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Prime bi-ideals and prime fuzzy bi-ideals in semirings

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Abstract: In this paper, we initiate the study of prime bi-ideals (fuzzy bi-ideals) in semirings. We define strongly prime, prime, semiprime, irreducible and strongly irreducible bi-ideals in semirings. We also define strongly prime, semiprime, irreducible, strongly irreducible fuzzy bi-ideals of semirings. We characterize those semirings in which each bi-ideal (fuzzy bi-ideal) is prime (strongly prime).

Keywords: Prime, strongly prime, semiprime, irreducible and strongly irreducible bi-ideals in semirings.

1. Introduction and historical background

Algebraic structures play very important role in mathematics with the concurrent applications in many knowledges for example, theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces etc. Semirings which are common generalization of associative ring and distributive lattice are found in abundance around us. Various semirings arise erupt in a natural manner in such variegated areas of mathematics as functional analysis, topology, graph theory, Euclidean geometry, probability theory, optimization theory, discrete event dynamical system, automata theory, formal language theory, and the mathematical designing of quantum physics and parallel computation systems. In 1934, Vandiver [1] introduced an algebraic system, which consisted of a non empty set S with two binary operations addition $(+)$ and multiplication $(\cdot)$ such that S was a semigroup under both operations. The system $(S,+,.)$ satisfied both distributive laws but did not satisfy the cancellation law of addition. The system he constructed was ring-like but not exactly a ring. Vandiver called this system a `semiring'.

In 1965, Zadeh [2] introduced the concept of fuzzy set. Many researchers used this concept to generalize some notions of mathematics. Ahsan et al. [3] initiated the study of fuzzy semirings. Many authors worked on the fuzzy ideal theory of semirings. In Ahsan et al. [4] studied prime fuzzy ideals and prime fuzzy subsemimodules of a semimodule over a semiring.

In [5] Shabir and Kanwal introduced prime bi-ideals, strongly prime bi-ideals and irreducible bi-ideals in semigroups and in [6] Shabir et al. studied prime fuzzy bi-ideals, strongly prime fuzzy bi-ideals and irreducible fuzzy bi-ideals in semigroups. In this paper we sketch out the concept of prime and semiprime bi-ideals of a semiring and prime and strongly prime fuzzy bi-ideals of a semiring.

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2. Preliminaries definitions and lemmas

A semiring is a non-empty set R together with two binary operations addition "+" and multiplication "." such that $(R,+)$ is a commutative semigroup and $(R,.)$ is a semigroup, where the two operations are connected by ring like distributive laws, that is $a(b+c)=ab+ac$ and $(a+b)c=ac+bc$ for all $a,b,c\in R$. An element $0\in R$ is called an absorbing element if $0+a=a=a+0$ and $0.a=0=a.0$ for all $a\in R$. We shall always assume that $(R,+,.)$ has an absorbing zero '0'. A semiring R is called a commutative semiring if multiplication is commutative. A non-empty subset B of a semiring R is called a subsemiring of R if for all $a,b\in B$, we have $a+b\in B$ and $ab\in B$. A left (right) ideal L of a semiring R is a non-empty subset of R such that $a+b\in L$ for all $a,b\in L$ and $xa\in L$ ($ax\in L$) for all $a\in L$ and $x\in R$. An ideal of a semiring R is a subset of R which is both a left ideal and a right ideal of R. By a quasi-ideal of R we mean a subsemigroup Q of $(R,+)$ such that $RQ\cap QR\subseteq Q$. A non-empty subset B of a semiring R is called a bi-ideal of R if B is a subsemiring of R and $BRB\subseteq B$. A semiring R is called von Neumann regular or simply regular if for each $a\in R$ there exists $x\in R$ such that $a=axa$. A semiring R is called an intra-regular semiring if for each $a\in R$ there exists $x_i, y_i\in R$ such that $a=\sum_{i=1}^{n} x_i a^2 y_i$.

It is well known that every quasi-ideal of a semiring is a bi-ideal but the converse is not true.

2.1. Theorem.

Let X be an arbitrary subset of a semiring R and B a bi-ideal of R. Then BX and XB are bi-ideals of R.

Proof. Straightforward.

2.2. Corollary.

The product of two bi-ideals of a semiring R is a bi-ideal of R.

2.3. Theorem. [7]

A semiring R is regular if and only if $A\cap B=AB$ for every right ideal A and left ideal B of R.

2.4. Theorem. [7]

The following conditions are equivalent for a semiring R:

(1) R is regular and intra-regular

(2) $BA=A\cap B\subseteq AB$ for every left ideal A and right ideal B of R.

(3) Every quasi-ideal of R is idempotent.

A function f from a non-empty set R to the unit closed interval $[0,1]$ of real numbers is called a fuzzy subset of R. A fuzzy subset f of R is non-empty if f is not the constant map which assumes the value 0. For fuzzy subsets f and g of R, $f\le g$ means that for all $a\in R$, $f(a)\le g(a)$. The symbols $f\wedge g$ and $f\vee g$ will mean the following fuzzy subsets of R

$$(f \wedge g)(a) = f(a) \wedge g(a)$$

and $$(f \vee g)(a) = f(a) \vee g(a) \quad \text{for all } a \in R.$$

Let X be a non-empty subset of R. Then the characteristic function of R denoted and defined by

$$f_X(a) = \begin{cases} 1 & \text{if } a \in \mathrm{X} \\ 0 & \text{otherwise} \end{cases}$$

If f and g are fuzzy subsets of a semiring R, then their product $f \circ g$ and sum $f + g$ are a fuzzy subsets of R defined by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{x=\sum\limits_{i=1}^{p} y_i z_i} \left[\bigwedge\limits_{1 \le i \le p} [f(y_i) \wedge g(z_i)] \right] & \text{if } x \text{ is expressible as } x = \sum\limits_{i=1}^{p} y_i z_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$(f+g)(x) = \bigvee_{x=y+z} \left[f(y) \wedge g(z) \right]$$

for all $x \in R$. The operation '$\circ$ ' is associative.

2.5. Lemma.

Let f, g, h be fuzzy subsets of a semiring R. If $f \le g$ then $f \circ h \le g \circ h$ and $h \circ f \le h \circ g$.

2.6. Definitions.

A fuzzy subset f of a semiring R is a fuzzy subsemiring of R if

(i) $f(a+b) \ge f(a) \wedge f(b)$ and

(ii) $f(ab) \ge f(a) \wedge f(b)$ for all $a, b \in R$.

A fuzzy subset f of a semiring R is a fuzzy left (right) ideal of R if

(i) $f(a+b) \ge f(a) \wedge f(b)$ and

(ii) $f(ab) \ge f(b)$ $\left(f(ab) \ge f(a) \right)$ for all $a, b \in R$.

If f is both a fuzzy left and a fuzzy right ideal of R, then it is called a fuzzy ideal of R.

2.7. Lemma.

For any non-empty subsets X and Y of a semiring R, we have

(1) $f_X \circ f_Y = f_{XY}$

(2) $f_X \wedge f_Y = f_{X \cap Y}$

(3) $f_X + f_Y = f_{X+Y}$.

2.8. Lemma.

(1) A fuzzy subset f of a semiring R is a fuzzy subsemiring of R if and only if $f + f \le f$

and $f \circ f \leq f$.

(2) A fuzzy subset f of a semiring R is a fuzzy left (right) ideal of R if and only if $f + f \leq f$ and $f_R \circ f \leq f$ $(f \circ f_R \leq f)$.

(3) If f and g are fuzzy left (right) ideals of a semiring R , then $f \wedge g$ is a fuzzy left (right) ideal of R .

(4) If f and g are fuzzy ideals of a semiring R, then $f \circ g$ is a fuzzy ideal of R .

2.9. Definition.

A fuzzy subset f of a semiring R is called a fuzzy bi-ideal of R if

(1) $f(a+b) \geq f(a) \wedge f(b)$

(2) $f(ab) \geq f(a) \wedge f(b)$

(3) $f(abc) \geq f(a) \wedge f(c)$ for all $a, b, c \in R$.

The proof of the following Lemma is straightforward.

2.10. Lemma.

(1) A fuzzy subset f of a semiring R is a fuzzy bi-ideal of R if and only if $f + f \leq f$, $f \circ f \leq f$ and $f \circ f_R \circ f \leq f$.

(2) A non-empty subset B of a semiring R is a bi-ideal of R if and only if the characteristic function f_B of B is a fuzzy bi-ideal of R .

(3) Let f and g be two fuzzy bi-ideals of a semiring R. Then $f \wedge g$ is a fuzzy bi-ideal of R .

(4) The product of two fuzzy bi-ideals of a semiring R is a fuzzy bi-ideal of R .

3. Prime, strongly prime and semiprime bi-ideals in semirings

3.1. Definition.

A bi-ideal B of a semiring R is called a prime bi-ideal of R if $B_1B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of R .

A bi-ideal B of a semiring R is called a strongly prime bi-ideal of R if $B_1B_2 \cap B_2B_1 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$ for any bi-ideals B_1, B_2 of R .

A bi-ideal B of a semiring R is called a semiprime bi-ideal of R if $B_1^2 \subseteq B$ implies $B_1 \subseteq B$ for any bi-ideal B_1 of R .

Obviously every strongly prime bi-ideal of a semiring is prime bi-ideal and every prime bi-ideal is semiprime bi-ideal but the converse is not true. The intersection of any family of prime bi-ideals of a semiring is semiprime bi-ideal.

3.2. Example.

Consider the semiring $R = \{0, a, b\}$ with binary operations addition and multiplication defined below:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

$.$	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

The bi-ideals of R are $\{0\}$, $\{0,a\}$, $\{0,b\}$ and $\{0,a,b\}$. The bi-ideal $\{0\}$ is a prime bi-ideal of R but not a strongly prime bi-ideal of R, because

$$\{0,a\}\{0,b\}\cap\{0,b\}\{0,a\}=\{0,a\}\cap\{0,b\}=\{0\}\subseteq\{0\}.$$

But neither $\{0,a\}$ nor $\{0,b\}$ is contained in $\{0\}$.

3.3. Example.

Let $R=\{\Phi,\{a\},\{b\},\{a,b\}\}$. Define addition and multiplication on R as $X+Y=X\Delta Y=(X\cup Y)-(X\cap Y)$ and $X.Y=X\cap Y$ for all $X,Y\in R$. The Cayley tables for the defined operations are given by

$+$	Φ	$\{a\}$	$\{b\}$	$\{a,b\}$
Φ	Φ	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\{a\}$	Φ	$\{a,b\}$	$\{b\}$
$\{b\}$	$\{b\}$	$\{a,b\}$	Φ	$\{a\}$
$\{a,b\}$	$\{a,b\}$	$\{b\}$	$\{a\}$	Φ

$.$	Φ	$\{a\}$	$\{b\}$	$\{a,b\}$
Φ	Φ	Φ	Φ	Φ
$\{a\}$	Φ	$\{a\}$	Φ	$\{a\}$
$\{b\}$	Φ	Φ	$\{b\}$	$\{b\}$
$\{a,b\}$	Φ	$\{a\}$	$\{b\}$	$\{a,b\}$

Then bi-ideals of R are $\{\Phi\},\{\Phi,\{a\}\},\{\Phi,\{b\}\}$ and $\{\Phi,\{a\},\{b\},\{a,b\}\}$. All these bi-ideals are semiprime. In particular $\{\Phi\}$ is a semiprime bi-ideal of R but not a prime bi-ideal of R, because $\{\Phi,\{a\}\}\cdot\{\Phi,\{b\}\}=\{\Phi\}\subseteq\{\Phi\}$. But neither $\{\Phi,\{a\}\}$ nor $\{\Phi,\{b\}\}$ is contained in $\{\Phi\}$.

3.4. Definition.

A bi-ideal B of a semiring R is called an irreducible bi-ideal of R if $B_1\cap B_2=B$ implies either $B_1=B$ or $B_2=B$ for any bi-ideals B_1,B_2 of R.

A bi-ideal B of a semiring R is called a strongly irreducible bi-ideal of R if $B_1\cap B_2\subseteq B$ implies either $B_1\subseteq B$ or $B_2\subseteq B$ for any bi-ideals B_1,B_2 of R.

Every strongly irreducible bi-ideal is irreducible but the converse is not true.

3.5. Proposition.

Every strongly irreducible semiprime bi-ideal of a semiring R is a strongly prime bi-ideal of R.

Proof. Let B be a strongly irreducible semiprime bi-ideal of a semiring R. Let B_1,B_2 be two bi-ideals of R such that $B_1B_2\cap B_2B_1\subseteq B$. As $B_1\cap B_2\subseteq B_1$ and $B_1\cap B_2\subseteq B_2$, we have $(B_1\cap B_2)^2\subseteq B_1B_2$ and $(B_1\cap B_2)^2\subseteq B_2B_1$. This implies $(B_1\cap B_2)^2\subseteq B_1B_2\cap B_2B_1\subseteq B$. Since $B_1\cap B_2$ is a bi-ideal and B is a semiprime bi-ideal of R, we have $(B_1\cap B_2)\subseteq B$. Since B is strongly irreducible, we have $B_1\subseteq B$ or $B_2\subseteq B$. This shows that B is a strongly prime bi-ideal of R.

3.6. Proposition.

Let B be a bi-ideal of a semiring R and $a \in R$ be such that $a \notin B$. Then there exists an irreducible bi-ideal I of R such that $B \subseteq I$ and $a \notin I$.

Proof. Let X be the collection of all bi-ideals of R which contains B but does not contain a, that is $X = \{Y_i : Y_i \text{ is a bi-ideal of } R, B \subseteq Y_i \text{ and } a \notin Y_i\}$. Then X is non-empty as $B \in X$. The collection X is a partially ordered set under inclusion. If $\{Y_i : i \in I\}$ is any totally ordered subset (chain) of X, then $\bigcup_{i \in I} Y_i = Y$ is also a bi-ideal of R containing B and $a \notin Y$. So Y is an upper bound of $\{Y_i : i \in I\}$. Thus every chain in X has an upper bound in X. Hence by Zorn's lemma, there exists a maximal element I (say) in X such that $B \subseteq I$ and $a \notin I$. Now we show that I is an irreducible bi-ideal of R. For this let C, D be two bi-ideals of R such that $I = C \bigcap D$. If both C and D properly contain I, then $a \in C$ and $a \in D$. Thus $a \in C \bigcap D = I$. Which is a contradiction to the fact that $a \notin I$. So either $I = C$ or $I = D$.

3.7. Theorem.

For a semiring R, the following assertions are equivalent:

(1) R is both regular and intra-regular.

(2) $B^2 = B$ for every bi-ideal B of R.

(3) $B_1B_2 \bigcap B_2B_1 = B_1 \bigcap B_2$ for any bi-ideals B_1, B_2 of R.

(4) Each bi-ideal of R is semiprime.

(5) Each proper bi-ideal of R is the intersection of irreducible semiprime bi-ideals of R which contain it.

Proof. $(1) \Rightarrow (2)$ Let R be both regular and intra-regular and B be a bi-ideal of R. Then $B^2 \subseteq B$. Let $a \in B$. Then there exist $x, y_i, z_i \in R$ such that $a = axa$ and $a = \sum_{i=1}^{n} y_i a a z_i$. Now, $a = axa = ax(axa) = ax(\sum_{i=1}^{n} y_i a a z_i)xa = \sum_{i=1}^{n} a(xy_i)aa(z_i x)a \in BRBBRB \subseteq BB = B^2$. This implies $B \subseteq B^2$. Hence $B^2 = B$ for every bi-ideal B of R.

$(2) \Rightarrow (1)$ Let Q be a quasi-ideal of R. Then Q is a bi-ideal of R. By hypothesis $Q^2 = Q$. Thus by Theorem 2.4, R is both regular and intra-regular semiring.

$(2) \Rightarrow (3)$ Let B_1, B_2 be any two bi-ideals of R. Then $B_1 \bigcap B_2$ is also a bi-ideals of R. Thus by hypothesis $B_1 \bigcap B_2 = (B_1 \bigcap B_2)^2 = (B_1 \bigcap B_2)(B_1 \bigcap B_2) \subseteq B_1B_2$. Similarly $B_1 \bigcap B_2 \subseteq B_2B_1$. Hence $B_1 \bigcap B_2 \subseteq B_1B_2 \bigcap B_2B_1$. Since B_1B_2 and B_2B_1 are bi-ideals of R, we have $B_1B_2 \bigcap B_2B_1$ is also a bi-ideal of R. Then by hypothesis $B_1B_2 \bigcap B_2B_1 = (B_1B_2 \bigcap B_2B_1)(B_1B_2 \bigcap B_2B_1) \subseteq B_1B_2B_2B_1 = B_1B_2^2B_1 \subseteq B_1B_2B_1 \subseteq B_1RB_1 \subseteq B_1$. Similarly $B_1B_2 \bigcap B_2B_1 \subseteq B_2$. Thus $B_1B_2 \bigcap B_2B_1 \subseteq B_1 \bigcap B_2$. Hence $B_1B_2 \bigcap B_2B_1 = B_1 \bigcap B_2$.

$(3) \Rightarrow (4)$ Let B be a bi-ideal of R such that $B_1^2 \subseteq B$ for any bi-ideal B_1 of R. Then by hypothesis, we have $B_1 = B_1 \bigcap B_1 = B_1B_1 \bigcap B_1B_1 = B_1^2 \subseteq B$. Which shows that B is a semiprime bi-ideal of R.

$(4) \Rightarrow (5)$ Let B be a proper bi-ideal of R. Then B is contained into the intersection of all irreducible bi-ideal of R which contains B. For the reverse inclusion, let $a \notin B$. Then by Proposition 3.6, there exists an irreducible bi-ideal which contains B but does not contain a. Thus intersection of all irreducible bi-ideals which contain B does not contain a. This shows that B is the intersection of all irreducible semiprime bi-ideals of R which contain it.

$(5) \Rightarrow (2)$ Let B be a bi-ideal of R. Then B^2 is also a bi-ideal of R. Thus by hypothesis $B^2 = \bigcap_a \{B_a : B_a \text{ is an irreducible semiprime bi-ideal of } R \text{ such that } B^2 \subseteq B_a \text{ for all } a\}$. Since each B_a is semiprime, we have $B \subseteq B_a$. Thus $B \subseteq \bigcap B_a = B^2$, but $B^2 \subseteq B$ always holds. Hence $B^2 = B$ for each bi-ideal B of R.

3.8. Theorem.

Let R be a regular and intra-regular semiring. Then the following assertions are equivalent for a bi-ideal B of R.

(1) B is strongly irreducible.

(2) B is strongly prime.

Proof. $(1) \Rightarrow (2)$ Let B, B_1, B_2 be bi-ideals of R such that $B_1B_2 \cap B_2B_1 \subseteq B$. As R is both regular and intra-regular semiring, therefore by Theorem 3.7, we have $B_1 \cap B_2 = B_1B_2 \cap B_2B_1 \subseteq B$. Thus by hypothesis, we have $B_1 \subseteq B$ or $B_2 \subseteq B$.

$(2) \Rightarrow (1)$ Let B, B_1, B_2 be bi-ideals of R such that $B_1 \cap B_2 \subseteq B$. As R is both regular and intra-regular semiring, therefore by Theorem 3.7, we have $B_1B_2 \cap B_2B_1 = B_1 \cap B_2 \subseteq B$. Thus by hypothesis, we have $B_1 \subseteq B$ or $B_2 \subseteq B$.

3.9. Theorem.

Each bi-ideal of a semiring R is strongly prime if and only if R is regular, intra-regular and the set of bi-ideals of R is totally ordered by inclusion.

Proof. Suppose each bi-ideal of R is strongly prime. This implies that each bi-ideal of R is semiprime. Then by Theorem 3.7, R is both regular and intra-regular.

Now we show that the set of bi-ideals of R is totally ordered by inclusion. For this, let B_1, B_2 be two bi-ideals of R. Then by Theorem 3.7, we have

$$B_1B_2 \cap B_2B_1 = B_1 \cap B_2$$

Since each bi-ideal is strongly prime, so $B_1 \cap B_2$ is strongly prime bi-ideal. Thus $B_1 \subseteq B_1 \cap B_2$ or $B_2 \subseteq B_1 \cap B_2$, that is $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of R is totally ordered by inclusion.

Conversely, assume that R is both regular and intra-regular and the set of bi-ideals of R is totally ordered by inclusion. We show that each bi-ideal of R is strongly prime. For this, let B, B_1, B_2 be any bi-ideals of R such that

$$B_1B_2 \cap B_2B_1 \subseteq B$$

As R is both regular and intra-regular, we have $B_1B_2 \cap B_2B_1 = B_1 \cap B_2$. Thus

$$B_1 \cap B_2 \subseteq B.$$

As the set of bi-ideals of R is totally ordered, we have $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$, that is either

$B_1 \cap B_2 = B_2$ or $B_1 \cap B_2 = B_1$. Thus either $B_1 \subseteq B$ or $B_2 \subseteq B_1$. Thus B is strongly prime.

3.10. Theorem.

If the set of bi-ideals of a semiring R is totally ordered, then R is both regular and intra-regular if and only if each bi-ideal of R is prime.

Proof. Let R be both regular and intra-regular. Let B, B_1, B_2 be any bi-ideals of R such that $B_1B_2 \subseteq B$. As the set of bi-ideals of R is totally ordered, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then $B_1^2 \subseteq B_1B_2 \subseteq B$. As R is both regular and intra-regular, so by Theorem 3.7, B is a semiprime bi-ideal of R. Thus $B_1^2 \subseteq B$ implies $B_1 \subseteq B$. Similarly, if $B_2 \subseteq B_1$, then $B_2 \subseteq B$. Thus B is a prime bi-ideal of R.

Conversely, suppose that each bi-ideal of R is prime, so semiprime. Thus by Theorem 3.7, R is both regular and intra-regular.

3.11. Proposition.

If the set of bi-ideals of a semiring R is totally ordered, then the concepts of primeness and strongly primeness coincide.

Proof. Let B be a prime bi-ideal of R. We show that B is a strongly prime bi-ideal of R. Let B_1 , B_2 be bi-ideals of R such that $B_1B_2 \cap B_2B_1 \subseteq B$. Since the set of bi-ideals of R is totally ordered, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then $B_1B_1 = B_1^2 = B_1^2 \cap B_1^2 \subseteq B_1B_2 \cap B_2B_1 \subseteq B$. This implies $B_1 \subseteq B$, because B is a prime bi-ideal of R. Similarly, if $B_2 \subseteq B_1$, then $B_2 \subseteq B$. This shows that B is a strongly prime bi-ideal of R. The converse is obvious.

3.12. Theorem.

For a semiring R, the following assertions are equivalent:

(1) The set of bi-ideals of R is totally ordered by set inclusion.

(2) Each bi-ideal of R is strongly irreducible.

(3) Each bi-ideal of R is irreducible.

Proof. $(1) \Rightarrow (2)$ Let B, B_1, B_2 be any bi-ideals of R such that $B_1 \cap B_2 \subseteq B$. By hypothesis, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. If $B_1 \subseteq B_2$, then $B_1 = B_1 \cap B_2 \subseteq B$. Similarly, if $B_2 \subseteq B_1$ then $B_2 = B_1 \cap B_2 \subseteq B$. So B is strongly irreducible.

$(2) \Rightarrow (3)$ Obvious.

$(3) \Rightarrow (1)$ Let B_1, B_2 be any bi-ideals of R. Then $B_1 \cap B_2$ is also a bi-ideal of R such that $B_1 \cap B_2 = B_1 \cap B_2$. Thus by hypothesis $B_1 = B_1 \cap B_2$ or $B_2 = B_1 \cap B_2$, that is $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Hence the set of bi-ideals of R is totally ordered by set inclusion.

3.13. Definition.

Let R be a semiring, $\mathbf{B}$ be the set of all bi-ideals of R and $\mathbf{P}$ be the set of all strongly prime proper bi-ideals of R. Define for each $B \in \mathbf{B}$

$$q_B = \{ J \in \mathbf{P} \ : \ B \not\subseteq J \}$$

$$t(\mathbf{P}) = \{ q_B \ : \ B \in \mathbf{B} \}$$

3.14. Theorem.

Let R be a regular and intra-regular semiring. Then $t(\mathbf{P})$ forms a topology.

Proof. Since $\{0\}$ is a bi-ideal of R and 0 belongs to every bi-ideal of R, therefore $q_{\{0\}}=\{J\in\mathbf{P}\ :\ \{0\}\not\subseteq J\}=j$ (the empty set) $\in t(\mathbf{P})$. Also R is a bi-ideal of R, then $q_R=\{J\in\mathbf{P}\ :\ R\not\subseteq J\}=\mathbf{P}\in t(\mathbf{P})$. Thus $j,\mathbf{P}\in t(\mathbf{P})$.

Now we show that the intersection of finite number of members of $t(\mathbf{P})$ belongs to $t(\mathbf{P})$. For this, let q_{B_1}, $q_{B_2}\in t(\mathbf{P})$. Then we have to show that $q_{B_1}\cap q_{B_2}\in t(\mathbf{P})$.

For this we show that $q_{B_1}\cap q_{B_2}=q_{B_1\cap B_2}$. Let $J\in q_{B_1}\cap q_{B_2}$. This implies $J\in q_{B_1}$ and $J\in q_{B_2}$. So $J\in\mathbf{P}$ and $B_1\not\subseteq J$ and $B_2\not\subseteq J$. Now suppose that $B_1\cap B_2\subseteq J$. Since A is both regular and intra-regular, then by Theorem 3.7, we have $B_1B_2\cap B_2B_1=B_1\cap B_2\subseteq J$. This implies $B_1\subseteq J$ or $B_2\subseteq J$, which is a contradiction. Hence $B_1\cap B_2\not\subseteq J$. This implies $J\in q_{B_1\cap B_2}$. So $q_{B_1}\cap q_{B_2}\subseteq q_{B_1\cap B_2}$. Now, let $J\in q_{B_1\cap B_2}$. This implies $B_1\cap B_2\not\subseteq J$, so $B_1\not\subseteq J$ and $B_2\not\subseteq J$. Thus $J\in q_{B_1}\cap q_{B_2}$, that is $q_{B_1\cap B_2}\subseteq q_{B_1}\cap q_{B_2}$. Hence $q_{B_1}\cap q_{B_2}=q_{B_1\cap B_2}\in t(\mathbf{P})$.

Now we show that union of any number of members of $t(\mathbf{P})$ belong to $t(\mathbf{P})$. For this, let $\{q_{B_a}\ :\ a\in I\}\subseteq t(\mathbf{P})$. Then we have to show that $\bigcup_a q_{B_a}\in t(r)$

$$\text{As}\quad \bigcup_a q_{B_a}=\{J\in r\ :\ B_a\not\subseteq J \text{ for some } a\in I\}$$

$$=\left\{J\in r\ :\ \widehat{\bigcup_a B_a}\not\subseteq J\right\}=q_{\widehat{\bigcup_a B_a}}\in t(\mathbf{P}).$$

Where $\widehat{\bigcup_a B_a}$ is the bi-ideal of R generated by $\bigcup_a B_a$. Thus $t(\mathbf{P})$ is a topology on $\mathbf{P}$.

4. Prime, strongly prime and semiprime fuzzy bi-ideals in semiring

In this section we study prime, semiprime, irreducible and strongly irreducible fuzzy bi-ideals in semiring.

4.1. Definition.

A fuzzy bi-ideal f of a semiring R is called a prime fuzzy bi-ideal of R if for any fuzzy bi-ideals $g,\ h$ of R, $g\circ h\le f$ implies $g\le f$ or $h\le f$.

A fuzzy bi-ideal f of a semiring R is called a strongly prime fuzzy bi-ideal of R if for any fuzzy bi-ideals g,h of R, $g\circ h\wedge h\circ g\le f$ implies $g\le f$ or $h\le f$.

A fuzzy bi-ideal g of a semiring R is said to be idempotent if $g=g\circ g=g^2$.

A fuzzy bi-ideal f of a semiring R is said to be a semiprime fuzzy bi-ideal of R if $g\circ g=g^2\le f$ implies $g\le f$ for every fuzzy bi-ideal g of R.

4.2. Proposition.

(1) Every strongly prime fuzzy bi-ideal of a semiring R is a prime fuzzy bi-ideal of R.

(2) Every prime fuzzy bi-ideal of a semiring R is a semiprime fuzzy bi-ideal of R.

(3) The intersection of any family of prime fuzzy bi-ideals of a semiring R is a semiprime fuzzy bi-ideal of R.

Proof. Straightforward.

4.3. Definition.

A fuzzy bi-ideal f of a semiring R is said to be an irreducible fuzzy bi-ideal of R if for any fuzzy bi-ideals g and h of R, $g \wedge h = f$ implies either $g = f$ or $h = f$.

A fuzzy bi-ideal f of a semiring R is said to be strongly irreducible fuzzy bi-ideal of R if for any fuzzy bi-ideals g and h of R, $g \wedge h \leq f$ implies either $g \leq f$ or $h \leq f$.

4.4. Proposition.

Every strongly irreducible semiprime fuzzy bi-ideal of a semiring R is a strongly prime fuzzy bi-ideal of R.

Proof. Let f be a strongly irreducible semiprime fuzzy bi-ideal of a semiring R. Let g, h be any fuzzy bi-ideals of R such that $g \circ h \wedge h \circ g \leq f$. Since $(g \wedge h) \circ (g \wedge h) = (g \wedge h)^2 \leq g \circ h$ and $(g \wedge h) \circ (g \wedge h) = (g \wedge h)^2 \leq h \circ g$, we have $(g \wedge h)^2 \leq (g \circ h) \wedge (h \circ g) \leq f$. Thus $g \wedge h \leq f$, because f is a semiprime fuzzy bi-ideal of R. This implies either $g \leq f$ or $h \leq f$, because f is a strongly irreducible fuzzy bi-ideal of R. Hence f is a strongly prime fuzzy bi-ideal of R.

4.5. Theorem.

Let f be a fuzzy bi-ideal of a semiring R with $f(a) = \alpha$, where $a \in R$ and $\alpha \in (0,1]$. Then there exists an irreducible fuzzy bi-ideal g of R such that $f \leq g$ and $g(a) = \alpha$.

Proof. Let $X = \{h : h \text{ is a fuzzy bi-ideal of } R,\ h(a) = \alpha \text{ and } f \leq h\}$. Then $X \neq \varphi$, because $f \in X$. The collection X is a partially ordered set under inclusion. If $Y = \{h_i : h_i \text{ is a fuzzy bi-ideal of } R,\ h_i(a) = \alpha \text{ and } f \leq h_i \text{ for all } i \in I\}$ is any totally ordered subset of X, then $\underset{i \in I}{\vee} h_i$ is a fuzzy bi-ideal of R such that $f \leq \underset{i \in I}{\vee} h_i$. Indeed, if $a, b, x \in R$ then

$$\left(\vee_{i\in I} h_i\right)(a+b) = \vee_{i\in I}\left(h_i(a+b)\right)$$
$$\geq \vee_{i\in I}\left(h_i(a) \wedge h_i(b)\right) \quad \text{because each } h_i \text{ is a fuzzy bi-ideal of } R.$$
$$= \left(\vee_{i\in I} h_i(a)\right) \wedge \left(\vee_{i\in I} h_i(b)\right)$$
$$= \left(\vee_{i\in I} h_i\right)(a) \wedge \left(\vee_{i\in I} h_i\right)(b).$$

And $\left(\vee_{i\in I} h_i\right)(ab) = \vee_{i\in I}\left(h_i(ab)\right)$

$$\geq \vee_{i\in I}\left(h_i(a) \wedge h_i(b)\right) \quad \text{because each } h_i \text{ is a fuzzy bi-ideal of } R.$$
$$= \left(\vee_{i\in I} h_i(a)\right) \wedge \left(\vee_{i\in I} h_i(b)\right)$$
$$= \left(\vee_{i\in I} h_i\right)(a) \wedge \left(\vee_{i\in I} h_i\right)(b).$$

Also $\left(\vee_{i\in I} h_i\right)(axb) = \vee_{i\in I}\left(h_i(axb)\right)$

$$\geq \vee_{i\in I}\left(h_i(a) \wedge h_i(b)\right) \quad \text{because each } h_i \text{ is a fuzzy bi-ideal of } R.$$
$$= \left(\vee_{i\in I} h_i(a)\right) \wedge \left(\vee_{i\in I} h_i(b)\right)$$
$$= \left(\vee_{i\in I} h_i\right)(a) \wedge \left(\vee_{i\in I} h_i\right)(b).$$

Hence $\vee_{i\in I} h_i$ is a fuzzy bi-ideal of R. As $f \leq h_i$ for all $i \in I$, we have $f \leq \vee_{i\in I} h_i$. Also $\left(\vee_{i\in I} h_i\right)(a) = \vee_{i\in I}(h_i(a)) = \boldsymbol{a}$. Thus $\vee_{i\in I} h_i \in X$ and $\vee_{i\in I} h_i$ is an upper bound of Y. Hence by Zorn's lemma, there exists a fuzzy bi-ideal g of R which is maximal with respect to the property $f \leq g$ and $g(a) = \boldsymbol{a}$. Now we show that g is an irreducible fuzzy bi-ideal of R. For this, suppose g_1, g_2 are fuzzy bi-ideals of R such that $g_1 \wedge g_2 = g$. This implies $g \leq g_1$ and $g \leq g_2$. We claim that $g = g_1$ or $g = g_2$. On the contrary, assume that $g \neq g_1$ and $g \neq g_2$. This implies $g < g_1$ and $g < g_2$. So $g_1(a) \neq \boldsymbol{a}$ and $g_2(a) \neq \boldsymbol{a}$. Hence $(g_1 \wedge g_2)(a) = g_1(a) \wedge g_2(a) \neq \boldsymbol{a}$. Which is a contradiction to the fact that $g_1(a) \wedge g_2(a) = g(a) = \boldsymbol{a}$. Hence either $g = g_1$ or $g = g_2$.

4.6. Theorem.

For a semiring R the following assertions are equivalent:

(1) R is both regular and intra-regular.

(2) $f \circ f = f$ for every fuzzy bi-ideal f of R.

(3) $g \wedge h = g \circ h \wedge h \circ g$ for all fuzzy bi-ideals g and h of R.

(4) Each fuzzy bi-ideal of R is fuzzy semiprime.

(5) Each proper fuzzy bi-ideal of R is intersection of irreducible semiprime fuzzy bi-ideals of R which contain it.

Proof. $(1)\Rightarrow(2)$ Let f be a fuzzy bi-ideal of R and $a\in R$. Then there exist elements x, y_i and z_i of R such that $a = axa$ and $a=\sum y_i a^2 z_i$. Then $a = axa = axaxa = ax\left(\sum y_i a^2 z_i\right)xa = \sum\left(axy_i a\right)\left(az_i xa\right)$. Thus we have

$$\begin{aligned}(f\circ f)(a) &= \bigvee_{a=\sum_{i=1}^{n} x_i y_i}\left[\bigwedge_{1\le i\le n}\{f(x_i)\wedge f(y_i)\}\right]\\ &\ge \bigwedge_{1\le i\le n}\{f(axy_i a)\wedge f(az_i xa)\}\\ &\ge f(a)\wedge f(a) = f(a).\end{aligned}$$

Thus $f\circ f\ge f$. But $f\circ f\le f$ is always true. Hence $f\circ f = f$.

$(2)\Rightarrow(1)$ Let B be a bi-ideal of R. Then by Lemma 2.10, f_B is a fuzzy bi-ideal of R. Thus by Lemma 2.7, and hypothesis $f_{BB} = f_B\circ f_B = f_B$. Hence $B^2 = B$. Then by Theorem 3.7, R is both regular and intra-regular.

$(2)\Rightarrow(3)$ Let g and h be two fuzzy bi-ideals of R. Then $g\wedge h$ is also a fuzzy bi-ideal of R. Thus by hypothesis, we have $g\wedge h = (g\wedge h)\circ(g\wedge h)\le g\circ h$. Similarly $g\wedge h\le h\circ g$. This implies $g\wedge h\le g\circ h\wedge h\circ g$.

Now, $g\circ h$ and $h\circ g$, are fuzzy bi-ideals of R, so $g\circ h\wedge h\circ g$ is also a fuzzy bi-ideal of R. Thus by hypothesis, we have

$$\begin{aligned}g\circ h\wedge h\circ g &= (g\circ h\wedge h\circ g)\circ(g\circ h\wedge h\circ g)\\ &\le (g\circ h)\circ(h\circ g)\\ &= g\circ h\circ g \quad \text{because } h\circ h = h \text{ (by hypothesis)}\\ &\le g\circ f_R\circ g \quad \text{because } h\le f_R\\ &\le g \quad \text{because } g \text{ is a fuzzy bi-ideal of } R.\end{aligned}$$

Similarly $g\circ h\wedge h\circ g\le h$. Thus $g\circ h\wedge h\circ g\le g\wedge h$. Hence $g\circ h\wedge h\circ g = g\wedge h$.

$(3)\Rightarrow(4)$ Let g, h be fuzzy bi-ideals of R such that $f^2\le g$. Then by hypothesis, $f = f\wedge f = f\circ f\wedge f\circ f = f\circ f = f^2\le g$. Thus g is a semiprime fuzzy bi-ideal of R. Hence every fuzzy bi-ideal of R is semiprime.

$(4)\Rightarrow(5)$ Let f be a proper fuzzy bi-ideal of R and $\{f_i : i\in I\}$ be the collection of all irreducible fuzzy bi-ideals of R which contains f. Then $f\le\bigwedge_{i\in I} f_i$. Let $a\in R$ and $a\in(0,1]$ be such that $f(a) = a$. Then by Theorem 4.5, there exists an irreducible fuzzy bi-ideal f_a of R such that $f\le f_a$ and $f(a) = f_a(a)$. This implies $f_a\in\{f_i : i\in I\}$. Thus $\bigwedge_{i\in I} f_i\le f_a$. So $\bigwedge_{i\in I} f_i(a)\le f_a(a) = f(a)$. This implies $\bigwedge_{i\in I} f_i\le f$. Hence $\bigwedge_{i\in I} f_i = f$. By hypothesis, each fuzzy bi-ideal of R is semiprime. Thus each fuzzy bi-ideal of R is the intersection of all irreducible semiprime fuzzy bi-ideals of R which contain it.

$(5) \Rightarrow (2)$ Let f be a fuzzy bi-ideal of R. Then $f^2 = f \circ f$ is a fuzzy bi-ideal of R. Thus by hypothesis $f^2 = \underset{i \in I}{\wedge} f_i$, where each f_i is an irreducible semiprime fuzzy bi-ideal of R such that $f^2 \le f_i$. This implies $f \le f_i$ for all $i \in I$, because each f_i is a semiprime fuzzy bi-ideal of R. Thus $f \le \underset{i \in I}{\wedge} f_i = f^2$. But $f^2 \le f$ is always true. Hence $f^2 = f$.

4.7. Proposition.

Let R be a regular and intra-regular semiring. Then the following assertions for a fuzzy bi-ideal f of R are equivalent:

(1) f is strongly irreducible.

(2) f is strongly prime.

Proof. The proof follows from the fact that in a regular and intra-regular semiring $g \wedge h = g \circ h \wedge h \circ g$ for any fuzzy bi-ideals g, h of R.

Next we characterize those semirings in which each fuzzy bi-ideal is strongly prime and also those semirings in which each fuzzy bi-ideal is strongly irreducible.

4.8. Theorem.

Each fuzzy bi-ideal of a semiring R is strongly prime if and only if R is regular and intra-regular and the set of fuzzy bi-ideals of R is totally ordered by inclusion.

Proof. Suppose that each fuzzy bi-ideal of R is strongly prime. Then each fuzzy bi-ideal of R is semiprime. Thus by Theorem 4.6, R is both regular and intra-regular. We show that the set of fuzzy bi-ideals of R is totally ordered by inclusion. For this, let g, h be any two fuzzy bi-ideals of R. Then by Theorem 4.6, $g \circ h \wedge h \circ g = g \wedge h$. Since each fuzzy bi-ideal of R is strongly prime, we have either $g \le g \wedge h$ or $h \le g \wedge h$. If $g \le g \wedge h$, then $g \le h$ and if $h \le g \wedge h$, then $h \le g$.

Conversely, assume that R is regular and intra-regular and the set of fuzzy bi-ideals of R is totally ordered by inclusion. Let f, g, h be fuzzy bi-ideals of R such that $g \circ h \wedge h \circ g \le f$. By Theorem 4.6,

$$g \wedge h = g \circ h \wedge h \circ g \le f.$$

Since the set of fuzzy bi-ideals of R is totally ordered by inclusion, so either $g \le h$ or $h \le g$. This implies either $g \wedge h = g$ or $g \wedge h = h$. Thus $g \le f$ or $h \le f$.

4.9. Theorem.

If the set of fuzzy bi-ideals of a semiring R is totally ordered by inclusion, then R is both regular and intra-regular if and only if each fuzzy bi-ideal of R is prime.

Proof. Suppose R is both regular and intra-regular. Let f, g, h be fuzzy bi-ideals of R such that $g \circ h \le f$. Since the set of fuzzy bi-ideals of R is totally ordered by inclusion, so either $g \le h$ or $h \le g$. If $g \le h$, then $g = g \circ g \le g \circ h \le f$. If $h \le g$, then $h = h \circ h \le g \circ h \le f$.

Conversely, suppose that every fuzzy bi-ideal of R is prime. Thus every prime fuzzy bi-ideal of R is semiprime, so by Theorem 4.6, R is both regular and intra-regular.

4.10. Proposition.

If the set of fuzzy bi-ideals of a semiring R is totally ordered, then the concept of primeness and strongly primeness coincides.

Proof. Let f be a prime fuzzy bi-ideal of R. Let f_1, f_2 be any two fuzzy bi-ideals of R such that $f_1 \circ f_2 \wedge f_2 \circ f_1 \leq f$. As the set of fuzzy bi-ideals of R is totally ordered, therefore either $f_1 \leq f_2$ or $f_2 \leq f_1$. If $f_1 \leq f_2$, then $f_1 \circ f_1 = f_1^2 = f_1^2 \wedge f_1^2 \leq f_1 \circ f_2 \wedge f_2 \circ f_1 \leq f$. Thus $f_1 \leq f$. Similarly, if $f_2 \leq f_1$, then $f_2 \leq f$. This implies that f is a strongly prime fuzzy bi-ideal of R. Every strongly prime fuzzy bi-ideal is obviously prime bi-ideal.

4.11. Theorem.

For a semiring R the following assertions are equivalent:

(1) Set of fuzzy bi-ideals of R is totally ordered by inclusion.

(2) Each fuzzy bi-ideal of R is strongly irreducible.

(3) Each fuzzy bi-ideal of R is irreducible.

Proof. $(1) \Rightarrow (2)$ Let f, g, h be any fuzzy bi-ideals of R such that $g \wedge h \leq f$. By hypothesis either $g \leq h$ or $h \leq g$, thus either $g \wedge h = g$ or $g \wedge h = h$. This implies either $g \leq f$ or $h \leq f$.

$(2) \Rightarrow (3)$ Straightforward.

$(3) \Rightarrow (1)$ Let g, h be two fuzzy bi-ideals of R. Then $g \wedge h$ is a fuzzy bi-ideal of R. Also $g \wedge h = g \wedge h$. This implies either $g = g \wedge h$ or $h = g \wedge h$. Thus $g \leq h$ or $h \leq g$.

4.12. Example.

Consider the semiring $R = \{0, a, b\}$.

+	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

.	0	a	b
0	0	0	0
a	0	a	a
b	0	b	b

It is evident that R is both regular and intra-regular.

Remark (1) *A fuzzy subset f of the semiring R is a fuzzy bi-ideal of R if and only if $f(0) \geq f(x)$ for all $x \in R$.*

Proof. Let f be a fuzzy bi-ideal of R. Then

$$f(0) = f(x0x) \geq f(x) \wedge f(x) = f(x) \quad \text{for all } x \in R.$$

Conversely, suppose that f satisfies $f(0) \geq f(x)$ for all $x \in R$. Then we show that f is a fuzzy bi-ideal of R. Let $x, y \in R$. Then

$f(x+y) = f(x) \wedge f(y)$ if $x = y$ or one of x, y is zero

$f(x+y) \geq f(x) \wedge f(y)$ if $x, y \in \{a, b\}$ and $x \neq y$

Thus $f(x+y) \geq f(x) \wedge f(y)$ for all $x, y \in R$.

Now as $xy = x$ if $x, y \in \{a, b\}$

and $xy = 0$ if one of x, y is zero.

Thus $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in R$.

Now as $xyz = x$ if $x, y, z \in \{a, b\}$

and $xyz = 0$ if one of x, y, z is zero.

Thus $f(xyz) \geq f(x) \wedge f(z)$ for all $x, y, z \in R$.

Remark (2) *By Theorem , every fuzzy bi-ideal of R is semiprime.*

Consider the fuzzy bi-ideals f , g and h of R given by

$$f(0) = .7 \qquad f(a) = .6 \qquad f(b) = .4$$
$$g(0) = 1 \qquad g(a) = .5 \qquad g(b) = .3$$
$$h(0) = .7 \qquad h(a) = .65 \qquad h(b) = .3$$

Then $g \circ h(0) = .7 \qquad g \circ h(a) = .5 \qquad g \circ h(b) = .3$

Then $g \circ h \leq f$ but neither $g \leq f$ nor $h \leq f$. Hence f is not a prime fuzzy bi-ideal of R .

4.13. Example.

Consider the semiring $R = \{0, x, 1\}$.

+	0	x	1
0	0	x	1
x	x	x	1
1	1	1	1

.	0	x	1
0	0	0	0
x	0	x	x
1	0	x	1

It is evident that R is commutative, regular and intra-regular. Bi-ideals of R are $\{0\}$, $\{0, x\}$ and R . All bi-ideals are strongly prime.

Remark. *A fuzzy subset f of the semiring R is a fuzzy bi-ideal of R if and only if $f(0) \geq f(x) \geq f(1)$.*

Proof. Let f be a fuzzy bi-ideal of R . Then

$$f(0) = f(a0a) \geq f(a) \wedge f(a) = f(a) \quad \text{for all } a \in R.$$

Also $f(x) = f(1x1) \geq f(1) \wedge f(1) = f(1)$.

Hence $f(0) \geq f(x) \geq f(1)$.

Conversely, assume that f satisfies $f(0) \geq f(x) \geq f(1)$. Then simple calculations show that f is a fuzzy bi-ideal of R .

Now we show that every fuzzy bi-ideal of R is not strongly prime fuzzy bi-ideal of R . Consider the fuzzy bi-ideals f , g and h of R given by

$$f(0)=.7 \qquad f(x)=.6 \qquad f(1)=.4$$
$$g(0)=1 \qquad g(x)=.5 \qquad g(1)=.3$$
$$h(0)=.7 \qquad h(x)=.65 \qquad h(1)=.3$$

Then $g\circ h(0)=h\circ g(0)=.7$

$$g\circ h(x)=h\circ g(x)=.5$$
$$g\circ h(1)=h\circ g(1)=.3$$

This implies $g\circ h(0)\wedge h\circ g(0)=.7$

$$g\circ h(x)\wedge h\circ g(x)=.5$$

and $g\circ h(1)\wedge h\circ g(1)=.3$

Thus $g\circ h\wedge h\circ g\leq f$ but neither $g\leq f$ nor $h\leq f$. Hence f is not a strongly prime fuzzy bi-ideal of R .

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