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Applications of the Exp-function Method for the MkdV-Sine-Gordon and Boussinesq-double Sine-Gordon Equations

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Abstract: In this paper, the Exp-function method is used to obtain generalized travelling wave solutions with free parameters of the MKdV-sine-Gordon and Boussinesq-double sine-Gordon equations. It is shown that the Exp-function method, with the help of any symbolic computation packages, provides an effective mathematical tool for nonlinear evolution equations arising in mathematical physics.

Key words: Exp-function method · Travelling wave solution · MkdV-sine-Gordon equation · Boussinesq-double sine-Gordon equation

INTRODUCTION

It is important to investigate new and more exact travelling wave solutions of nonlinear evolution equations (NLEEs) arising in various fields of physics and engineering. In recent years, with the rapid development of computer symbolic systems such as Mathematica or Maple allowing us to perform the complicated and tedious algebraic calculations on computer, many direct and effective methods for constructing exact solutions of NLEEs have become more and more attractive. Up to now, many authors have presented various powerful methods to deal with this subject, such as the inverse scattering transform method [1, 2], the Bäcklund transform [3], Darboux transforms [4–6], homotopy perturbation method [7- 9], variational iteration method [10-14], the parameter-expansion method [15-17] and so on. Recently, He and Wu [18] developed a straightforward method, called Exp-function method, for seeking analytic solutions of nonlinear partial differential equations. Applications of the method can be found in [19-26] and by the reference therein. The last development of the Exp-function method is reviewed in Ref. [27].

In this paper, we will apply Exp-function method to the MKdV-sine-Gordon and Boussinesq-double sine-Gordon equations in the form

$$u_{xt} + \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} = \sin u \tag{1}$$

and

$$u_{tt} - \alpha u_{xx} + u_{xxxx} = \sin u + \frac{3}{2}\sin(2u) \tag{2}$$

recpectively.

Exp-Function Method: Suppose we have a nonlinear partial differential equation for $u(x,t)$ in the form

$$\phi(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0 \tag{3}$$

where ϕ is a polynomial function with respect to the indicated variables. Using the wave transformation

$$u = u(\eta), \quad \eta = kx + wt \tag{4}$$

Eq. (3) reduces to an ordinary differential equation in the form

$$\psi(u, wu', ku', w^2u'', wku'', k^2u'', \ldots) = 0 \tag{5}$$

where k and w are constants to be determined later.

According to Exp-function method, the wave solution can be expressed in the form

$$u(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)} = \frac{a_c \exp(c\eta) + \ldots + a_{-d}\exp(-d\eta)}{a_p \exp(p\eta) + \ldots + a_{-q}\exp(-q\eta)} \tag{6}$$

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where *c, d, p* and *q* are positive integers which could be freely chosen, a_n and b_m are unknown constants to be determined later. To determine the values of *c* and *p*, we balance the linear term of the highest order in Eq. (5) with the highest order nonlinear term. Similarly to determine the values of *d* and *q*, we balance the linear term of the lowest order in Eq. (5) with the lowest order nonlinear term. So by means of the Exp-function method, we obtain the generalized travelling wave solutions for NLEEs arising in mathematical physics. To illustrate the effectiveness and convenience of the method, we consider the MKdV-sine-Gordon (1) and Boussinesq-double sine-Gordon equation (2) in the following section.

Application of the Exp-Function Methods

MKdV-Sine-Gordon Equation: We first consider, MKdV-sine-Gordon equation arising as nonlinear wave propagation in one-dimensional mono-atomic lattice in which the anharmonic potential competes with the dislocation potential and which can be solved by the inverse scattering transform [28-30]. Substituting the wave transformation (4) into Eq. (1), we obtain

$$wku'' + \frac{3}{2}k^4u'^2u'' + k^4u^{(iv)} = \sin(u) \tag{7}$$

where the prime denotes the derivative with respect to η. Next, we use the transformations

$$v = e^{iu}, \quad i = \sqrt{-1} \tag{8}$$

so that

$$\sin u = \frac{v - v^{-1}}{2i}, \quad \cos u = \frac{v + v^{-1}}{2}, \tag{9}$$

that also gives

$$u = \arccos\left(\frac{v + v^{-1}}{2}\right). \tag{10}$$

Using this transformation into Eq. (7), we obtain

$$\begin{aligned} &-v^3 + v^5 + 2kwv^2v'^2 + 9k^4v'^4 - 2kwv^3v'' \\ &\quad -21k^4vv'^2v'' + 6k^4v^2v''^2 + 8k^4v^2v'v''' \\ &\quad\quad -2k^4v^3v^{(iv)} = 0. \end{aligned} \tag{11}$$

Since there is no linear term in Eq. (11), in order to determine the values of *c, d, p* and *q*, we balance the nonlinear term of the highest order $v^3v^{(iv)}$ with the nonlinear term v^5. Using the ansatz (6), for the terms $v^3v^{(iv)}$ and by v^5 simple calculation, it follows

$$v^3v^{(iv)} = \frac{c_1\exp[(4c+15p)\eta] + \ldots}{c_2\exp(19p\eta) + \ldots} = \frac{c_1\exp[(4c+p)\eta] + \ldots}{c_2\exp(5p\eta) + \ldots} \tag{12}$$

and

$$v^5 = \frac{c_3\exp(5c\eta) + \ldots}{c_4\exp(5p\eta) + \ldots} \tag{13}$$

where c_1 are coefficients for simplicity. By balancing the highest order of Exp-function in Eqs. (12) and (13), we have $4c + p = 5c$ which leads to the result $c = p$.

Proceeding in the same manner as illustrated above, we can determine the values of *d* and *q* by balancing the lowest order terms of $v^3v^{(iv)}$ and v^5. Thus, we have

$$v^3v^{(iv)} = \frac{\ldots + d_1\exp[-(4d+15q)\eta]}{\ldots + d_2\exp(-19q\eta)} = \frac{\ldots + d_1\exp[-(4d+q)\eta]}{\ldots + d_2\exp(-5q\eta)} \tag{14}$$

and

$$v^5 = \frac{\ldots + d_3\exp(-5d\eta)}{\ldots + d_4\exp(-5q\eta)} \tag{15}$$

where d_i are coefficients for simplicity. By balancing the lowest order of Exp-function in Eqs. (14) and (15), we have $4d + q = 5d$ which leads to the result $d = q$.

In view of the obtained results, we can freely choose the values of *c* and *d*. For simplicity, we set $c = p = 1$ and $d = q = 1$, then Eq. (6) becomes

$$v(\eta) = \frac{a_1\exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{16}$$

Substituting Eq. (16) into Eq. (11), equating the coefficients of $\exp(n\eta)$ to zero yields a set of algebraic equations for $a_{-1}, a_0, a_1, b_{-1}, b_0, k$ and *w*. Solving the system of algebraic equations with the aid of Maple, we obtain

$$a_{-1} = -\frac{1}{4}b_0^2, a_0 = b_0, a_1 = -1, b_{-1} = \frac{1}{4}b_0^2, k = \pm\frac{1}{\sqrt{3}}, w = \mp\frac{10}{3\sqrt{3}} \tag{17}$$

$$a_{-1} = -\frac{1}{4}b_0^2, a_0 = b_0, a_1 = -1, b_{-1} = \frac{1}{4}b_0^2, k = \pm\frac{i}{\sqrt{3}}, w = \pm\frac{10i}{3\sqrt{3}} \tag{18}$$

and

$$a_{-1} = \frac{1}{4}b_0^{\ 2}, a_0 = -b_0, a_1 = 1, b_{-1} = \frac{1}{4}b_0^{\ 2}, k = k, w = \frac{1}{k} - k^3. \tag{19}$$

Inserting (17) into (16) with Eq. (10), we get the following generalized solitary solutions of Eq. (1)

$$u(x,t) = \arccos\left\{\frac{1}{2}\left(\frac{-e^{\eta} + b_0 - \frac{b_0^{\ 2}}{4}e^{-\eta}}{e^{\eta} + b_0 + \frac{b_0^{\ 2}}{4}e^{-\eta}} + \frac{e^{\eta} + b_0 + \frac{b_0^{\ 2}}{4}e^{-\eta}}{-e^{\eta} + b_0 - \frac{b_0^{\ 2}}{4}e^{-\eta}}\right)\right\}. \tag{20}$$

Simplifying Eq.(20) yields

$$u(x,t) = \arccos\left\{-\frac{16e^{2\eta} + 24b_0^{\ 2} + b_0^{\ 4}e^{-2\eta}}{(-4e^{\eta} + b_0^{\ 2}e^{-\eta})^2}\right\} \tag{21}$$

where b_0 is a non zero free parameter and $\eta = \pm\frac{x}{\sqrt{3}} \mp \frac{10t}{3\sqrt{3}}$. If we set $b = 2$, we obtain solution

$$u(x,t) = \arccos\left\{-1 - 2\operatorname{csch}^2\left(-\frac{x}{\sqrt{3}} + \frac{10t}{3\sqrt{3}}\right)\right\}. \tag{22}$$

Substituting (18) into (16) with Eq. (10), we get the same generalized solution as (21) for $\eta = \mp\frac{ix}{\sqrt{3}} \mp \frac{10it}{3\sqrt{3}}$. If we set $b_0 = 2$, then we obtain solution

$$u(x,t) = \arccos\left\{-1 + 2\csc^2\left(\frac{x}{\sqrt{3}} + \frac{10t}{3\sqrt{3}}\right)\right\}. \tag{23}$$

Inserting (19) into (16) with Eq. (10) and simplifying, we get

$$u(x,t) = \arccos\left\{\frac{16e^{2\eta} + 24b_0^{\ 2} + b_0^{\ 4}e^{-2\eta}}{(-4e^{\eta} + b_0^{\ 2}e^{-\eta})^2}\right\} \tag{24}$$

where b_0 is a non zero free parameter and $\eta = kx + \frac{t}{k} - k^3t$. If we set $b_0 = 1$, we obtain solution

$$u(x,t) = \arccos\left\{1 + 2\operatorname{csch}^2(kx + \frac{t}{k} - k^3t)\right\}. \tag{25}$$

To our knowledge, solutions (22), (23) and (25) are new solutions for the MKdV sine- Gordon equation and have not been reported yet.

Boussinesq-Double Sine-Gordon Equation: We secondly consider Boussinesq-double sine-Gordon equation [31] given by Eq.(2). Substituting the wave transformation (4) into Eq. (2), we obtain

$$(w^2 - \alpha k^2)u'' + k^4u^{(iv)} = \sin(u) + \frac{3}{2}\sin(2u) \tag{26}$$

Using the transformation (9) into Eq. (26), we get

$$\begin{aligned}&-3v^2 - 2v^3 + 2v^5 + 3v^6 + (4w^2 - 4\alpha k^2)v^2v'^2 \\ &+24k^4v'^4 - 4w^2v^3v'' + 4\alpha k^2v^3v'' - 48k^4vv'^2v'' \\ &+12k^4v^2v''^2 + 16k^4v^2v'v''' - 4k^4v^3v^{(iv)} = 0.\end{aligned} \tag{27}$$

Since there is no linear term in Eq. (27), in order to determine the values of c, d, p and q, we balance the nonlinear term of highest order $v^3v^{(iv)}$ with the nonlinear term v^6.

Using the ansatz (6), for the terms $v^3v^{(iv)}$ and by v^6 simple calculation, it follows

$$v^3v^{(iv)} = \frac{c_1\exp[(4c + 15p)\eta] + ...}{c_2\exp(19p\eta) + ...} = \frac{c_1\exp[(4c + 2p)\eta] + ...}{c_2\exp(6p\eta) + ...} \tag{28}$$

and

$$v^6 = \frac{c_3\exp(6c\eta) + ...}{c_4\exp(6p\eta) + ...} \tag{29}$$

where c_1 are determined coefficients only for simplicity. Balancing the highest order of Exp-function in Eqs. (28) and (29), we have $4c = 2p = 6c$ which leads to the result $c = p$.

Similarly, we can determine the values of d and q by balancing the lowest order terms of $v^3v^{(iv)}$ and v^6. Thus, we have

$$v^3v^{(iv)} = \frac{... + d_1\exp[-(4d + 15q)\eta]}{... + d_2\exp(-19q\eta)} = \frac{... + d_1\exp[-(4d + 2q)\eta]}{... + d_2\exp(-6q\eta)} \tag{30}$$

and

$$v^6 = \frac{... + d_3\exp(-6d\eta)}{... + d_4\exp(-6q\eta)} \tag{31}$$

where d_i are determined coefficients only for simplicity. Balancing the lowest order of Exp-function in Eqs. (30) and (31), we have $4d + 2q = 6d$ which leads to the result $d = q$.

In view of the obtained results, we can freely choose the values of c and d. For simplicity, we will examine case $c = p = 1$ and $d = q = 1$.

For $c = p = 1$ and $d = q = 1$, Eq. (6) can be expressed as

$$v(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1}\exp(-\eta)}{\exp(\eta) + b_0 + b_{-1}\exp(-\eta)}. \tag{32}$$

Substituting Eq. (32) into Eq. (27), equating the coefficients of $\exp(n\eta)$ to zero and solving the system of algebraic equations with the aid of Maple, we obtain

$$a_{-1} = \frac{1}{4}b_0{}^2, a_0 = -b_0, a_1 = 1, b_{-1} = \frac{1}{4}b_0{}^2, k = \pm 1, w = \pm\sqrt{\alpha+3} \tag{33}$$

and

$$a_{-1} = \frac{1}{4}b_0{}^2, a_0 = -b_0, a_1 = 1, b_{-1} = \frac{1}{4}b_0{}^2, k = \pm i, w = \pm i\sqrt{\alpha-3} \tag{34}$$

Inserting (33) into (32) with Eq. (10), we get the following new generalized solitary solutions of Eq. (26)

$$u(x,t) = \arccos\left\{\frac{1}{2}\left(\frac{e^{\eta} - b_0 + \frac{b_0{}^2}{4}e^{-\eta}}{e^{\eta} + b_0 + \frac{b_0{}^2}{4}e^{-\eta}} + \frac{e^{\eta} + b_0 + \frac{b_0{}^2}{4}e^{-\eta}}{e^{\eta} - b_0 + \frac{b_0{}^2}{4}e^{-\eta}}\right)\right\}. \tag{35}$$

Simplifying Eq.(35) yields

$$u(x,t) = \arccos\left\{\frac{16e^{2\eta} + 24b_0{}^2 + b_0{}^4 e^{-2\eta}}{(-4e^{\eta} + b_0{}^2 e^{-\eta})^2}\right\} \tag{36}$$

where b_0 is a non-zero free parameter and $\eta = \pm x \mp t\sqrt{\alpha+3}$. If we set $b_0 = 2$, then we obtain solution

$$u(x,t) = \arccos\left\{1 + 2\operatorname{csch}^2(x \mp t\sqrt{\alpha+3})\right\}, \alpha \geq -3. \tag{37}$$

Substituting Eq. (34) into (32) with Eq. (10), we obtain the same solution as (36) for $\eta = \pm ix \mp i\sqrt{\alpha-3}t$. If we set $b_0 = 1$, we get solution

$$u(x,t) = \arccos\left\{1 - 2\csc^2(x \mp t\sqrt{\alpha-3})\right\}, \alpha \geq 3. \tag{38}$$

Again, solutions (37) and (38) are new solutions for the Boussinesq-double sine-Gordon equation and have not been reported yet.

CONCLUSION

In this paper, the Exp-function method has been successfully used to obtain some new generalized travelling wave solutions of the MKdV-sine-Gordon and Boussinesq-double sine-Gordon equations. The results show that the Exp-function method is an effective mathematical tool for searching the exact travelling wave solutions of NLEEs arising in mathematical physics.

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