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Domains like Krull Domains and Their Factorization Properties

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Abstract: Recently we defined pseudo-valuation maps and by using these maps we reproduced and discussed pseudo-valuation domains. In this article we introduce P-Krull domains, while construction of these domains pseudo-valuation maps play an important role. Finally by using pseudo-valuation maps as a length function wecharacterize newly created structures as half factorial and bounded factorial domains.

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INTRODUCTION

Let R be an integral domain with quotient field K. A prime ideal P of R is called strongly prime if $xy\in P$; where $x, y\in K$; then $x\in P$ or $y\in P$. An integral domain R is said to be a pseudo-valuation domain (PVD) if every prime ideal of R is a strongly prime [1, Definition page 2]. An integral domain R is a PVD if and only if for each nonzero $x\in K\backslash R$ and for each nonunit$a\in R$; we have $ax^{-1}\in R$ [1, Theorem 1.5(3)]. Every valuation domain is a PVD [1, Proposition 1.1] but converse is not true. A quasi-local domain (R, M) is a PVD if and only if $x^{-1}M\subset M$ whenever $x\in K\backslash R$ [1, Theorem 1.4]. A Noetherian domain R with quotient field K is a PVD if and only if $x^{-1}\in R'$ (integral closure of R in K) whenever $x\in K\backslash R$[1, Theorem 3.1]. Let R be an integrally closed domain with quotient field K and $F = \{V_\lambda\}_{\lambda\in\gamma}$ be a family of valuation overrings of R. Then R is said to be a Krull domain if,$R = \cap\{V_\lambda\}_{\lambda\in\gamma}$ each V_λ is a DVR; the family F has finite character (that is, if $0\neq x\in K$, then x is a non-unit in only finitely many of the valuation rings in the family F) and each V_λ is essential for R (A valuation over ring of integral domain R is said to be essential for R if V_λ is fraction ring of R).

Recently much literature has been presented on the factorization properties weaker than UFD are HFDs and BFDs. An integral domain R is a bounded factorization domain (BFD) if it is atomic and for each nonzero nonunit of R, there is a bound on the length of factorization into products of irreducible elements. Examples of BFDs are UFDs and a Noetherian domains or a Krull domain. We define R to be half-factorial domain (HFD) if R is atomic and whenever $x_1 \dots x_m = y_1 \dots y_n$, where $x_1, x_2, \dots x_m, y_1, y_2, \dots y_n$ are irreducible in R; then $m = n$. A UFD is obviously an HFD, but the converse fails, since any Krull domain R with divisor class group, $CI(R) \cong Z$, is an HFD [2] but not a UFD. Moreover a polynomial extension of an HFD is not respond affirmatively, for example by [3, Theorem 2.2], $Z[\sqrt{-3}][X]$ is not an HFD, as $Z[\sqrt{-3}]$ is an HFD which is not integrally closed. Since every Krull domain is BFD, therefore it is not necessary that a BFD is an HFD.

In [4] author[2]has been discussed the stability of some domains including the pseudo-valuation domain for a domains extension relative to a condition*. In [5] it has been proved that a Krullmonoid domain $R[S]$ is an HFD if$S$has trivial class group while $CI(R) \cong Z_2$. Further in [6] authors[1,3] have been discussed the factorization properties, particularly the half-factorial and bounded factorization of Krullmonoid domain $R[S]$ through the length functions and class groups of monoid$S$ and domain R respectively.

We (authors[1,3]) gave brief introduction of generalized form of valuation maps (i.e. pseudo-valuation map) in [6] and have been discussed pseudo-valuation domain and its characteristics with the help of pseudo-valuation map.

In this study we establish new structures and generalized the work contained in [6] for newly established structures by characterizing newly born domains as HFD and BFD.

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P-Krull Domains: Let K be a field and R be a subring of K with identity, $K^*= K\backslash\{0\}$, the multiplicative group and $U(R)$ or U represent the units of R, which is the subgroup of K. $G= K^*/ U(R)$, the factor group with binary operation addition, $xU + yU = xyU$, where $xU, yU \in$ G and partial ordering on G is defined as; $xU \leq yU$ if and only if $x \mid y$ in R. The positive subset $G_+ = \{xU: xU \geq U\} = \{xU: x \in R\}$ is a cone with respect to relation $\leq$ containing 0 and closed under addition.

- Hereafter we add a property $*$ in G as follows:

Property$_*$: A partially ordered group G in which each $g \in G$, either $g \geq 0$ or $g<h$ for all $h \in G$ with $h>0$. A partially ordered group G with property $*$ will be denoted by G^*.

Let G be a partially ordered group and K be a field and $K^*= K\backslash\{0\}$, the multiplicative group. We initiate the following definition.

Definition 1: By [6], let $\omega : K^* \to G$ be an onto map, which has the following properties. For $x, y \in K$;

- $\omega(xy) = \omega(x) + \omega(y)$.
- $\omega(x) < \omega(y)$ implies $\omega(x + y) = \omega(x)$.
- $\omega(x) = g \geq 0$ or $\omega(x) < \omega(y) = h$, where $g, h \in G$ and $h > 0$.

In Definition 1, the map ω is the extension of semi-valuation map. No doubt Definition 1(b) implies that it is a quasi-local domain as discussed in [7, page 180]. Moreover condition (c) plays an important role and it induce property in G. Hereafter we call ω the pseudo-valuation map.

Now we construct a close but weaker structure to a Krull domain, i.e. a P-Krull domain, with the help of pseudo-value map ω. Before we start the discussion about P-Krull domain we feel necessary to define few terminologies.

Remark 1: Let G be a partially ordered group and X be a set of bounded elements of G; then X is convex subsemigroup of G. The subgroup $B(G)$ of G generated by X is a convex subgroup of G. If G is lattice ordered, then $B(G)$ is a sublattice subgroup of G [7, Proposition 19.10].

Remark 2: [6, Remark2] Let P be a strongly prime ideal in D and G be a group of divisibility of D, then there is one to one correspondence between strongly prime ideals and a convex subsets X in G which generate the convex subgroup $B(G)$ as in remark 1. By the definition of a strongly prime ideal (i.e. P is a strongly prime if and only if $x^{-1}P$, whenever $x \in K\backslash R$), we have a convex set C and for $x \in G\backslash C$ such that $-x + C \subset C$. We call such C a strongly convex set.

Definition 2:

- A family of pseudo-valuations of the field K is said to be of finite P-character if for every $x \in K$, $x \neq 0$, the set $\{\omega \in \Omega : w(x) \neq 0\}$ is finite.
- Corresponding to $\omega \in \Omega$, the PVD R_ω with a maximal ideal (which is strongly prime) P, the ring $A = \cap_{\omega \in \Omega} R_\omega$ is said to be defined by ω and $P \cap A$ is a (strongly) prime ideal in A, called centre of ω on A and we denote it by $Z(\omega)$. If $R_\omega = A_{Z(\omega)}$, then ω is said to be essential pseudo-valuation for A.
- Rank of a PVD D is the rank of pseudo-value group of D, i.e. G^*. Rank of pseudo-value group G depend upon the existence of ordinal type of set of proper strongly convex sets which are described in remark2 under inclusion.

Definition 3: Rank of a PVD R is the rank of the pseudo-value group of R, i.e. G^*. Rank of pseudo-value group G depend upon the existence of ordinal type of set of proper strongly convex sets which are described in remark1 under inclusion. If pseudo-value group G is isomorphic to $Z \oplus Z$, then it is P-rank one P-discrete.

Remark 3: The family of pseudo-valuations $V_{\lambda's}$ satisfying the conditions of definition 3, will call now P-rank one P-discrete pseudo-valuation overrings.

We define P-Krull domain as:

Definition 4: Let $F= \{V_\lambda\}_{\lambda \in \gamma}$ be the family of P-discrete pseudo-valuation overrings of integral domain R such that each V_λ is a root closed and;

- $R = \cap V_\lambda$
- Each V_λ is of finite P-character (i.e. every non-zero element is contain in atmost finitely many maximal ideals of V_λ for all $\lambda \in \gamma$).
- Each V_λ has P-rank one.
- Each V_λ is essential for R (PVD is essential for R if it is fraction ring of R).

Example 1: If we look upon P-Krull domain, then we observe that it is a pullback overV_λ/M, where M is unique common maximal ideal in each overring V_λ of domain $R = \phi^{-1}(D)$; where each V_λ is P-discrete. Where D is a root closed subring of V_λ/M, and $R = D + M$ is a P-Krull domain.

Example 2: If D is a Noetherian root closed domain then it has discrete valuation overrings (cf. [8]), thus it has pseudo-discrete pseudo-valuation (p-discrete valuations) overrings. Let these overrings of the form $B = K + M$. Also $k_1 \subset k_2 \subset \ldots K$ where each k_i is root closed in K. Let k_1 be a root closed subfield of K, then the subring $D = k_1 + M$, clearly D is a Noetherian root closed PVD and can be written as an intersection of $k_i + M$ where $i \triangleright 1$. Clearly D is a P-Krull domain (a PVD).

Remark 4: A P-Krulldomain is closed to a quasilocalKrull domain.

P-Krull domain may or may not be a PVD. If a P-Krull domain is a PVD then Krull domain has the following relation with it.

Proposition 1Krull domain having every ideal a strongly prime is a P-Krull domain (a PVD).

Proof: Let D be a Krull domain no doubt it is integrally closed hence root closed, also each Krull domain can be written as an intersection of discrete valuation over-rings. As discrete valuation overrings implies valuation overrings which implies pseudo-valuation overrings thus definition4(a) is satisfied, similarly easy to prove (b), (c) and (d), which completes the proof.

On the other hand if a P-Krull domain is not a PVD then Krull domain ⇒ P-Krull domain as in the following proposition.

Proposition 2: Krull domain is a P-Krull domain (not a PVD).

Proof: It is very clear from the definitions of Krull domain and P-Krull domain (not a PVD).

We can make the implication of newly established structures with the existing structures in the literature as;

$$\begin{array}{c} \text{DV R} \Rightarrow \text{PDPV R} \Rightarrow \text{P-Krull domain} \\ \Downarrow \\ \text{Krull domain} \end{array}$$

After introducing P-Krull domains now we discuss their class group (group of divisibility). Since these domains are not integrally closed, we may discuss about group of divisibility (which is in fact play a role as a class group) of any domain without distinction of integrally closedness. The construction of class groups of said domains are similar as discussed in [7, Page 69]. We construct a class group of P-Krull domains as follows.

Remark 5: Let k = Z/2Z and k ⊆ F be a subfield with four elements such as GF (2^2). Let $V_\lambda$be a P-discrete pseudo-valuations overrings of the form $V_\lambda = F + M$, where $M = X V_\lambda$ and each $V_\lambda$is a root closed. $R = k + M$ is a P-Krull domain and M : M = V_λ, a P-discrete pseudo-valuations overrings. Here $G\ (V_\lambda) \approx Z$ and $V^*/R^* \approx F^*/k^* \approx Z/3Z$, where Z=3Z is a trivially ordered semi-value group and Z is totally ordered. Also by [9, Remark 3.5] there is lexicographical exact sequence

$$\alpha\beta$$

$$1 \rightarrow F^*/\ k^* \approx Z/3Z \rightarrow G(R) \rightarrow G\ (V) \approx Z \rightarrow 1.$$

Here imα = kerβ, $G(R) \approx Z/3Z \oplus$ Z is a non-torsion free group of divisibility of R. Indeed $G(R)$ satisfying property and the positive cone G_+ = {(e, b): e $\in Z/3Z$; $0<$ b $\in Z$} induced partial ordering in $G(R)$. Any element g $\in G(R)$, where $g = (e, b) > 0$ if and only if $b > 0$.

Characterization of P-krull Domains: In this section we established a length function on generalized P-Krull domains via pseudo-valuation maps and class groups. Furthermore we characterized P- Krull domains as HFDs and BFDs. We defined length function which is based upon the pseudo-valuation map. Indeed, our defined pseudo-valuation map is also a length function (by Definition1 (a)). Throughout in this work we denote length functions by L_R and l_R, we mean by this notation a pseudo- value map. In general,

$$\text{UFD} \Rightarrow \text{HFD} \Rightarrow \text{BFD} \Rightarrow \text{ACCP} \Rightarrow \text{Atomic.}$$

Following [10, Pages 933 and 934], let R be an integral domain and let x be an element of R. We define;

- $l_R(x) = \inf\{x: x = x_1x_2\ldots x_n$, where each x_i is irreducible and $1 \le i \le n\}$ and
- $L_R(x) = \sup\ \{x: x = x_1x_2\ldots x_n$, where each x_i is irreducible and $1 \le i \le n\}$.

However $l_R(x) = L_R(x) = 0$ if x is unit element of R and $l_R(x) = L_R(x) = 1$ if x is irreducible. Moreover;

$l_R(xy) \leq l_R(x) + l_R(y)$ and $L_R(xy) \geq L_R(x) + L_R(y)$.

We are very clear about atomic pseudo-valuation domains (HFDs). We may observe the following possibilities for a P-Krull domain:

- Atomic PVD.
- A non atomic PVD.
- Non PVD atomic.
- Non atomic non PVD.

Two of them are important from our point of view that are an atomic PVD (an HFD) and non PVD atomic. Generally if we look upon example 1, one can easily judge when a P-Krull domains are atomic or non atomic (cf. [9, Proposition 3.2]).

Example 3: P-Krull domains which are atomic but not a PVD are in the form of $K + LX + X^2F[[X]]$, where $K \subset L \subset F$.

Remark 6: If P-Krull domains are PVDs and atomic then by [11] these are HFDs.

Now we discuss about those P-Krull domains which are atomic but not PVDs. We are proving in proposition3 that an atomic P-Krull domain R (not a PVD) is a BFD, since every strongly prime ideal is a prime ideal so we will freely use the terminology as used for a Krull domains in the literature.

Proposition 3: Let R be an atomic P-Krull domain (not a PVD) and $X^{(1)}(R)$, the set of height one strongly prime ideals with family $\{v_P : P \in X^{(1)}(R)\}$ of essential pseudo-discrete pseudo-rank one valuations. Then V: $R^* \to Z_+$ by $V(x) = \sum v_P(x)$, where $x \in R^*$ characterize P-Krull domain as a BFD.

Proof: Let us define,

- V: $R^* \to Z_+$ by $V(x) = \sum v_P(x)$, where $x \in R^*$.
- Thus $V(x) = n \geq 1$ if and only if $xR = (P_1 . P_2 \ldots P_n)$ for some $P_i \in X^{(1)}(R)$.

Then V defines a length function on R such that $V(x) = 1$ if and only if x is irreducible. Note that $L_R(x) \leq V(x)$ for each $x \in R$, so we can say there exist a bound on the factorization into irreducible of the elements of the P-Krull domain R (not a PVD). Hence a P-Krull domain (not a PVD) is a BFD.

Proposition 4: Let R be an atomic P-Krull domain (not a PVD). If class group (i.e. group of divisibility) of R is isomorphic to $Z \oplus Z_2$; then R is an HFD.

Proof: Suppose R is an atomic P-Krull domain (not a PVD) such that $\mathrm{Cl}(R) \cong Z \oplus Z_2$. By partial ordering induced by $G_+ = \{(e, b)$: $e \in Z/3Z$, $0 < b \in Z\}$ in $\mathrm{Cl}(R)$, i.e. any element $g \in G(R)$, where $g = (e, b) > 0$ if and only if $b > 0$. We can also defind a length function as;

- L_R: $R \to Z_+$ defined as: $l_R(x) = n_i \|[P_i]\|$,

where $\|[P_i]\|$ represent the order of $[P_i]$; l_R is a length function which shows that R is a Half-factorization domain (HFD).

Now finally we define a length function on additive P-Krullmonoids with the help of pseudo-valuation map and characterize them. But here first we define useful terminologies.

Definition 5: A monoid H is called pseudo-valuation monoid (PVM) if $x \in G \backslash H$ (G be a quotient groupoid of H) and $a \in H \backslash H^{\times}$ (where $H^{\times}$ is a set of invertible elements of H) implies $x^{-1}a \in H$ [12, Definition 16.7].

In the above definition5, the monoid H is multiplicative monoid, we can redefine it for addition as:

Definition 6:

- G be a quotient groupoid of an additive monoid H if $H \subset G$ is a submonoid and $G = \{c + a$: $a \in H$; $c \in H \backslash \{0\}\}$.
- An additive monoid H is called pseudo-valuation monoid if $x \in G \backslash H$ (G be aquotientgroupoid of H) and $a \in H \backslash H^{\times}$ (where $H^{\times}$ is a set of invertible elements of H) implies $-x + a \in H$.
- Let S be a cancellative additive monoid with quotient group G. A fractional ideal of S is a nonzero subset I of G such that there exist nonzero element $s \in S$ with $s + I \subseteq S$ and $S + I \subseteq I$. A fractional ideal need not be a subsemigroup of G. A principal fractional ideal of S is a subset I of G such that $I = x + S = (x)$ for some $x \in G$ [13, page 1460].
- The torsion free cancellativemonoid S is a P-Krullmonoid if there exists afamily $(v_\alpha)_{\alpha \in A}$ of rank-one pseudo-discrete valuations on G, the quotient group of S, such that S is the intersection of the pseudo-valuation semigroups of the $v_{\alpha's}$ and for every $x \in S$, $\{\alpha \in A$: $v_\alpha(x) > 0\}$ is finite.

- An atomic monoid$M$ is said to be bounded factorization monoid(BFM) if there exist a bound $N(x) \in Z^{+}$ on factorization of any nonzero nonunit$x \in M$, that is, if $x = \sum_{i=1}^{t} x_i$ where $x_{i's}$ are irreducible elements in M, then $t \leq N(x)$ [6, Definition3].

Proposition 5: Let S be an additive atomic P-Krullmonoid (not a PVM) having zero is the only invertible element, with torsion class group and for some $P_1, P_2, \ldots, P_t$, the height one strongly prime ideals, $x \in S$ be a nonunit such that $(x) = (n_1P_1 + n_2P_2 + \ldots + n_tP_t)$. Then $L_S : S \to Z_+$ defined as;

$L_S(x) = \sum n_i \|[P_i]\|$, where $\|[P_i]\|$ represent the order of $[P_i]$, $L_S$is a length function S is a bounded factorization momoid.

Proof: For nonunits$x, y \in S$, we have;

- $L_S(x) = \sum n_i\|[P_i]\|$ and $L_S(y) = \sum m_i\|[P_i]\|$,
- $L_S(x) = 0$ if $x \in U(S)$ ($U(S)$ represent units of S).
- Consider$L_S(x) + L_S(y) = \sum (n_i + m_i)\|[P_i]\|$, where each $n_i, m_i \in Z_+$.
- Since $(x) = (n_1P_1 + n_2P_2 + \ldots + n_tP_t)$ and $(y) = (m_1P_1 + m_2P_2 + \ldots + m_tP_t)$, therefore, $(x) + (y) = (n_1P_1 + n_2P_2 + \ldots + n_tP_t) + = (m_1P_1 + m_2P_2 + \ldots + m_tP_t)$.

As prime ideals are linearly ordered and also by the definition of length function we have,

$$L_S(x) + L_S(y) \leq L_S(x + y).$$

Hence L is a length function. Since each L_S is finite which shows that additive P-Krullmonoid (not a PVM) is a bounded factorization monoid.

Proposition 6: A monoid domain $R[S]$ be a P-Krullmonoid domain (not a PVD) if R is a P-Krull domain (not a PVD), S is a P-Krullmonoid (not a PVM).

Proof: If K is the quotient field of R then $R = K \cap R[S]$ and hence R is a P-Krull domain. Let G be the quotient groupoid of S and $(v_{\alpha})_{\alpha \in \lambda}$ be a family of P-rank one P-discrete pseudo-valuations defining $R[S]$ as a P-Krull domain. For $\alpha \in \lambda$define $\omega_{\alpha} : G \to Z \oplus Z$ by $\omega_{\alpha}(g) = v_{\alpha}(X^{g})$ and let $A = \{\alpha \in \lambda : \omega_{\alpha}$ has P-rank one$\}$, thus family $(\omega_{\alpha})_{\alpha \in A}$ has a finite P-character. Now for $g \in G$, $\omega_{\alpha}(g) \geq 0$ for each $\alpha \in A$ $\Leftrightarrow v_{\alpha}(X^{g})$ for each $\alpha \in A \Leftrightarrow X \in R[S] \Leftrightarrow g \in S$. Thus S is a P-Krullmonoid

Remark 7: If $R[S]$ is a P-Krull domain then $R[G]$ is also a P-Krull domain, as discussed in [13, Theorem 1] for a Krull domain.

Remark 8: Let $R[S]$ be an atomic P-Krull domain (not a PVD) and $(v_{\alpha})_{\alpha \in I}$ be a defining family of P-discrete pseudo-valuations on $R[G]$ such that each nonzero element of $R[G]$ has nonzero value at only finitely many of the pseudo-valuations v_{α}. If $f \in R[S]$, then $v_i(f) \geq 0$ for all $i \in I$, $v_i(f) = 0$ if f is invertible.

Example 4: If R[S] is an atomic P-Krull domain (not a PVD) and $S_0 = \{s \in S: v(s)\ 0\}$ is a sbsemigroup of P-Krullsemigroup$S$. If we take $R[S_0]$ such that $Cl(R[S_0])$ $\cong Z \oplus Z_2$ with S_0 has trivial class group, then we can define a length fuction$l : R[S] \to R_+$ as;

- $l(f) = \sum v_i(f) = \sum \inf\{v_i(g) : g \in Supp(f)\}$ which shows that $R[S]$ is an HFD.

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