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The Generalized Finite Difference Method for Solving Elliptic Equation on Irregular Mesh

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Abstract: In this paper, the Generalized Finite Difference Method (GFDM) is used for solving elliptic equation on irregular grids or irregular domains.This Method is applied to 3D Poisson's equation with Dirichlet boundary conditions on irregular grids in a cuboid. This method is also used for more general linear PDS's on 3D domains. The partial derivatives are approximated by using the least squares method and Taylor expansion. The results show the performance and efficiency of method.

Key words: Generalized finite differences . advection-diffusion equation . irregular grids

INTRODUCTION

Numerical solution of Partial Differential Equations (PDE) on irregular grids is an important field in applied sciences and engineering. Solving PDE problems on irregular domains using finite difference formula is not a new approach, but in recent years it has been attracted special attention and it is generalized to irregular meshes. For instance Urena *et al*. [1] applied twenty seven node star and least squares method to solve PDE's. Benito *et al*. [2-4] have made interesting contributions to the development of this method. This latter applications are concerned solving parabolic equations. Urena *et al*. [1] used a twenty seven points star to solve advection-diffusion on irregular grids. In this paper the latter method is applied to solve three dimentional linear PDE's, in particular Poisson's equation on 3D irregular grids. The 2D Poisson's equation have been solved by a special method using finite differences on irregular meshes and irregular domains by Izadian *et al*. [5]. This paper is organized as follows. In section 2 a description of method is given. In section 3 the numerical results are presented. Finally, in section 4 the conclusion terminate the paper.

DESCRIPTION OF THE METHOD

Consider the following linear partial differential equation

$$a_1 u_x + a_2 u_y + a_3 u_z + b_1 u_{xx} + b_{:} + b_3 u_{zz} + c_1 u_{xy} + c_2 u_{xz} + c_3 u_{yz} + du = f,\ \ on\ \Omega \subset \mathbb{R} \qquad (2.1)$$

subject to the following Dirichlet boundary condition

$$u(x,y,z) = f(x,y,z), \qquad (x,y,z) \in \partial\,\Omega \qquad (2.2)$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c$ and are given functions of Suppose is discretized by a set of discrete points i.e.

$$\Omega_h = \{(x_i, y_i, z_i) \in \Omega \mid i = 1, 2, \ldots, i$$

suppose

$$\Omega_h^o = \{(x_i, y_i, z_i) \in \Omega_h \mid (x_i, y_i, z_i) \in int(\Omega)\}$$

$$\Gamma_h = \{(x_i, y_i, z_i) \in \Omega_h \mid (x_i, y_i, z_i) \in \partial(\Omega)\}.$$

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Accepting $N \in \mathbb{N}$ is a given positive integer, then at each corresponds a set which is called star set, that contains points adjacent to the point and itself. Then

$$S^{(i)} = \left\{ (x_i^{(j)}, y_i^{(j)}, z_i^{(j)}) \in \Omega_h | j = 1,2,\ldots,N, i = 1,2,\ldots,N_i \right\},$$

where is the number of elements of and is the number of elements of In order to solve (2.1) and (2.2), it is desirable that the first and second order partial derivatives of are approximated by the following formulae:

$$\frac{\partial u}{\partial x}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \lambda_j^{(k)} u(X_k^{(j)}),$$

$$\frac{\partial u}{\partial y}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \mu_j^{(k)} u\left(X_k^{(j)}\right),$$

$$\frac{\partial u}{\partial y}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \eta_j^{(k)} u\left(X_k^{(j)}\right),$$

$$\frac{\partial^2 u}{\partial x^2}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \rho_j^{(k)} u(X_k^{(j)}),$$

$$\frac{\partial^2 u}{\partial y^2}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \gamma_j^{(k)} u\left(X_k^{(j)}\right),$$

$$\frac{\partial^2 u}{\partial x^2}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \tau_j^{(k)} u(X_k^{(j)}),$$

$$\frac{\partial^2 u}{\partial x \partial y}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \delta_j^{(k)} u\left(X_k^{(j)}\right),$$

$$\frac{\partial^2 u}{\partial x \partial z}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \sigma_j^{(k)} u(X_k^{(j)}),$$

$$\frac{\partial^2 u}{\partial y \partial z}\left(X_k^{(1)}\right) = \sum_{j=1}^{N} \vartheta_j^{(k)} u\left(X_k^{(j)}\right) \qquad (2.3)$$

where $X_k^{(j)} - (x_k^{(j}$ for and are the points of The parameters λ, μ and are the constant real parameters that can be determined by using the method of [1]. In particular this parameters satisfy the following relations:

$$\lambda_1^{(k)} = -\sum_{j=2}^{N} \lambda_j^{(k)}, \quad \mu_1^{(k)} = -\sum_{j=2}^{N} \mu_j^{(k)},$$

$$\eta_1^{(k)} = -\sum\nolimits_{j=2}^{N} \eta_j^{(k)}, \qquad \rho_1^{(k)} = -\sum_{j=2}^{N} \rho_j^{(k)},$$

$$\tau_1^{(k)} = -\sum_{j=2}^{N} \tau_j^{(k)}, \quad \delta_1^{(k)} = -\sum_{j=1}^{N} \delta_j^{(k)},$$

$$\sigma_1^{(k)} = -\sum_{j=2}^{N} \sigma_j^{(k)}, \vartheta_1^{(k)} = -\sum_{j=2}^{N} \vartheta_j^{(k)} \tag{2.4}$$

In fact these formulae are valid, because each formula of numerical differentiating must be satisfied for constant functions. For determining the above partial derivatives, the Taylor expansion of   in each point of star-set with center

$$\left(x_i^{(1)}, y_i^{(1)}, z_i^{(1)}\right), i = 1,$$

is applied. Next a linear least squares problem is constituted to determine the approximation of partial derivatives. For more details it is recommended to see [1]. In practice the parameters appeared in (2.3) are not computed directly, instead, the partial derivatives are calculated by solving the following symbolic system of linear equation of nine unknowns

$$A^{(i)} D_u^{(i)} = B^{(i)}, \quad i = 1,2 \tag{2.5}$$

where   and   are given as follows

$$A_j^{(i)} = \begin{bmatrix} h_j & h_jk_j & h_jl_j & h_j^3/2 & h_jk_j^2/2 & h_jk_j^2/2 & h_j^2k_j & h_j^2l_j & h_jk_jl_j \\ h_jk_j & k_j^2 & k_jl_j & h_j^2k_j/2 & k_j^3/2 & k_jl_j^2/2 & k_j^2h_j & h_jk_jl_j & k_j^2l_j \\ h_jk_j & k_jl_j & l_j^2 & h_j^2l_j/2 & l_jk_j^2/2 & l_j^3/2 & h_jk_jl_j & h_jl_j^2 & k_jl_j^2 \\ h_j^3/2 & h_j^2k_j/2 & h_j^2l_j/2 & h_j^4/4 & h_j^2k_j^2/4 & h_j^2l_j^2/4 & h_j^3k_j/2 & h_j^3l_j/2 & h_j^2k_jl_j/2 \\ h_jk_j^2/2 & k_j^3/2 & k_j^2l_j/2 & h_j^2k_j^2/4 & k_j^4/4 & k_j^2l_j/4 & h_jk_j^3/2 & h_jk_j^2l_j/2 & k_j^3l_j/2 \\ h_jl_j^2/2 & k_jl_j^2/2 & l_j^3/2 & h_j^2l_j^2/2 & k_j^2l_j^2/4 & l_j^4/4 & h_jk_jl_j^2/2 & h_jl_j^3/2 & k_jl_j^3/2 \\ k_jh_j^2 & k_j^2h_j & h_jk_jl_j & k_jh_j^3/2 & h_jk_j^3/2 & h_jk_jl_j^2/2 & h_j^2k_j^2 & h_j^2k_jl_j & h_jk_j^2l_j \\ h_j^2l_j & k_jh_jl_j & h_jl_j^2 & l_jh_j^3/2 & h_jl_jk_j^2/2 & h_jl_j^3/2 & h_j^2k_jl_j & h_j^2l_j^2 & h_jk_jl_j^2 \\ l_jh_jk_j & l_jk_j^2 & k_jl_j^2 & l_jk_jh_j^2/2 & l_jk_j^3/2 & k_jl_j^3/2 & l_jh_jk_j^2 & k_jh_jl_j^2 & k_j^2l_j^2 \end{bmatrix}$$

where A′ and

$$D_u^{(i)} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y \partial z}, \frac{\partial^2 u}{\partial x \partial z}\right) \left(x_i^{(1)}, y_i^{(1)}, z_i^{(1)}\right),$$

$$B^{(i)} = \begin{bmatrix} \sum_{j=1}^{N-1} h_j \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} k_j \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} l_j \left(u_i^{(1)} \quad u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} h_j^2 \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} k_j^2 \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} l_j^2 \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} h_jk_j \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} h_jl_j \left(u_i^{(1)} - u_i^{(j+1)}\right) \\ \sum_{j=1}^{N-1} k_jl_j \left(u_i^{(1)} - u_i^{(j+1)}\right) \end{bmatrix}, \quad i = 1,2,\ldots,N_L$$

The linear system (2.5) is the normal equation of least squares problem used by Urena [1] with constants weights. In order to solve the problem (2.1) and (2.2), the following equation in internal mesh points are considered:

$$\begin{aligned}&a_1(x_i,y_i,z_i)\frac{\partial u^{(i)}}{\partial x}+a_2(x_i,y_i,z_i)\frac{\partial u^{(i)}}{\partial y}+a_3(x_i,y_i,z_i)\frac{\partial u^{(i)}}{\partial z}\\&+b_1(x_i,y_i,z_i)\frac{\partial^2 u^{(i)}}{\partial x^2}+b_2(x_i,y_i,z_i)\frac{\partial^2 u^{(i)}}{\partial y^2}+b_3(x_i,y_i,z_i)\frac{\partial^2 u^{(i)}}{\partial z^2}\\&+c_1(x_i,y_i,z_i)\frac{\partial^2 u^{(i)}}{\partial x\partial y}+c_2(x_i,y_i,z_i)\frac{\partial^2 u^{(i)}}{\partial x\partial z}+c_3(x_i,y_i,z_i)\frac{\partial^2 u^{(i)}}{\partial y\partial z}\\&+d(x_i,y_i,z_i)u^{(i)}-f(x_i,y_i,z_i),\quad i=1,2,\dots,N_i\end{aligned}\tag{2.6}$$

where , are the derivatives in point By substituting the obtained approximation of partial derivatives in (2.6), a system of linear equation is constituted. The solution of this equation, that can be determined by one of the known iterative methods, is the desired approximate solution on mesh points.

NUMERICAL EXPERIMENTS

In this section 3 examples of 3D linear partial differential equations of poisson type are given, the numerical results are compared with analytic solution. In the Tables, is the approximate solution and error is

Example: Consider the following Poisson's equation

$$u_{xx}+u_{yy}+u_{zz}-6\,,on\ \Omega\in[0,1]^3,$$

with the boundary condition

$$u(x,y,z)=x^2+y^2+z^2,\quad (x,y,z)\in\partial\Omega.$$

Table 3.1: Some results for

x	y	z		error
0.00	0.33	0.23	0.2178	0
0.33	0.66	0.33	0.6534	2
0.66	1.00	0.33	1.5445	0
1.00	0.334	1.00	2.1089	0
0.00	0.66	0.66	0.8712	0
0.33	0.66	0.33	0.6535	0
0.66	1.00	0.33	1.5445	0
1.00	0.00	0.66	1.4356	0
0.00	1.00	1.00	2.0000	0
1.00	1.00	0.00	2.0000	0

Table 3.2: Some results for N_i

x	y	z	u^*	error
0.0	1.0	0.3	1.09	-7.0
0.1	0.9	0.4	0.98	1.2
0.2	0.8	0.5	0.93	1.0
0.3	0.1	0.6	0.46	-4.0
0.4	0.6	0.6	0.88	-2.0
0.5	1.0	0.7	1.74	-4.6
0.6	0.4	0.9	1.33	-1.0
0.7	0.3	0.9	1.39	1.0
0.8	0.7	0.9	1.94	2.0
0.9	0.2	1.0	1.82	0.0
1.0	1.0	0.9	2.81	0.0

The exact solution of problem is u = : This problem are solved for and N_b Some results are given in Table 3.1 and 3.2. The maximum of absolute value of errors in mesh points is and the cpu time is 7.52 seconds (for the second case).

Example: Consider the following linear partial differential equation

$$2u_x + 3u_y - 2u_z + u_{xx} + u_{yy} + u_{zz} - 6 + 4x + 6y - 4z, \qquad (x, y, z) \in \Omega - [0,1]^3.$$

with the boundary condition:

$$u(x, y, z) = x^2 + y^2 + z^2, \qquad (x, y, z) \in \partial\Omega.$$

The exact solution of problem is u = : In this problem two cases are considered: and . Some results are given in Table 3.3 and 3.4. The maximum of absolute value of errors in mesh points is and cpu time is 7.9 second (for second case).

Example: Consider the following Poisson's partial differential equation

$$2u_x + 3u_y - 2u_z + u_{xx} + u_{yy} + u_{zz} - 6y = \exp(-x^2 - y^2 - z^2)$$
$$(-4x \quad + 4z - 6 + 4x^2 + 4y^2 + 4, \ (x, y, z) \in \Omega = [0,1]^3,$$

with the boundary condition

Table 3.3: The some results for

x	y	z	u^*	error
0.00	0.33	0.33	0.2178	0
0.33	1.00	0.66	1.5440	2
0.66	0.00	1.00	1.4356	0
1.00	0.33	1.00	2.1089	0
0.00	1.00	1.00	2.0000	0
0.33	0.33	0.00	0.2178	0
0.66	0.66	0.00	0.8712	0
1.00	1.00	0.00	2.0000	0
0.00	1.00	1.00	2.0000	0
0.66	0.00	1.00	1.4356	0

Table 3.4: Some results for N_i

x	y	z	u^*	error
0.0	0.5	0.1	0.26	-2.100
0.1	0.9	0.2	0.86	9.212
0.2	0.3	0.3	0.22	-1.912
0.3	0.2	0.4	0.29	-3.025
0.4	0.0	0.7	0.65	-1.500
0.5	1.0	0.7	1.74	-1.370
0.6	0.4	0.8	1.16	-1.479
0.7	0.8	0.8	1.77	1.250
0.8	0.2	0.9	1.49	-1.005
0.9	0.6	0.9	1.98	8.576
1.0	1.0	0.9	2.81	0.000

Table 3.5: Some results for

x	y	z	u^*	Error
0.00	0.00	0.00	1.0000	0.00
0.33	0.66	0.33	0.5200	0.021
0.66	0.00	1.00	0.2300	0.00
0.00	0.66	0.66	0.4184	0.00
0.33	0.33	0.00	0.8043	0.00
0.66	1.00	0.33	0.2134	0.02
0.30	0.66	0.66	0.4184	0.00
0.66	0.00	1.00	0.2300	0.00
1.00	0.33	1.00	0.1214	0.00

Table 3.6: The some results for

x	y	z	u^*	Error
0.0	0.0	0.1	0.9900	-2.00
0.1	0.4	0.2	0.8105	1.046
0.2	0.3	0.5	0.6838	1.178
0.3	0.2	0.6	0.6126	9.024
0.4	0.0	0.7	0.5220	3.300
0.5	0.5	0.7	0.3715	6.676
0.6	0.4	0.8	0.3134	3.976
0.7	0.3	0.9	0.2253	1.296
0.8	0.6	0.9	0.2253	2.180
0.9	0.5	2.0	0.1000	8.576

$$u(x,y,z) = \exp(-x^2 - y^2 - z^2), \qquad (x,y,z) \in \partial\Omega.$$

The exact solution of problem is given by $u(x,y,z) = \exp(-x^2 - y^2 - z$

This problem is solved for and N. Some results are given in Table 3.5 and 3.6. The maximum of absolute value of errors in mesh points is and cpu time is 7.8 seconds (for second case).

CONCLUSION

In this paper, GFDM is applied to solve the linear partial differential equations of three independent variables and second order. Three test examples are examined. The numerical results are presented in section 3, which demonstrate efficiency of the proposed method.

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