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Tensorial approximate identities for vector valued functions

M. Akram

Department of Mathematics, GC University, Katchery Road Lahore, 54000, Pakistan

dr.makram@gcu.edu.pk, makram71@yahoo.com

Abstract: In this article, we introduce a tensorial kernel for vector valued functions which is compactly supported on a bounded region V in $\mathbb{R}^3$. It is proved that the convolutions of the derived kernels with a vector valued function f having continuous first derivatives converges to f , i.e. the defining property of an approximate identity, is proved. Further, some numerical tests are shown.

Key words: Green's function, weak convergence, approximate identity, tensorial kernel.

INTRODUCTION

The rule of an approximate identity is as follows: A function f which has to be approximated is convolved with a kernel K_δ for some δ. If the kernel satisfies certain conditions then the convolutions converge in a certain limit such as $\delta \to 0+$ to f. Note that approximate identities for scalar function on balls in $\mathbb{R}^3$ are studied e.g. in [14]. Further works on localizing kernels, like scaling functions and wavelets, on the 3D ball can e.g. be found in [3, 5, 8, 12, 13, 15]. In this paper we show how tensorial kernels on a 3-dimensional bounded region can be constructed. We prove that these kernels establish an approximate identity for all vector valued functions with continuous first derivative. The convergence established in this case is in the sense of a distribution. Particular practical relevance is the case when the considered bounded region is a ball. An example of an application of approximating structures on a three-dimensional bounded region can be found in geophysics. There, the choice of appropriate tools for describing features of the Earth's interior, such as the mass density, the speed of propagation of seismic P and S waves and other rheological quantities, is still a field of research. Moreover, the approximation of vectorial fields such as currents on a ball is also relevant in medical imaging.

2. TEST FUNCTIONS AND DISTRIBUTIONS

Throughout the sequel, $\mathbb{R}^n, n \in \mathbb{N}$ represents the n-dimensional Euclidean space. The following introduction to the theory of distributions is based on [4], [9] and [10], where further details about this subject can be found.

Let V be an open and bounded subset of $\mathbb{R}^3$, we denote the set of functions $F : V \to \mathbb{R}$ with continuous derivatives of all orders by $\mathrm{C}^\infty(V)$, the set of functions $f : V \to \mathbb{R}^3$ with continuous derivatives of all orders by $\mathrm{c}^\infty(V)$ and and for every compact subset K of V, $\mathrm{C}_K^\infty(V)$ is the set of all functions in $\mathrm{C}^\infty(V)$ with support in K. $\mathrm{C}_K^\infty(V)$ is a linear space and the topology which makes it into a Fréchet space is induced by the seminorms

$$P_m(\phi) = \sup\left\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \le m\right\},$$

$\phi \in \mathrm{C}_K^\infty(V),\ m \in \mathbb{N}_0$, with the sets

$$D_m(r) = \left\{\phi \in \mathrm{C}_K^\infty(V) : P_m(\phi) < r\right\},$$

as a local base (see [4]). Where, a local base at a point x, we means, a collection of base open sets in which every open set containing x contains one of these base sets, which contains x. Moreover, $\mathrm{C}_K^\infty(V)$ is also a closed subspace of $\mathrm{C}_0^\infty(V)$, where we define the topology of

$$\mathrm{C}_0^\infty(V) := \cup_{K\subset V} \mathrm{C}_K^\infty(V)$$

to be the finest locally convex topology for which every linear functional defined on $\mathrm{C}_0^\infty(V)$ is continuous if and only if its restriction to $\mathrm{C}_K^\infty(V)$ is continuous for every $K \subset V$. This topology is known as the inductive limit of the topologies on $\mathrm{C}_K^\infty(V)$, i.e. the topologies of $\mathrm{C}_K^\infty(V)$ are "pieced together" to form the topology of the union $\mathrm{C}_0^\infty(V)$ (see [16]).
The locally convex space $\mathrm{C}_0^\infty(V)$ endowed with the inductive limit topology, is called the space of test functions and is commonly denoted by $\mathscr{D}(V)$, in accordance with Schwartz' notation in [17]. A distribution on V is a continuous linear functional on $\mathscr{D}(V)$.

We shall denote the linear space of all distributions on V by $\mathscr{D}'(V)$, the topological dual of $\mathscr{D}(V)$.
A function F defined on V is locally integrable on V if

$$\int_E |F(x)|dx$$

is finite on every compact subset E of V and we shall use $\mathrm{L}^1_{\mathrm{loc}}(V)$ to denote the space of all locally integrable functions on V. All continuous functions on $\mathbb{R}^n$, for example, are locally integrable, although some of them, such as polynomials, are not integrable on $\mathbb{R}^n$. Clearly,

$$\mathrm{L}^1(V) \subset \mathrm{L}^1_{\mathrm{loc}}(V).$$

If $F \in \mathrm{L}^1_{\mathrm{loc}}(V)$ then the linear functional T_F defined on $\mathscr{D}(V)$ by

$$T_F(\phi) = \int_V F(x)\phi(x)dx, \quad \phi \in \mathscr{D}(V) \tag{1}$$

is a distribution. A distribution T is said to be regular if there is a locally integrable function f on V such that

$$T(\phi) = \int_V F(x)\phi(x)dx, \quad \phi \in \mathscr{D}(V).$$

Otherwise it is singular. Among the topologies that can be defined on the vector space $\mathscr{D}'(V)$, the most important is the one known as the weak topology. This is the locally convex topology defined by the family of seminorms

$$P_\phi(T) = |T(\phi)|$$

with $\phi \in \mathscr{D}(V)$ and $T \in \mathscr{D}'(V)$, and it leads to the following definition of (weak) convergence in $\mathscr{D}'(V)$. The sequence $\{T_k\}$ in $\mathscr{D}'(V)$ converges to 0 if and only if, for every $\phi \in \mathscr{D}(V)$, the sequence $T_k(\phi)$ converges to 0 in $\mathbb{C}$.
This is really a "pointwise" convergence on $\mathscr{D}(V)$ and, as usual, we shall write

$$T_k \to T \ \text{ in } \mathscr{D}'(V)$$

if the sequence $(T_k - T)$ converges to 0 in the sense of the above definition. The space of distributions $\mathscr{D}'(V)$ is (sequentially) complete.

Theorem 1. *If $T_k \in \mathscr{D}'(V)$ for every $k \in \mathbb{N}$ and $\lim T_k = T$, then $\lim \partial^\alpha T_k = \partial^\alpha T$ for every multi-index $\alpha \in \mathbb{N}_0^n$, where the derivatives are understood in the distributional sense.*

3. INTERCHANGE OF LIMIT WITH DIFFERENTIAL AND INTEGRAL OPERATOR

Definition 2. *The function G_0 defined by*

$$G_0(x,y) := \frac{1}{4\pi|x-y|}$$

for all $x, y \in \mathbb{R}^3$ with $x \neq y$ is a Green's function for the operator $-\nabla^2$ in the space $\mathbb{R}^3$ (see [6]).

Definition 3. *For $\delta \in (0,1)$, we may define the regularised Green's function with respect to $-\nabla^2$ for all $x, y \in \mathbb{R}^3$ by*

$$G_0^\delta(x,y) := \frac{1}{4\pi\left(|x-y|^2+\delta\right)^{\frac{1}{2}}}.$$

Theorem 4. *Let V be a bounded region in $\mathbb{R}^3$ and $\overline{V}$ represent the closure of V. Let f be a vector valued function, which is defined in the domain $\overline{V}$ such that f is continuous on $\overline{V}$ and has continuous first order derivatives in V. Then $f = -\nabla\rho + \nabla \times \Lambda$ with*

(i)

$$\nabla_x \rho = \lim_{\delta\to 0+} \nabla_x \rho^\delta \textit{ in } \mathscr{D}'(V)$$

with

$$\rho^\delta(x) = \int_V \nabla_x G_0^\delta(x,y) \cdot f(y)\, dy$$

and

$$\rho(x) = \int_V \nabla_x G_0(x,y) \cdot f(y)\, dy$$

(ii)

$$\nabla_x \times \Lambda = \lim_{\delta\to 0+} \nabla_x \times \Lambda^\delta \textit{ in } \mathscr{D}'(V)$$

with

$$\Lambda^\delta(x) = \int_V \nabla_x G_0^\delta(x,y) \times f(y)\, dy$$

and

$$\Lambda(x) = \int_V \nabla_x G_0(x,y) \times f(y)\, dy$$

for all $x \in V$, where G_0^δ is the regularised Green's function for $-\nabla^2$ on $\mathbb{R}^3$ and G_0 is the Green's function for $-\nabla^2$ on $\mathbb{R}^3$.

Proof. From Corollary 3.4 (i) of [2], we have

$$\rho^\delta \to \rho \text{ as } \delta \to 0+ \ \text{ in } \mathscr{D}'(V),$$

i.e. the sequence of distributions T_{ρ^δ} defined by

$$T_{\rho^\delta}(\phi) := \int_V \rho^\delta(x)\phi(x)dx$$

converges to

$$T_\rho(\phi) := \int_V \rho(x)\phi(x)dx \text{ as } \delta \to 0+$$

for all ϕ in $\mathscr{D}(V)$. Therefore, we can say $T_{\rho^\delta} \to T_\rho$ as $\delta \to 0+$ in $\mathscr{D}'(V)$. Due to Theorem 1 we have

$$\lim_{\delta\to 0+} \nabla_x T_{\rho^\delta} = \nabla_x \lim_{\delta\to 0+} T_{\rho^\delta} = \nabla_x T_\rho$$

in $\mathscr{D}'(V)$. This implies that

$$\lim_{\delta\to 0+} \nabla_x \rho^\delta = \nabla_x \lim_{\delta\to 0+} \rho^\delta = \nabla_x \rho$$

in $\mathscr{D}'(V)$. This proves part (i).
For (ii) once again, we use (ii) of Corollary 3.4 of [2], which gives

$$\Lambda = \lim_{\delta\to 0+} \Lambda^\delta \text{ in } \mathscr{D}'(V).$$

It means that the real components $\Lambda_1^\delta, \Lambda_2^\delta$ and Λ_3^δ of Λ^δ converge to the corresponding real components Λ_1, Λ_2 and Λ_3 of Λ. To prove

$$\nabla_x \times \Lambda = \lim_{\delta\to 0+} \nabla_x \times \Lambda^\delta \text{ in } \mathscr{D}'(V),$$

we have to show that

$$(\nabla_x \times \Lambda)_i = \lim_{\delta\to 0+} (\nabla_x \times \Lambda^\delta)_i \text{ for } i = 1, 2, 3 \text{ in } \mathscr{D}'(V).$$

Here we show that $(\nabla_x \times \Lambda^\delta)_1 \to (\nabla_x \times \Lambda)_1$ in $\mathscr{D}'(V)$. For the other components the proof is similar to this one. We know that

$$(\nabla_x \times \Lambda^\delta)_1 = \frac{\partial}{\partial x_2}\Lambda_3^\delta - \frac{\partial}{\partial x_3}\Lambda_2^\delta$$

and

$$(\nabla_x \times \Lambda)_1 = \frac{\partial}{\partial x_2}\Lambda_3 - \frac{\partial}{\partial x_3}\Lambda_2.$$

As we know that

$$\Lambda_2^\delta \to \Lambda_2 \text{ in } \mathscr{D}'(V)$$

and

$$\Lambda_3^\delta \to \Lambda_3 \text{ in } \mathscr{D}'(V)$$

therefore using Theorem 1 we have

$$\frac{\partial}{\partial x_3}\Lambda_2^\delta \to \frac{\partial}{\partial x_3}\Lambda_2 \text{ in } \mathscr{D}'(V)$$

and

$$\frac{\partial}{\partial x_3}\Lambda_2^\delta \to \frac{\partial}{\partial x_2}\Lambda_3 \text{ in } \mathscr{D}'(V).$$

From the information given above, we get

$$\left(\nabla_x \times \Lambda^\delta\right)_1 \to \left(\nabla_x \times \Lambda\right)_1 \text{ in } \mathscr{D}'(V).$$

Similarly

$$\left(\nabla_x \times \Lambda^\delta\right)_2 \to \left(\nabla_x \times \Lambda\right)_2 \text{ in } \mathscr{D}'(V)$$

and

$$\left(\nabla_x \times \Lambda^\delta\right)_3 \to \left(\nabla_x \times \Lambda\right)_3 \text{ in } \mathscr{D}'(V).$$

Combining all these we get

$$\left(\nabla_x \times \Lambda^\delta\right) \to \left(\nabla_x \times \Lambda\right) \text{ in } \mathscr{D}'(V).$$

■

Theorem 5. *([11], p. 335) Let X be an open subset of $\mathbb{R}^m$ and Y be a measure space. Suppose that the function $f : X \times Y \to \mathbb{R}$ satisfies the following conditions:*
(i) $f(x, y)$ is a measurable function of y for each $x \in X$.
(ii) For almost all $y \in Y$, the derivative $\frac{\partial f(x,y)}{\partial x_i}$ exists for all $x \in X$.
(iii) There is an integrable function $G : Y \to \mathbb{R}$ such that $\left|\frac{\partial f(x,y)}{\partial x_i}\right| \leq G(y)$ for all $x \in X$.
Then

$$\frac{\partial}{\partial x_i}\int_Y f(x, y)dy = \int_Y \frac{\partial}{\partial x_i} f(x, y)dy.$$

Theorem 6. *Let f, ρ^δ, Λ^δ and G_0^δ be the same as in Theorem 4. Then*

1.

$$\int_V \nabla_x \left(\nabla_x G_0^\delta(x, y) \cdot f(y)\right) dy = \nabla_x \int_V \nabla_x G_0^\delta(x, y) \cdot f(y)\, dy$$

$$= \nabla_x \rho^\delta(x)$$

2.

$$\int_V \nabla_x \times \left(\nabla_x G_0^\delta(x, y) \times f(y)\right) dy = \nabla_x \times \int_V \nabla_x G_0^\delta(x, y) \times f(y)\, dy$$

$$= \nabla_x \times \Lambda^\delta(x)$$

Proof. 1. To prove the result, we have to show that
(i) $\nabla_x G_0^\delta(x, y) \cdot f(y) =: H(x, y)$ is a measurable function of y for each $x \in V$.
(ii) For almost all $y \in V$, the expression $\nabla_x H(x, y)$ exists for all $x \in V$.
(iii) There is an integrable function $G : V \to \mathbb{R}$ such that

$$|\nabla_x H(x, y)| \leq G(y),$$

for all $x \in V$.
Due to Equations (11) and (12) of [2], we conclude that (i) is satisfied. Further, from Equation (A.1.5) of [1], p. 101, we obtain (ii). To show that (iii) is satisfied, we use

Equation (A.1.5) of [1], p. 101 as

$$\left|\nabla_x\left(\nabla_x G_0^\delta(x,y)\cdot f(y)\right)\right|^2=\frac{1}{(4\pi)^2}\times \left(\left(\frac{\left(|x-y|^2+\delta\right)F_1(y)}{(|x-y|^2+\delta)^{\frac{5}{2}}}-3\frac{((x-y)\cdot f(y))\,(x_1-y_1)}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right)^2 +\left(\frac{\left(|x-y|^2+\delta\right)F_2(y)}{(|x-y|^2+\delta)^{\frac{5}{2}}}-3\frac{((x-y)\cdot f(y))\,(x_2-y_2)}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right)^2 +\left(\frac{\left(|x-y|^2+\delta\right)F_3(y)}{(|x-y|^2+\delta)^{\frac{5}{2}}}-3\frac{((x-y)\cdot f(y))\,(x_3-y_3)}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right)^2\right)$$

where $f=(F_1,F_2,F_3)^t$. After simplying, we obtain

$$\left|\nabla_x\left(\nabla_x G_0^\delta(x,y)\cdot f(y)\right)\right|\le\sqrt{3}\left(\frac{\delta+16R^2}{4\pi\delta^{\frac{5}{2}}}\right)|f(y)| =: G(y), \tag{2}$$

where R is the radius of the ball containing $\overline{V}$. Since f is an integrable function of y therefore G is also an integrable function of y. Hence, (iii) is satisfied. Finally, using Theorem 5 we get **1.**

2. Due to Equation (17) of [2], we can say that $\nabla_x G_0^\delta(x,y)\times f(y)$ is a measurable function of y for all $x\in V$. From Equations (A.1.8), (A.1.9) and (A.1.10) of [1], p. 103, we conclude that $\nabla_x\times\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)$ exists for all $x\in V$. Further, from Equation (A.1.8) of [1], p. 103, we get

$$\left|\left(\nabla_x\times\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)\right)_1\right|\le\frac{1}{4\pi}\times \left(\frac{2\left(|x-y|^2+\delta\right)|F_1(y)|}{(|x-y|^2+\delta)^{\frac{5}{2}}} +3\frac{|((x_1-y_1)F_2(y)-(x_2-y_2)F_1(y))\,(x_2-y_2)|}{(|x-y|^2+\delta)^{\frac{5}{2}}} +3\frac{|((x_3-y_3)F_1(y)-(x_1-y_1)F_3(y))\,(x_3-y_3)|}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right). \tag{3}$$

After simplifying, we have

$$\left|\left(\nabla_x\times\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)\right)_1\right|\le\left(\frac{\delta+28R^2}{2\pi\delta^{\frac{5}{2}}}\right)\times|f(y)|, \tag{4}$$

where R is the radius of the ball containing $\overline{V}$. Similarly, from Equations (A.1.9) and (A.1.10) of [1], p. 103, we get

$$\left|\left(\nabla_x\times\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)\right)_2\right|\le\left(\frac{\delta+28R^2}{2\pi\delta^{\frac{5}{2}}}\right)\times|f(y)| \tag{5}$$

$$\left|\left(\nabla_x\times\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)\right)_3\right|\le\left(\frac{\delta+28R^2}{2\pi\delta^{\frac{5}{2}}}\right)\times|f(y)|, \tag{6}$$

where R is the radius of the ball containing $\overline{V}$. From the discussion above, we can conclude that

$$\left|\left(\nabla_x\times\left(\nabla_x G_0^\delta(x,y)\times f(y)\right)\right)\right| \le\left(\frac{\delta+28R^2}{2\pi\delta^{\frac{5}{2}}}\right)\sqrt{3}|f(y)| =: \tilde{G}(y). \tag{7}$$

Since $|f|$ is an integrable function of y, therefore, $\tilde{G}$ is also an integrable function of y. Hence, applying Theorem 5, we finally obtain **2.** ∎

4. TENSORIAL KERNELS

let $c(\overline{V})$ denote the set of continuous functions $f:\overline{V}\to\mathbb{R}^3$ and using the canonical orthonormal basis $\varepsilon^1,\varepsilon^2,\varepsilon^3$ of $\mathbb{R}^3$, a tensor **f** of rank-2 can be written as

$$\mathbf{f}=\sum_{i,j=1}^{3}F_{ij}\varepsilon^i\otimes\varepsilon^j \tag{8}$$

where $\otimes$ is the dyadic product and $F_{ij}\in\mathbb{R}$. Clearly, $\mathbf{f}\in\mathbb{R}^3\otimes\mathbb{R}^3$.

Let $\mathbf{k}:\overline{V}\times\overline{V}\to\mathbb{R}^3\otimes\mathbb{R}^3$ and $f\in c(\overline{V})$, we define the convolution of f and **k** by

$$\mathbf{k}\circledast f:=\int_{\overline{V}}\mathbf{k}\,(.,y)\,f(y)dy \tag{9}$$

or

$$f\circledast\mathbf{k}:=\int_{\overline{V}}f^{\mathrm{t}}(y)\mathbf{k}^{\mathrm{t}}\,(.,y)\,dy, \tag{10}$$

where $f^{\mathrm{t}}(y)$ is the transpose of the vector $f(y)$ and $\mathbf{k}^{\mathrm{t}}\,(x,y)$ is the transpose of the tensor $\mathbf{k}\,(x,y)$. As we use Lebesgue integrals therefore

$$\int_{\overline{V}}\mathbf{k}\,(.,y)\,f(y)dy=\int_{V}\mathbf{k}\,(.,y)\,f(y)dy.$$

Theorem 7. *Let $\boldsymbol{k}_\delta:\overline{V}\times\overline{V}\to\mathbb{R}^3\otimes\mathbb{R}^3$ be a tensorial function defined by*

$$\boldsymbol{k}_\delta\,(x,y)=\sum_{i,j=1}^{3}K_{ij}^\delta\,(x,y)\,\varepsilon^i\otimes\varepsilon^j \tag{11}$$

for all $x, y \in \overline{V}$ and $\delta > 0$. For $i, j = 1, 2, 3$, we define $K^{\delta}_{ij} : \overline{V} \times \overline{V} \to \mathbb{R}$ as below

$$K^{\delta}_{ij}(x,y) = \frac{3}{4\pi}\left(\frac{1}{(|x-y|^2+\delta)^{\frac{3}{2}}} - \frac{|x-y|^2}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right),$$

if $i = j$ and for all $x, y \in \overline{V}$. If $i \neq j$ we define

$$K^{\delta}_{ij}(x,y) = 0 \tag{12}$$

for all $x, y \in \overline{V}$.

Let f be a vector valued function which is defined in the domain $\overline{V}$ such that f is continuous on $\overline{V}$ and has continuous first order derivatives in V. Then $f \circledast \mathbf{k}_\delta \to f$ as $\delta \to 0+$ in $\mathscr{D}'(V)$.

Proof. From Equation (10) we have

$$\begin{aligned}(f \circledast \mathbf{k}_\delta)(x) &= \int_V f^t(y)\mathbf{k}^t_\delta(x,y)dy \\ &= \int_V (F_1(y), F_2(y), F_3(y)) \left(K^{\delta}_{ji}(x,y)\right)_{3\times 3} dy,\end{aligned}$$

where $F_i,\ i = 1, 2, 3$ are the components of f and $\left(K^{\delta}_{ji}(x,y)\right)$ is a 3×3 matrix with components $K^{\delta}_{ji}(x,y)$. Now just performing the multiplication of matrices, we get

$$\begin{aligned}(f \circledast \mathbf{k}_\delta)(x) = \int_V \big(&F_1(y)K^{\delta}_{11}(x,y) + F_2(y)K^{\delta}_{21}(x,y) \\ &+F_3(y)K^{\delta}_{31}(x,y), \\ &F_1(y)K^{\delta}_{12}(x,y) + F_2(y)K^{\delta}_{22}(x,y) + F_3(y)K^{\delta}_{32}(x,y), \\ &F_1(y)K^{\delta}_{13}(x,y) + F_2(y)K^{\delta}_{23}(x,y) + F_3(y)K^{\delta}_{33}(x,y)\big)\, dy.\end{aligned} \tag{13}$$

Let us investigate the components one by one. Using Equations (12) and (12) we obtain

$$\begin{aligned}&F_1(y)K^{\delta}_{11}(x,y) + F_2(y)K^{\delta}_{21}(x,y) + F_3(y)K^{\delta}_{31}(x,y) \\ &= \frac{1}{4\pi}\left(\frac{3F_1(y)}{(|x-y|^2+\delta)^{\frac{3}{2}}} - 3F_1(y)\frac{|x-y|^2}{(|x-y|^2+\delta)^{\frac{5}{2}}} \right. \\ &\left. \qquad +F_2(y)\cdot 0 + F_3(y)\cdot 0\right).\end{aligned}$$

Now adding and subtracting $3F_2(y)\frac{(x_1-y_1)(x_2-y_2)}{(|x-y|^2+\delta)^{\frac{5}{2}}}$ and $3F_3(y)\frac{(x_1-y_1)(x_3-y_3)}{(|x-y|^2+\delta)^{\frac{5}{2}}}$ in the third and the fourth term on the right hand side of the equation given above respectively we get

$$\begin{aligned}&F_1(y)K^{\delta}_{11}(x,y) + F_2(y)K^{\delta}_{21}(x,y) + F_3(y)K^{\delta}_{31}(x,y) \\ &= \frac{1}{4\pi}\left(\frac{3F_1(y)}{(|x-y|^2+\delta)^{\frac{3}{2}}} - 3F_1(y)\frac{|x-y|^2}{(|x-y|^2+\delta)^{\frac{5}{2}}} \right. \\ &-3F_2(y)\frac{(x_1-y_1)(x_2-y_2) - (x_1-y_1)(x_2-y_2)}{(|x-y|^2+\delta)^{\frac{5}{2}}} \\ &\left. -3F_3(y)\frac{(x_1-y_1)(x_3-y_3) - (x_1-y_1)(x_3-y_3)}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right).\end{aligned}$$

After simplifying the equation given above, we get

$$\begin{aligned}&F_1(y)K^{\delta}_{11}(x,y) + F_2(y)K^{\delta}_{21}(x,y) + F_3(y)K^{\delta}_{31}(x,y) = \frac{1}{4\pi} \\ &\times\left(\frac{F_1(y)}{(|x-y|^2+\delta)^{\frac{3}{2}}} - 3\frac{((x_1-y_1)F_1(y) + (x_2-y_2)F_2(y))}{(|x-y|^2+\delta)^{\frac{5}{2}}} \right. \\ &-3\frac{((x_3-y_3)F_3(y))\,(x_1-y_1)}{(|x-y|^2+\delta)^{\frac{5}{2}}} + \frac{2F_1(y)}{(|x-y|^2+\delta)^{\frac{3}{2}}} \\ &+3\frac{((x_1-y_1)F_2(y) - (x_2-y_2)F_1(y))\,(x_2-y_2)}{(|x-y|^2+\delta)^{\frac{5}{2}}} \\ &\left. -3\frac{((x_3-y_3)F_1(y) - (x_1-y_1)F_3(y))\,(x_3-y_3)}{(|x-y|^2+\delta)^{\frac{5}{2}}}\right).\end{aligned}$$

Now, by comparing the equation given above with Equations (A.1.5) and (A.1.8) of [1], p. 101-103, we have

$$\begin{aligned}&F_1(y)K^{\delta}_{11}(x,y) + F_2(y)K^{\delta}_{21}(x,y) + F_3(y)K^{\delta}_{31}(x,y) \\ &= -\left(\nabla_x\left(\nabla_x G^{\delta}_0(x,y)\cdot f(y)\right)\right)_1 \\ &\quad + \left(\nabla_x \times \left(\nabla_x G^{\delta}_0(x,y)\times f(y)\right)\right)_1.\end{aligned} \tag{14}$$

$$\begin{aligned}&F_1(y)K^{\delta}_{21}(x,y) + F_2(y)K^{\delta}_{22}(x,y) + F_3(y)K^{\delta}_{23}(x,y) \\ &= -\left(\nabla_x\left(\nabla_x G^{\delta}_0(x,y)\cdot f(y)\right)\right)_2 \\ &\quad + \left(\nabla_x \times \left(\nabla_x G^{\delta}_0(x,y)\times f(y)\right)\right)_2.\end{aligned} \tag{15}$$

$$\begin{aligned}&F_1(y)K^{\delta}_{13}(x,y) + F_2(y)K^{\delta}_{23}(x,y) + F_3(y)K^{\delta}_{33}(x,y) \\ &= -\left(\nabla_x\left(\nabla_x G^{\delta}_0(x,y)\cdot f(y)\right)\right)_3 \\ &\quad + \left(\nabla_x \times \left(\nabla_x G^{\delta}_0(x,y)\times f(y)\right)\right)_3.\end{aligned} \tag{16}$$

Putting the values of the components from Equations (14), (15) and (16) into Equation (13) we obtain

$$\begin{aligned}(f \circledast \mathbf{k}_\delta)(x) \;=\; \int_V -\Big(&\left(\nabla_x\left(\nabla_x G^{\delta}_0(x,y)\cdot f(y)\right)\right)_1 \\ &+ \left(\nabla_x \times \left(\nabla_x G^{\delta}_0(x,y)\times f(y)\right)\right)_1, \\ &-\left(\nabla_x\left(\nabla_x G^{\delta}_0(x,y)\cdot f(y)\right)\right)_2 \\ &+ \left(\nabla_x \times \left(\nabla_x G^{\delta}_0(x,y)\times f(y)\right)\right)_2, \\ &-\left(\nabla_x\left(\nabla_x G^{\delta}_0(x,y)\cdot f(y)\right)\right)_3 \\ &+ \left(\nabla_x \times \left(\nabla_x G^{\delta}_0(x,y)\times f(y)\right)\right)_3\Big)\, dy.\end{aligned}$$

The equation given above can be written as

$$\begin{aligned}(f \circledast \mathbf{k}_\delta)(x) &= \int_V -\left(\nabla_x \left(\nabla_x G_0^\delta(x,y) \cdot f(y)\right)\right. \\ &\quad \left. +\nabla_x \times \left(\nabla_x G_0^\delta(x,y) \times f(y)\right)\right) dy \\ &= -\int_V \nabla_x \left(\nabla_x G_0^\delta(x,y) \cdot f(y)\right) dy \\ &\quad + \int_V \nabla_x \times \left(\nabla_x G_0^\delta(x,y) \times f(y)\right) dy.\end{aligned}$$

Due to Theorem 6, the equation given above takes the form

$$\begin{aligned}(f \circledast \mathbf{k}_\delta)(x) &= -\nabla_x \int_V \nabla_x G_0^\delta(x,y) \cdot f(y) dy \\ &\quad +\nabla_x \times \int_V \nabla_x G_0^\delta(x,y) \times f(y) dy \\ &= -\nabla_x \rho^\delta(x) + \nabla_x \times \Lambda^\delta(x).\end{aligned}$$

Taking the limit $\delta \to 0+$ on both hand sides, we get

$$\begin{aligned}\lim_{\delta \to 0+} (f \circledast \mathbf{k}_\delta)(x) &= -\lim_{\delta \to 0+} \nabla_x \rho^\delta(x) \\ &\quad + \lim_{\delta \to 0+} \nabla_x \times \Lambda^\delta(x).\end{aligned}$$

Using Theorem 4 we can write the equation given above as follows

$$\begin{aligned}\lim_{\delta \to 0+} (f \circledast \mathbf{k}_\delta)(x) &= -\nabla_x \rho(x) + \nabla_x \times \Lambda(x) \\ &= f(x).\end{aligned}$$

$\blacksquare$

5. SOME NUMERICAL TESTS

In this section, we present some numerical tests in which we reconstruct the synthetic vectorial function f by using the kernels given in Theorem 7 for different parameters $\delta = 0.01$, $\delta = 0.001$ and $\delta = 0.0001$ We have also calculated the rooted mean square error for these values of δ which are given in Table 1. For all values of the parameter δ we used $100 \times 256 \times 256$ grid points i.e. 100 point along radius, 256 points along longitude and 256 points along latitude for the integration on $B_1(0) = V$. To compute our convolution on $V = B_1(0)$, we use the standard separation of an integral on $B_1(0)$

$$\int_{B_1(0)} F(x) dx = \int_0^1 r^2 \int_{\partial B_1(0)} F(r\eta) d\omega(\eta) dr. \tag{17}$$

For the integral over the sphere, the most commonly used sampling measures have support on either an equiangular grid or a Gaussian grid. We use an equiangular grid and the quadrature theorem given in [7] and for the line integral we use the composite Simpson's rule. Our vectorial synthetic function is given by

δ	Rooted mean square in reconstructed f
10^{-2}	0.0300
10^{-3}	0.0100
10^{-4}	0.0056

Table 1: Rooted mean square error in the reconstruction of the synthetic vectorial function f for different values of the parameter δ.

$$f(y) = f(r\eta) = r^3 \eta Y_{3,2}(\eta) \sin(\eta_2) \cos(\eta_3) \tag{18}$$

where $r = |y|$ and $\eta = (\eta_1, \eta_2, \eta_3) = \frac{y}{|y|}$ for $y \in \overline{B_1(0)}$. Moreover, $Y_{3,2}$ is a spherical harmonic of degree 3 and order 2. Analysing the graphs in Figures 3 to 4 and the errors in Table 1, we can say that we get a very good approximation of the vectorial function f.

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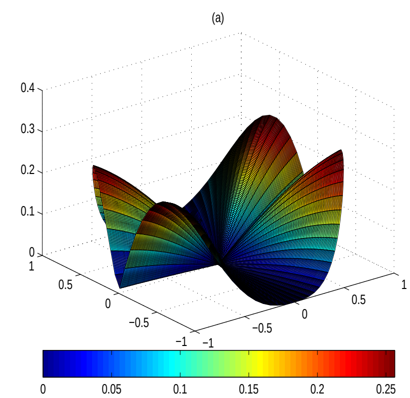

Figure 1: (a) is the graph of the absolute values of the vector function f plotted in the $y_1 = 0$ plane.

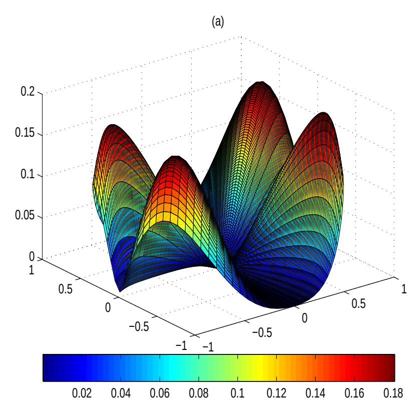

Figure 2: (a) is the graph of the absolute values of the reconstructed vector function f using the kernel of Theorem 7 with the parameter $\delta = 0.01$. The function is plotted in the $y_1 = 0$ plane.

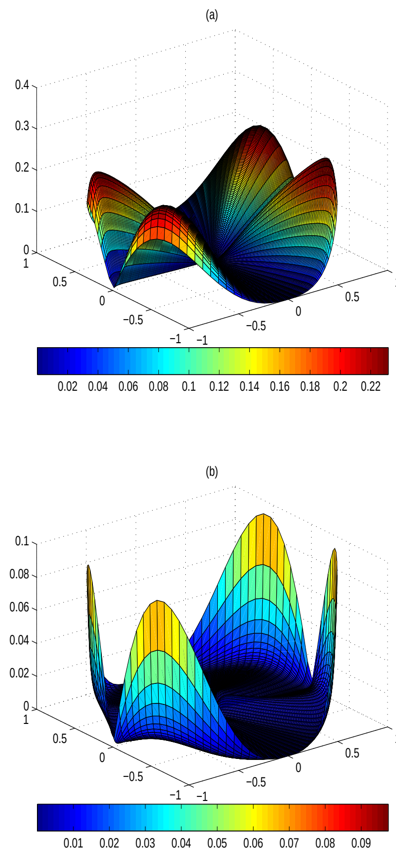

Figure 3: (a) is the graph of the absolute values of the reconstructed vector function f using the kernel of Theorem 7 with the parameter $\delta = 0.001$ and (b) is the graph of the absolute values of the difference of the actual vector function f and the reconstructed vector function f using the kernel of Theorem 7 with the parameter $\delta = 0.001$. In both cases, the function is plotted in the $y_1 = 0$ plane.

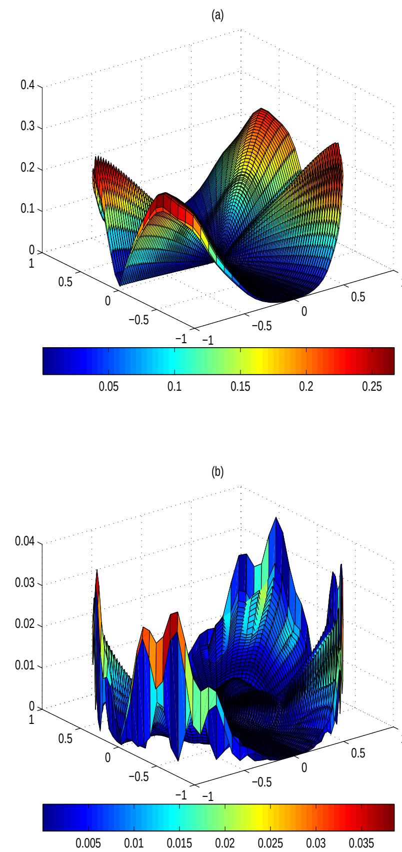

Figure 4: (a) is the graph of the absolute values of the reconstructed vector function f using the kernel of Theorem 7 with the parameter $\delta = 0.0001$ and (b) is the graph of the absolute values of the difference of the actual vector function f and the reconstructed vector function using the kernel of Theorem 7 with the parameter $\delta = 0.0001$. In both cases, the function is plotted in the $y_1 = 0$ plane.