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The Extention of Mean Value Theorem for Set-valued Lipschitz Mappings

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Abstract: In this article, we research on a kind of mean value theorem. we prove that this theorem for set-valued mappings under invexity of domein in Banach spaces.

MSC: 2010

Key words: Invex analysis . set-valued map . Lipschitz map . mean value theorem

INTRODUCTION

The mean value theorem is an important theorem in different fields. This theorem is intesting in smooth and non-smooth analysis forms [1, 5, 7]. In this decade many researcher work on set-value mapping [6] and non smooth form [1, 5-7] also the extend of this theorem in non smooth case by using the concept of subdiffrential, introduced by Clark,be obtained.In all of these theorems the related function is defined on a line segment (interval) and the results eare obtained under a convex domain. Antczak [7] by using the concept of Clarke's generalized gradient established a mean value theorem under invexity of domain for locally Lipschitz mappings [1]. R. Burachik proved an extended version of mean value theorem for set-valued functions. In this paper we extend Burachik's proof under invexity of domains.

PRELIMINARIES

Let X and Y be two nonempty sets and let $F: X \rightrightarrows Y$ be a mapping defined on X which takes values in the family of subsets of Y; that is F(x) is a subset of Y with the possibility that F(x) = ∅ for some x∈X is admitted. In this case F is characterized by its graph, which is defined by

$$Gph(F):-\{(x,y)\in(X,Y): y\in F(x)\}.$$

The projection of Gph(F) onto its first argument is the domain of F, denoted by D(F); i.e.,

$$D(F):-\{x\in X: F(x)\neq\emptyset\}.$$

The projection of Gph(F) onto its second argument is the range of F, denoted by R(F) and hence

$$R(F):-\{y\in Y: \exists x\in D(F)$$

such that y∈F(x)}.

Assume S is a nonempty open set in $X\ and\ f: S\rightarrow\mathbb{R}$ be a real-valued mapping. The Clarke's generalized directional derivative of f at x∈S in the direction v∈X is defined by

$$f^{\circ}(x;v)=\limsup_{y\to x,\,t\downarrow 0}\frac{f(y+tv)-f(y)}{t}.$$

Also, the Clarke's generalized gradient of f at x, denoted by $\partial f(x)$, is the subset of X^* given by

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$$\partial f(x) = \{x^* \in X^* : \langle x^*, u\rangle \leq f^\circ(x;u) \geq \text{for all} u \in X\}.$$

It is well known that

(i) $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$.
(ii) $\partial(tf)(x) = t\,\partial f(x)$, where t is any scalar.
(iii) If f attains a local extremum at x, then $0\in\partial f(x)$.

For a nonempty subset K of a Banach space X, the normal cone $N_K: X \rightrightarrows X^*$ is defined by

$$N_K(x) = \begin{cases} \{x^* \in X^* : \langle x^*, y-x\rangle \leq 0 \ \ \forall y \in K\} & \text{if} \quad x \in K, \\ \emptyset & \text{otherwise,} \end{cases}$$

the tangent cone $T_K: X\bar{\Lambda}X$ of K is defined by

$$T_K(x) = \{y \in X : \langle x^*, y\rangle \leq 0 \forall x^* \in N_K(x)\}$$

and $S_K: X\bar{\Lambda}X$ is defined as

$$S_K(x) = \begin{cases} \left\{\frac{y-x}{\lambda} : y \in K, \lambda > 0\right\} & \text{if} \quad x \in K, \\ \emptyset & \text{otherwise.} \end{cases}$$

For a convex subset K of X, it is well known that $S_K(x) \subseteq T_K(x)$ for all $x\in K$ [6].

Definition 2.1: The derivative $DF(x_\circ, y_\circ): X \rightrightarrows Y$ of F at a point $(x_\circ, y_\circ) \in Gph(F)$ is the set-valued map whose graph is the tangent cone $T_{Gph(F)}(x_\circ, y_\circ)$, as introduced in definition

$$Gph(DF(x_\circ, y_\circ)) = T_{Gph(F)}(x_\circ, y_\circ).$$

Definition 2.2: Let S be a nonempty subset of $\mathbb{R}^n$, $\eta: S\times S \to \mathbb{R}^n$ and $f: S \to \mathbb{R}$ betwo mappings. The set S is said to be

(i) Invex at $u\in S$ with respect to η if $u + \lambda\eta(x,u) \in S$ for all $x\in S$ and all $\lambda\in[0,1]$.
(ii) Invex with respect to η if S is invex at each $u\in S$ with respect to the same η.

Let S be a nonempty invex set with respect to η and $x,u\in S$. A set P_{uv} is said to be a closed η-path joining the points u and $v = u+\eta(x,u)$ (contained in S) if

$$P_{uv} := \{y = u + \lambda\eta(x,u) : \lambda \in [0,1]\}.$$

Analogously, an open η-path joining the points u and $v = u+\eta(x,u)$ (contained in S) we call a set of the form

$$P^\circ_{uv} := \{y = u + \lambda\eta(x,u) : \lambda \in (0,1)\}.$$

A function f is said to be pre-invex with respect to η if, there exists a vector-valued function η such that

$$f(u + \lambda\eta(x,u)) \leq \lambda f(x) - (1-\lambda) f(u) \forall x,u \in S, \forall\lambda \in [0,1]. \tag{1}$$

holds. If function f be differentiable said to be invex with respect to η if there exists a vector-valued function η such that

$$f(x) - f(u) \geq [\eta(x,u)]^T \nabla f(u) \quad \forall x,u \in S. \tag{2}$$

Definition 2.3: Let $S \subseteq \mathbb{R}^n$be a nonempty invex set with respect to η. A function $f: S \to \mathbb{R}$is said to be Q-pre-invex with respect to η and β, if there exist vector-valued functions η and $\beta: C\times C \to \mathbb{R}$ (where C is open subset of $\mathbb{R}$) such that the relation

$$f(u+\lambda\eta(x,u)) \leq f(u)+\lambda\beta(f(x),f(u))\forall x,u \in S, \forall\lambda \in [0,1]$$

holds also differentiable function f is said to be invex with respect to η and β if there exists a vector-valued functions η and β such that the relation

$$\beta(f(x),f(u)) \geq [\eta(x,u)]^T \nabla f(u) \quad \forall x,u \in S \tag{3}$$

Definition 2.4: Let X and Y be Banach spaces. $F: X \rightrightarrows Y$ is called

(i) Convex [7] if Gph(F) is convex.
(ii) η-convex if for all α∈[0,1] and all x_1, x_2∈D(F) we have

$$F(x_2)+\alpha(F(x_1)-F(x_2)) \subseteq F(x_2+\alpha\eta(x_1,x_2))$$

(iii) Q-η-convex if for all α∈[0,1] and all x_1, x_2∈D(F) we have

$$F(x_2)+\alpha\beta(F(x_1),F(x_2)) \subseteq F(x_2+\alpha\eta(x_1,x_2)).$$

Note that $F: X \rightrightarrows Y$ is Convex if and only if

$$\alpha F(x_1)+(1-\alpha)F(x_2) \subseteq F(\alpha x_1+(1-\alpha)x_2)$$

∀α∈[0,1] and x_1,x_2∈D(F) [7]. Also, a mapping F is η-convex if Gph(F) is η-convex.

MAIN RESULT

Theorem 3.1: Suppose $F: X\bar{\Lambda}Y$ is η-convex graph, $(x_\circ, y_\circ) \in Gph(F)$ and x∈D(F). Then

$$F(x)-y_\circ \subseteq DF(x_\circ,y_\circ)\eta(x,x_\circ).$$

Proof: By η-convexity of Gph(F) we have

$$(x_\circ+t\eta(x,x_\circ), y_\circ+t(y-y_\circ)) \in Gph(F)$$

where y∈F(x) and t∈[0,1]. Hence,

$$y-y_\circ \in \frac{F(x_\circ+t\eta(x,x_\circ))-y_\circ}{t}.$$

Then

$$(\eta(x,x_\circ), y-y_\circ) \in S_{Gph}(F)(x_\circ,y_\circ) \subset T_{Gph}(F)(x_\circ,y_\circ) = Gph\ (DF(x_\circ,y_\circ)).$$

which it implies that $y-y_\circ \in DF(x_\circ,y_\circ)\eta(x,x_\circ)$ for all y∈F(x).

Theorem 3.2: Suppose C is open subset of Y, $F: X\bar{\Lambda}Y$ is Q,η-convex graph, $(x_\circ,y_\circ) \in Gph(F), x \in D(F)$ and $\beta: C \times C \rightarrow Y$, Then

$$\beta(F(x),y_\circ) \subseteq DF(x_\circ,y_\circ)\eta(x,x_\circ).$$

Proof: By Q,η-convexity of Gph(F) we get

$$(x_\circ+t\eta(x,x_\circ), y_\circ+t\beta(y,y_\circ)) \in Gph(F)$$

for all y∈F(x) and all t∈[0,1]. Hence,

$$\beta(y, y_\circ) \in \frac{F(x_\circ + t\eta(x, x_\circ)) - y_\circ}{t}.$$

Then

$$(\eta(x, x_\circ), \beta(y, y_\circ)) \in S_{Gph}(F)(x_\circ, y_\circ) \subset T_{Gph}(F)(x_\circ, y_\circ) = Gph(DF(x_\circ, y_\circ)),$$

which it implie that $\beta(y, y_\circ) \in DF(x_\circ, y_\circ)\eta(x, x_\circ)$ holds for all $y \in F(x)$

Theorem 3.3 [5]: For any (real valued) locally Lipschitzian function f defined on P_{uv} contained in an open invex set $S \subseteq X$ with respect to η there exist $z \in P_{uv}^o$ and $x^* \in \partial f(z)$ such that

$$f(v) - f(u) - \langle x^*, \eta(x, u)\rangle.$$

Theorem 3.4: For any (real valued) locally Lipschitzian function f defined on P_{uv} contained in an open invex set $S \subseteq X$ with respect to η and $\beta: C \times C \to \mathbb{R}$ (Cis open subset of $\mathbb{R}$) be locally Lipschitzian function. Let

$$f(u) - f(v) - \beta(f(u), f(v))$$

Then there exist $z \in P_{uv}^o$ and $x^* \in \partial f(z)$ such that

$$\beta(f(v), f(u)) - < x^*, \eta(x, u) >.$$

Proof: Set $u_t: - u + t\eta(x, u)$ and define the function $\mu: [0,1] \to \mathbb{R}$ as $\mu(t) = f(u_t)$. Since f is locally Lipschitz, μ is locally Lipschitz too. First we consider two sets $\partial\mu(t)$ and $\langle \partial f(u_\mu), \eta(x,u)\rangle$. These two sets are intervals in $\mathbb{R}$. For each $\lambda = \pm 1$,

$$\max\{\partial\mu(t)\lambda\} - \mu^\circ(t; \lambda) = \limsup_{s \to t, \delta \downarrow 0} \frac{\mu(s+\delta\lambda) - \mu(s)}{\delta} = \limsup_{s \to t, \delta \downarrow 0} \frac{f(u + (s+\delta\lambda)\eta(x,u)) - f(u + s\eta(x,u))}{\delta}.$$

By setting $y - u + s\eta(x, u)$, we have $y \to u_t$ and hence

$$\max\{\partial\mu(t)\lambda\} \leq \limsup_{s \to u_t, \delta \downarrow 0} \frac{f(y + \delta\lambda\eta(x,u)) - f(y)}{\delta} - f^\circ(u_t; \eta(x, u)\lambda) - \max\langle \partial f(u_t), \eta(x, u)\lambda\rangle$$

Therefore, $\partial\mu(t) \subseteq \langle \partial f(u_t), \eta(x, u)\rangle$. Now define the function $g: [0,1] \to \mathbb{R}$ as

$$g(t) - f(u_t) + t\beta(f(u), f(v)).$$

Since g is continuous, it attains a local extremum at a point $\bar{t} \in (0,1)$, which it implies that $0 \in \partial g(\bar{t})$. Thus $0 \in \partial g(\bar{t}) = \partial\mu(\bar{t}) + \beta(f(u), f(v)) \subseteq < \partial g f(u_t); \eta(x, u) > + \beta(f(u), f(v))$

By setting $c - u_{\bar{t}} - u + \bar{t}\eta(x, u)$, $c \in P_{uv}^\circ$ and so there exists $x^* \in \partial f(c) = \partial f(u_{\bar{t}})$ such that

$$\beta(f(v), f(u)) - \langle x^*, \eta(x, u)\rangle.$$

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