4, u_3) is a δ^*-arc.*

Remark 11 *A strongest path in G may contain δ^*- arc. In Figure 2, $u_0 - u_4 - u_3 - u_2$ is a strongest $u_0 - u_2$ path in which (u_4, u_3) is a δ^*- arc.*

Remark 12 *A $\delta-$ arc in a strongest path in G need not be a δ^*- arc. In Figure 3, $u - x - y - v$ is a strongest $u - v$ path having the $\delta-$ arc(x, y) , which is not a δ^*- arc.*

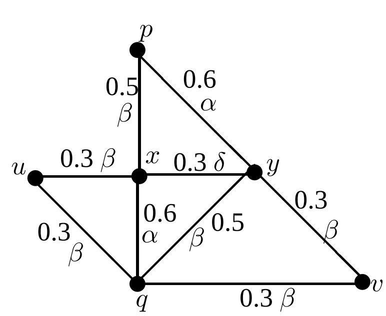

Figure 3. Strongest path containing $\delta-$ arc.

$\delta-$ ARCS IN A FUZZY GRAPH

Note that in a fuzzy graph G, a $\delta-$arc need not be a weakest arc of G. In Figure 2, the arc $(u_3 - u_4)$ is a $\delta-$arc which is not the weakest arc of G. In the following theorem a necessary and sufficient condition for an arc in a fuzzy graph to be a $\delta-$arc is discussed.

Theorem 13 *An arc (u, v) in a fuzzy graph $G : (V, \sigma, \mu)$ is a $\delta-$ arc if and only if (u, v) is the unique weakest arc of at least one cycle in G.*

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Let (u, v) be $\delta-$ arc in G. Then by Definition 3, $\mu(u, v) < CONN_{G-(u,v)}(u, v)$. i.e ; There exists at least one path P joining u and v and not containing the arc (u, v) such that Strength of $P > \mu(u, v)$. This path P together with the arc (u, v) forms a cycle in which (u, v) is the unique weakest arc.

Conversely, let (u, v) be the unique weakest arc of a cycle C in G. Let P be the $u - v$ path in C not containing the arc (u, v). Then,

$\mu(u, v) <$ Strength of P ...(1)

Suppose (u, v) is not a $\delta-$ arc in G. Then we have by Definition 3,

$\mu(u, v) \geq CONN_{G-(u,v)}(u, v)$ ...(2)

Also note that Strength of $P \leq CONN_{G-(u,v)}(u, v)$ ...(3)

From (2) and (3) we get $\mu(u, v) \geq$ Strength of P , which contradicts (1).

$\therefore (u, v)$ is a $\delta-$ arc in G.

□

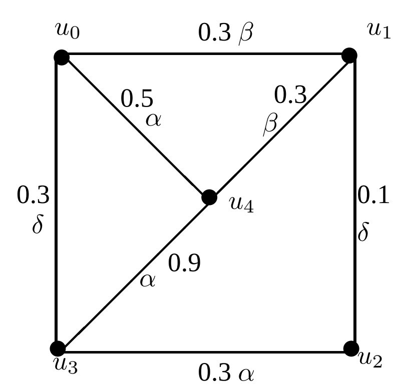

Figure 4. $\delta-$ arc as unique weakest arc of a cycle.

Example 14

In Figure 4, (u_0, u_3) is a $\delta-$ arc since it is the unique weakest arc in the cycle u_0 - u_4 - u_3 - u_0.

Lemma 15 *Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Any fuzzy cutnode w of G reduces strength of connectedness only* *between u, v ($u, v \neq w$) where (u, v) is either $\delta-$ arc or $\mu(u, v) = 0$.*

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Let w be a fuzzy cutnode of G. Let u, v ($u, v \neq w$) be any pair of nodes such that strength of connectedness between u and v is reduced by removal of w. To prove (u, v) is either $\delta-$ arc or $\mu(u, v) = 0$ it is enough to prove that arc (u, v) is neither $\alpha-$strong nor $\beta-$strong. Suppose arc (u, v) is either $\alpha-$strong or $\beta-$strong. Then by Theorem 8, we have $CONN_G(u, v) = \mu(u, v)$ and note that arc (u, v) $\in G - w$. $\therefore$ Removal of w will not reduce strength of connectedness between u and v. So the only possibility is that either (u, v) is $\delta-$ arc or $\mu(u, v) = 0$. $\square$

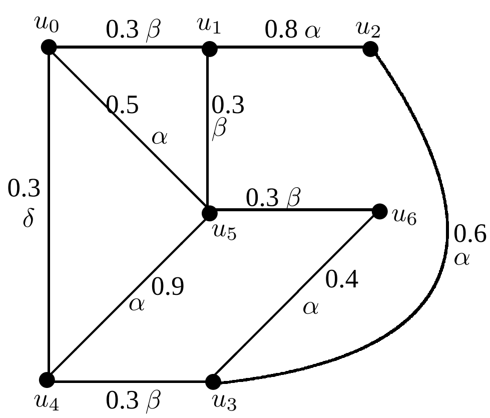

Figure 5. Effect of f-cutnode on $\delta-$ arc.

Example 16 *In Figure 5, u_5 is a fuzzy cut node and removal of u_5 reduces strength of connectedness between the nodes u_0 and u_4 and between the nodes u_0 and u_6, where (u_0, u_4) is a $\delta-$ arc and $\mu(u_0, u_6) = 0$.*

Remark 17 *Note that from Lemma 15, it follows that existence of an f-cutnode in G implies existence of at least one $\delta-$ arc provided G^* is a complete graph but the converse does not hold in general. In Figure 6, (u, v) and (x, w) are $\delta-$ arcs but there is no fuzzy cutnode.*

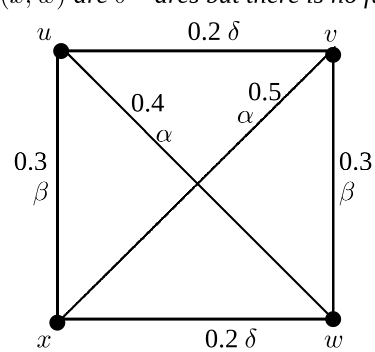

Figure 6. Fuzzy graph with $\delta-$arc but no f-cutnodes.

Proposition 18 *An arc (u, v) in $G : (V, \sigma, \mu)$ is $\beta-$ strong iff G has atleast one cycle C containing (u,v) and no $\delta-$ arcs.*

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph. Let (u, v) be a $\beta-$ strong arc in G. Then by Definition 2, $CONN_{G-(u,v)}(u, v) = \mu(u, v)$. ie. There exist a strongest $u - v$ path P not containing the arc (u, v) such that Strength of $P = \mu(u, v)$. ie. All arcs in P have strength atleast $\mu(u, v)$ with atleast one arc in P having strength $\mu(u, v)$. This path P together with the arc (u, v) forms a cycle C. If all arcs in C are either $\alpha-$ strong or $\beta-$ strong the result follows.

Suppose there is a $\delta-$ arc say (x, y) in the cycle C. Then by Theorem 13, there exists at least one cycle say C' in which (x, y) is the weakest arc. If all arcs in $C' - (x, y)$ are either $\alpha-$ strong or $\beta-$ strong, considering the cycle $C \cup C' - (x, y)$ the result follows. If $C' - (x, y)$ has a $\delta-$ arc, considering the cycle containing this $\delta-$ arc and proceeding similarly as above the result follows. The process is terminated since V is finite.

Conversely let C be a cycle in G having no $\delta-$ arc. For all arcs (x, y) in C we have $\mu(x, y) \geq CONN_{G-(x,y)}(x, y)$...(1)

Also the weakest arc in C is not unique for, otherwise it will be a $\delta-$ arc . Let (u, v) be a weakest arc in C. Consider the $u - v$ path P in C not containing the arc (u, v). Since the weakest arc is not unique in C , we have Strength of $P = \mu(u, v)$ From (1), $\mu(u, v) = CONN_{G-(u,v)}(u, v)$.
$\therefore (u, v)$ is a $\beta-$ strong arc. $\square$

FUZZY CUTNODES AND FUZZY BONDS IN A FUZZY GRAPH

In a connected fuzzy graph $G : (V, \sigma, \mu)$, a set of strong arcs $E = e_1, e_2, ...e_n$ with $e_i = (u_i, v_i)$,$i = 1, 2, 3, ...n$ is said to be a fuzzy arc cut (FAC) if either $CONN_{G-E}(x, y) < CONN_G(x, y)$ for some pair of nodes $x, y \in \sigma^*$ with at least one of x or y is different from both u_i and v_i, $i = 1, 2, 3, ...n$, or $G - E$ is disconnected. If there are n arcs in E then it is called an $n - FAC$ and a $1 - FAC$ is called a fuzzy bond(f-bond)[8].

Note that every f-bond is an f-bridge and not the converse. Also at least one of the end nodes of an f-bond is an f-cutnode (Remark 6.4.3 and Proposition 6.4.6 of [8]).

Example 19 *In Figure 7 there are two f-bonds ($1-FAC$) namely arcs (v, w) and (x, w). Removal of (v, w) and (x, w) reduces strength of connectedness between v and*

z. Note that arc (x, y) is an f-bridge which is not an f-bond.

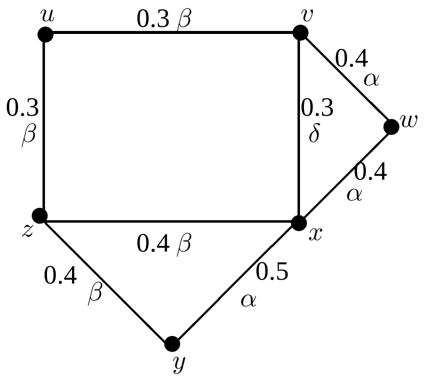

Figure 7. Fuzzy graph with f-bonds.

Theorem 20 *Let G : (V,σ,μ) be an f-graph. An arc (x, y) is an f-bond if and only if there exist a pair of nodes u and v with at least one of them different from both x and y such that (x, y) is on every strongest $u - v$ path where (u, v) is a $\delta-$ arc or $\mu(u, v) = 0$.*

Proof. Let G : (V,σ,μ) be a fuzzy graph and let (x, y) be an f-bond. Then by definition ,there exist a pair of nodes u and v with at least one of them different from both x and y such that (x, y) is on every strongest $u - v$ path. Also note that either x or y is an f-cutnode. Then by Lemma 15, the result follows.
Conversly, suppose that there exist a pair of nodes u and v with at least one of them different from both x and y such that the arc (x, y) is on every strongest $u - v$ path where (u, v) is a $\delta-$ arc or $\mu(u, v) = 0$. Then (x, y) is an f-bridge and removal of (x, y) reduces strength of connectedness betweeen u and v, with at least one of u or v different from both x and y and hence (x, y) is an f-bond. □

Remark 21 *An f-bond is an f-bridge with at least one of its end node as an f-cutnode but an f-bridge with one of its end node as an f-cutnode need not be an f-bond whereas an f-bridge with both its end node as f-cutnode is an f-bond. In Figure 7, x and w are f-cutnodes and arcs (x, y), (x, w) and (w, v) are f-bridges. Arc (x, y) having x as f-cutnode is not an f-bond where as arcs (x, w) and (w, v) are f-bonds. Arc (x, w) is an f-bond with both end nodes as f-cutnodes and arc (w, v) is an f-bond with only one end node (say) w as f-cutnode.*

Theorem 22 *Let G : (V,σ,μ) be an f-graph with at least one f-cutnode. If G^* is either K_3 or K_4 then G contains at least one $\alpha-$ strong arc.*

Proof. Let G : (V,σ,μ) be a fuzzy graph with at least one f-cutnode. Assume that G^* is K_3. Theorem 2 of [9] states that in a fuzzy graph G : (V,σ,μ) such that G^* :(σ^*, μ^*) is a cycle, a node is a fuzzy cut node if and only of it is a common of two fuzzy bridges. Hence the result. Now assume that G^* is K_4 and let $V = \{u, v, w, x\}$ be the vertex set of G.

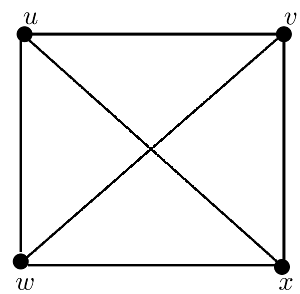

Figure 8. A Fuzzy graph where G^* is K_4 .

Let w be an f-cutnode of G. Then by Lemma 15, since G^* is complete, $\exists$ a $\delta-$arc say (u, v) in G, such that w is on every strongest $u - v$ path. The possible strongest $u - v$ paths are of length 2 and 3 say $P_1 : u-w-v$, $P_2 : u-w-x-v$ and $P_3 : u-x-w-v$.

Claim 1 *All the 3 paths P_1,P_2 and P_3 cannot be strongest $u - v$ paths simultaneously.*

Case 1 : P_1 and P_2 are strongest $u - v$ paths and P_3 is not a strongest $u - v$ path.
Suppose P_3 is also a strongest $u - v$ path. Since (u, x) is an arc of P_3 , $\mu(u, x) \geq CONN_G(u, v)$ ...(1) Similarly since (x, v) is an arc of P_2, $\mu(x, v) \geq CONN_G(u, v)$ ...(2) (1) and (2) implies that the path $P : u - x - v$ has strength at least $CONN_G(u, v)$, which means that P is a strongest $u - v$ path not containing w , which contradicts that w is an f-cutnode.

Case 2 : P_1 and P_3 are strongest $u - v$ paths and P_2 is not a strongest $u - v$ path.
Proof similar as in Case 1.

Claim 2 The paths P_2 and P_3 cannot be strongest $u - v$ paths simultaneously.
Suppose not,let P_2 and P_3 be strongest $u - v$ paths and P_1 is not a strongest $u - v$ path. Since P_1 is not a strongest $u - v$ path , either $\mu(u, w) < CONN_G(u, v)$ or $\mu(w, v) < CONN_G(u, v)$...(1) But (u, w) is an arc in P_2 and (w, v) is an arc in P_3. Hence by (1) Strength of $P_2 < CONN_G(u, v)$ and Strength of $P_3 < CONN_G(u, v)$. Therefore P_2 and P_3 are not strongest $u - v$ paths, contradiction. From the above two claims

the only possible cases of strongest $u-v$ paths are as follows.

Case 1 : P_1 and P_2 are the only strongest $u-v$ paths and (u,w) is an arc common in both P_1 and P_2 and hence (u,w) is an f-bridge.

Case 2 : P_1 and P_3 are the only strongest $u-v$ paths and (w,v) is an arc common in both P_1 and P_3 and hence (w,v) is an f-bridge.

Case 3: Pi is the only strongest $u-v$ path for $i=1,2$ or 3.

Since P_i is the unique strongest $u-v$ path, all arcs in P_i are f-bridges, by definition of f-bridge. Hence the result.

□

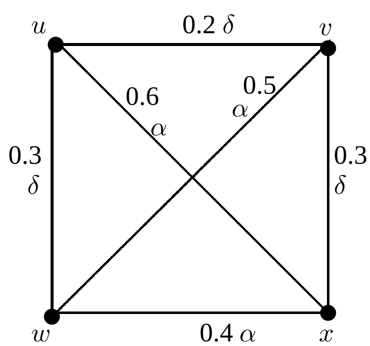

Figure 9. Fuzzy graph with f-cutnodes.

Example 23 *In Figure 9 nodes w and x are f-cutnodes. Removal of w reduces strength of connectedness between the nodes u and v and between the nodes v and x.Removal of x reduces strength of connectedness between the nodes u and v and between the nodes u and w.*

Remark 24 *If G^* is neither K_3 nor K_4 existence of f-cutnode need not imply $\alpha-$ strong arc.*

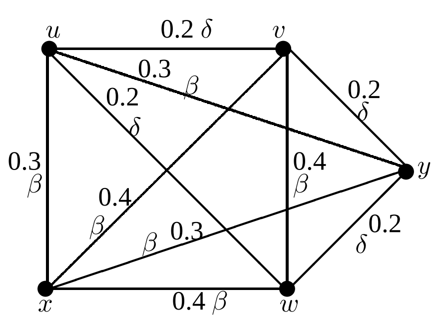

Figure 10. Fuzzy graph with f-cutnode but no $\alpha-$ strong arc.

In Figure 10 G^* is K_5 and there are no $\alpha-$ strong arcs. x is an f-cutnode , reduces strength of connectedness between u and v.

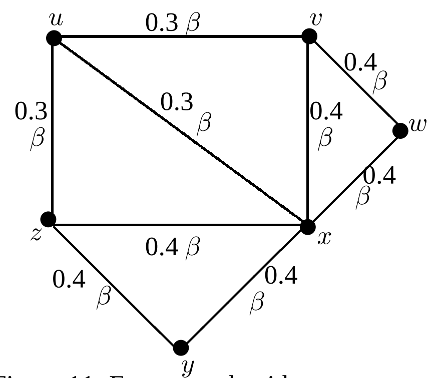

Figure 11. Fuzzy graph with no $\alpha-$strong arc and G^* not complete.

In Figure 11 x is an f-cutnode , reduces strength of connectedness between v and z.

CHARACTERIZATION OF BLOCKS

In [10] M.S. Sunitha and A. Vijayakumar proved that in a block, there exist two internally disjoint strongest paths joining every pair of nodes u and v such that (u,v) is not a fuzzy bridge and in [8] Sunil Mathew and M.S. Sunitha established that in a block, there exist two internally disjoint strongest strong paths joining every pair of nodes u and v such that (u,v) is not a fuzzy bridge . The following theorem gives a necessary and sufficient condition for a fuzzy graph G to be a block in general.

Theorem 25 *A fuzzy graph $G:(V,\sigma,\mu)$ is a block if and only if for every pair of nodes u,v, such that either (u,v) is a $\delta-$ arc or $\mu(u,v)=0$, there exists at least two internally disjoint strongest $u-v$ paths.*

Proof. Let $G:(V,\sigma,\mu)$ be a block. Theorem 11 of [10] states that for every pair of nodes u and v of G, such that (u,v) is not a fuzzy bridge there exists two internally disjoint strongest $u-v$ paths. Also note that (u,v) is not a fuzzy bridge means that (u,v) is a $\beta-$ strong arc or (u,v) is a $\delta-$ arc or $\mu(u,v)=0$. Hence the result follows.

Conversely assume that for every pair of nodes u,v such that either (u,v) is a $\delta-$ arc or $\mu(u,v)=0$, then there exists at least two internally disjoint strongest $u-v$ paths. On the contrary assume that G is not a block. Let w be a fuzzy cut node of G. Then by lemma 15, there exists at least one pair of nodes x, y $(x, y \neq w)$where (x, y) is either $\delta-$ arc or $\mu(x, y) = 0$ such that w is on all strongest paths joining x and y, which contradicts the assumption. Hence the Theorem.

□

Theorem 26 *A fuzzy graph $G : (V, \sigma, \mu)$ without $\delta-$ arc is a block provided G^* is a complete graph.*

Proof. Let $G : (V, \sigma, \mu)$ be a fuzzy graph such that G^* is a complete graph and G has no $\delta-$ arcs. By Lemma 15 and Remark 17 G has no f-cutnodes.
Hence the Theorem. □

Remark 27 *Converse of Theorem 26 is not true. The fuzzy graph in Figure 6 is a block containing $\delta-$ arcs.*

Remark 28 *A fuzzy graph G without $\beta-$ strong arcs is an f-tree by Theorem 2.4.3 of [8] and hence not a block. Note that the converse of this statement need not be true. ie.A fuzzy graph G which is not a block may contain $\beta-$ strong arcs. The f-graph in Figure 7 is not a block since x is an f-cutnode. But G contains $\beta-$ strong arcs namely $(x, v)$and (x, w).*

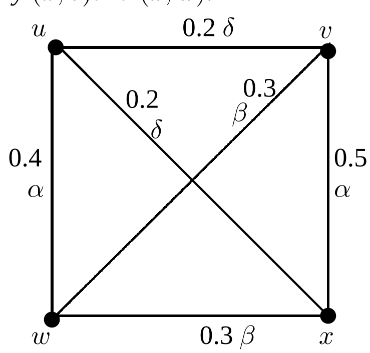

Figure 12. Fuzzy graph which is not a block containing $\beta-$ strong arcs.

Remark 29 A strongest path in a block may contain $\delta-$ arc or δ^*- arc. The fuzzy graph in Figure 3 is a block, in which (x, y) is a $\delta-$ arc in the strongest path $u - x - y - v$. Note that if we put $\mu(x, y)$ = 0.4 then still it is a block, but (x, y) becomes δ^*- arc in the strongest $u - x - y - v$ path.

ARCS INCIDENT ON A NODE OF A FUZZY GRAPH

In this section the types of arcs incident on a node is studied.

Theorem 30 *Let $G : (V, \sigma, \mu)$ be an f-graph. Let u_0 be a node in G such that for any path P with initial node u_0 , the arc incident on u_0 is the unique weakest arc of the path P and for any two nodes u_i,u_j, in G there exist at least one cycle in G containing the nodes u_i and u_j .Then we have*

(1) If all arcs incident on u_0 have same strength, then all these arcs are $\beta-$strong arcs.

*(*2) Among the arcs incident on u_0, if (u_0, u_n) is the unique arc with maximum strength, then arc (u_o, u_n) is $\alpha-$strong and all remaining arcs are $\delta-$ arcs).

3) Among the arcs incident on u_0 if there exist more than one arc with maximum strength, then all arcs with maximum strength arc are $\beta-$strong arcs and all the remaining arcs are $\delta-$ arcs.

Proof. 1) Let C : $u_0 - u_i - ... - u_j - u_0$ be any cycle in G. Now by hypothesis all the arcs incident on u_0 have strength less than all other arcs of G. By assumption (u_0, u_i) and (u_0, u_j) have same strength and they are the weakest arcs of the cycle C. Hence C is an f-cycle and (u_0, u_i) and (u_0, u_j) are $\beta-$strong arcs.

2) Let $u_1, u_2, u_3, ..., u_n$ be the nodes adjacent to u_0 and let (u_0, u_n) be the only arc with maximum strength. Then we have, $\mu(u_0, u_1) \leq \mu(u_0, u_2) \leq \mu(u_0, u_3) \leq ... \leq \mu(u_0, u_{n-1}) < \mu(u_0, u_n)$.
Consider a cycle C : $u_0 - u_i ... - u_n - u_0$,$i = 1, 2, ...n-1$. Now by hypothesis all the arcs incident on u_0 have strength less than all other arcs of G, we have all arcs in C except (u_0, u_i) have strength greater than $\mu(u_0, u_n)$. Also since $\mu(u_0, u_i) < \mu(u_0, u_n)$,$(u_0, u_i)$ is the unique weakest arc in C and hence (u_0, u_i) is a $\delta-$ arc, by Theorem 13. Also we have $\mu(u_0, u_n) > CONN_{G-(u_0,u_n)}(u_0, u_n)$. Hence (u_0, u_n) is $\alpha-$strong arc.

3) If all arcs incident on u_0 have same strength, then by case (1) all arcs are $\beta-$strong.
Let m be the maximum of strength of arcs incident on u_0. Now suppose there exist at least one arc say (u_0, u_i) such that $\mu(u_0, u_i) < m$, $i = 1, 2..., n$.
Consider the cycle C : $u_0 - u_i - ... - u_j - u_0$, $i \neq j$. Now by hypothesis all the arcs incident on u_0 have strength less than all other arcs of G, then as in case(2) (u_0, u_i) is the unique weakest arc in C and hence (u_0, u_i) is a $\delta-$ arc, by Theorem 13. Now consider a cycle C through u_0 containing the arcs with maximum strength. Then as in case(1), arcs incident on u_0 are $\beta-$strong arcs. □

CONCLUSION

The concept of strongest paths is studied using strong arcs. The relationship between f-cutnode and $\delta-$ arcs is explored using which blocks are characterized. Some properties of fuzzy bonds is analyzed and a characterization for an arc to be a fuzzy bond is obtained. Also an observation is made on the types of arcs incident on a node of a fuzzy graph.

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