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Optimal Homotopy Asymptotic Solution for Thin Film Flow of a Third Grade Fluid with Partial Slip

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Abstract: The aim of this communication is to investigate thin film flow of an incompressible third grade fluid on an inclined plane with partial slip. The governing equations of this flow are nonlinear and solved for the velocity field using analytical method, name the optimal homotopy asymptotic method (OHAM). Results obtained by OHAM are compared with exact solution and a close agreement was found. Finally the graphs are plotted to discuss the effect of different parameters.

Key words: Optimal homotopy asymptotic method · Partial slip · Inclined plane

INTRODUCTION

Most of the scientific, engineering and commercial problems are generally described by non-Newtonian fluids. The governing equations of non-Newtonian fluids are highly non-linear as compared to Newtonian fluids. Due to complexity of non-Newtonian fluids, it becomes difficult to suggest a single model which exhibits all properties of non-Newtonian fluids, therefore various empirical and semi empirical models have been proposed [1-3].

In the study of fluid dynamics, slip boundary conditions have great importance. It ranges from technological applications to medical applications, especially in polishing artificial heart valves. Although the no slip condition is widely used for flows of non-Newtonian fluids but it is inadequate in problems involving thin film flow, multiple interfaces and rarefied fluid flows etc. In the literature, much attention is given to slip effect, especially from polymer industry, when polymer melts flow under the application of pressure gradient which exhibits a macroscopic wall slip [4-8].

The purpose of present attempt is to analyze the slip effect on the thin film flow of a third grade fluid down an inclined plane. The problem was first studied by Siddiqui *et al* [9] in which they studied third grade flow over an inclined plane. But no attention has been paid on exploring the slip effects on the thin film flow of a third grade fluid down an inclined plane.

Most real world problems and phenomena occur nonlinearly. In view of their potential applications in industry and technology, number of numerical and analytical techniques has been proposed by various researchers [4-8]. But the complete understanding of a non-linear problem is difficult because when numerical results are plotted, they give discontinues points of a curve. Similarly the analytical methods for solving non-linear problem have limitations at the same time they have their own advantages too.

Since there is a paucity of exact solution of non-linear problems, we may go for approximate analytic solutions. Many asymptotic techniques are used for solving non-linear problems. Keeping this fact in mind, we have applied a new powerful technique OHAM [10-17] which is generalization of HPM and HAM. OHAM is simple, straightforward technique and does not require the existence of any small or large parameter as does traditional perturbation. OHAM has successfully applied to a number of nonlinear problems arising in the science and engineering by various researchers. This proves the validity and acceptability of OHAM as a useful technique.

Here, it is important to mention that the previous results of Siddiqui *et al*. [9] can easily be recovered by substituting the slip parameter equal to zero. By substituting the slip parameter equal to zero, graph is plotted to visualize the comparison between the exact [18] and OHAM solution. This comparison proves the confidence and reliability of OHAM.

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The distribution of this paper is in six sections. Section 2 contains the basic idea of OHAM and in Section 3, we formulate the problem of thin film flow down an inclined plane with slip conditions. OHAM solution is obtained in Section 4. Discussion on results and concluding remarks are given in next two sections.

Basic Formulation of OHAM: We apply the OHAM to the following differential equation:

$$L(u(x)) + g(x) + N(u(x)) = 0,\ B\left(u, \frac{du}{dx}\right) = 0 \tag{2.1}$$

Where L is a linear operator, $u(x)$is known function and $g(x)$ is known function, $N(u(x))$ is a non-linear operator and B is boundary operator.

By means of OHAM one first constructs a family of equation [10];

$$(1-p)\left[L(u(x,p)) + g(x)\right] = H(p)\left[L(u(x,p)) + g(x) + N(u(x,p))\right], \tag{2.2}$$

$$B\left(u(x,p), \frac{\partial u(x,p)}{\partial x}\right) = 0$$

where $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a non-zero auxiliary function for $p \neq 0$ and $H(0) = 0$, $u(x, p)$ is an unknown function. Obviously, when $p = 0$ and $p = 1$ it holds

$$u(x,0) = u_0(x),\ u(x,1) = u(x) \tag{2.3}$$

respectively. Thus as p increases from 0 to 1, the solution $u(x,p)$ varies from $u_0(x)$ to the function $u(x)$, where $u_0(x)$ is obtained from Eq. (2.1) for $p = 0$:

$$L(u_0(x)) + g(x) = 0,\ B\left(u_0, \frac{du_0}{dx}\right) = 0 \tag{2.4}$$

We choose auxiliary function $H(p)$ in the form

$$H(p) = pc_1 + p^2c_2 + p^3c_3 + \cdots \tag{2.5}$$

Where $c_1, c_2, \ldots$ are constants, which can be determined later. Let us consider the solution of Eq. (2.2) in the form

$$u(x;p,c_i) = u_0(x) + \sum_{k\geq 1} u_k(x,c_i)p^k,\quad i = 1,2,\ldots \tag{2.6}$$

Now substituting Eq. (2.6) in Eq. (2.2) and equating the coefficients of like powers of p, we obtain the governing equation of $u_0(x)$, given by Eq. (2.4) and the governing equation of $u_k(x)$ i.e.

$$L(u_1(x)) = c_1 N_0(u_0(x)),\ B\left(u_1, \frac{du_1}{dx}\right) = 0 \tag{2.7}$$

$$L(u_k(x) - u_{k-1}(x)) = c_k N_0(u_0(x)) + \sum_{i=1}^{k-1} c_i\left[L(u_{k-i}(x)) + N_{k-i}(u_0(x),\ldots,u_{k-1}(x))\right] \tag{2.8}$$

$$B\left(u_k, \frac{du_k}{dx}\right) = 0,\ k = 2,3,\ldots$$

where $N_m(u_0(x),u_1(x),\ldots,u_m(x))$ is the coefficient of p^m, obtained by expanding $N(u(x; p,c_i))$ in series with respect to the embedding parameter p:

$$N(u(x;p,c_i)) = N_0(u_0(x)) + \sum_{m\geq 1} N_m(u_0,u_1,\ldots,u_m)p^m,\ i = 1,2,\ldots \tag{2.9}$$

where $u(x; p, c_i)$ is given by Eq. (2.6).

It should be emphasized that u_k for $k \geq 0$ are governed by the linear Eqs. (2.4), (2.7) & (2.8) with the linear boundary conditions that come from original problem, which can be easily solved. The convergence of the series Eq. (2.6) depends upon the auxiliary constants $c_1, c_2 \ldots$ if it isconvergent at $p=1$, one has

$$u(x,c_i) = u_0(x) + \sum_{k\geq 1} u_k(x,c_i), \tag{2.10}$$

Generally speaking, the solution of Eq. (2.1) can be determined approximately in the form:

$$u^m(x,c_i) = u_0(x) + \sum_{k=1}^{m} u_k(x,c_i),\quad i = 1,2,\ldots,m \tag{2.11}$$

Substituting Eq. (2.11) into Eq. (2.1) it results the following residual

$$R(x,c_i) = L\left(u^m(x,c_i)\right) + g(x) + N\left(u^m(x,c_i)\right),\ i = 1,2,\ldots,m. \tag{2.12}$$

If $R(x,c_i) = 0$ then $u^m(x, c_i)$ happens to be the exact solution. Generally such case will not arise for nonlinear problems, but we can minimize the functional

$$J(c_i) = \int_a^b R^2(x,c_i)\,dx \tag{2.13}$$

Where a and b are two values, depending on the given problem. The unknown constants $c_i(i = 1,2,...,m)$ can be optimally identified from the conditions

$$\frac{\partial J}{\partial c_i} = \frac{\partial J}{\partial c_2} = \cdots = \frac{\partial J}{\partial c_m} = 0. \tag{2.14}$$

With these known constants, the approximate solution (of order m) Eq. (2.11) is well determined. The constants c_i can be determined in another forms, for example, if $k \in (a, b)$, $i = 1,2...,m$ and substituting k_i into Eq. (2.12), we obtain the equation

$$R(k_1,c_i) = R(k_2,c_i) = \cdots = R(k_m,c_i) = 0, \quad i = 1,2,\ldots,m. \tag{2.15}$$

Formulation of Problem: The governing equations of third grade uni-directional thin film flow down an inclined plane of inclination $\alpha \neq 0$ consisting of incompressibility condition are

$$\nabla.V = 0 \tag{3.1}$$

$$\rho\frac{DV}{Dt} = -\nabla p + \rho B + \nabla.T \tag{3.2}$$

where ρ is fluid density, V is velocityvelocity, p is pressure, B is body force, T is Cauchy stress tensor and $\frac{D}{Dt}$ denoting the material time derivative. The Cauchy stress tensor in a third grade fluid is given by

$$T = -pI + \mu A_1 + a_1A_2 + +a_2A_1^2 + S,$$

where I is the identity tensor, μ is coefficient of viscosity and $a_i(i = 1,2)$ are material coefficients. The kinematical tensors $A_k(k = 1,2,3)$ are Rivlin-Ericksen tensors and S is extra stress tensor. The RivlinEricksen tensor $A_k(k = 1,2,3)$ and extra stress tensor S for third grade fluid is given by

$$S = \beta_1A_3 + \beta_2\left(A_2A_1 + A_1A_2\right) + \beta_3\left(trA_1^2\right)A_1$$

where $A_0 = I$, $A_1 = L + L^T$,

$$A_n = \frac{dA_{n-1}}{dt} + A_{n-1}L + L^TA_{n-1}, \quad n = 2,3,\ldots$$

$$L = \nabla V$$

where ∇ is the gradient operator and $\frac{d}{dt}$ is the material time derivative defined by

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (V.\nabla)$$

For simplicity, some assumptions are made

- The ambient air is stationary
- Surface tension is negligible
- Thin film is of uniform thickness δ
- Thermal effects are negligible
- Pressure gradient is absent.

We have a velocity field of the form $V = (u(y),0,0)$. By using the above assumptions and substituting the values of V and T in Eqs. (3.1) and (3.2), we get the following non-linear second order differential equation

$$\mu\frac{\partial^2u}{\partial y^2} + 6(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^2\left(\frac{\partial^2u}{\partial y^2}\right) + \rho g\sin\alpha = 0 \tag{3.3}$$

The boundary conditions on u are

$$u - \gamma\left[\mu\left(\frac{\partial u}{\partial y}\right) + 2(\beta_2+\beta_3)\left(\frac{\partial u}{\partial y}\right)^3\right] = 0 \text{ at } y = 0 \tag{3.4}$$

$$\frac{du}{dy} = 0 \text{ at } y = \delta \tag{3.5}$$

Eq. (3.4) is slip condition, where γ is coefficient of slip and Eq. (3.5) comes from $\tau_{yx} = 0$ and $y = \delta$. In order to carry out the non-dimensional analysis, we define the following variables

$$u = \frac{u^*v}{\delta},\; y = \delta y^*,\; \beta^* = \frac{(\beta_2+\beta_3)v^2}{\delta^4\mu},\; m^* = \frac{\delta^3 g\sin\alpha}{v^2} \tag{3.6}$$

Using Eq. (3.6) in Eqs. (3.3)-(3.5), we get

$$\frac{d^2u^*}{dy^{*2}} + 6\beta^*\left(\frac{du^*}{dy^*}\right)^2\left(\frac{d^2u^*}{dy^{*2}}\right) + m^* = 0 \tag{3.7}$$

$$u^* - \gamma \frac{\mu}{\delta}\left[\left(\frac{\partial u^*}{\partial y^*}\right) + 2\beta^*\left(\frac{\partial u^*}{\partial y^*}\right)^3\right] = 0 \text{ at } y^* = 0 \tag{3.8}$$

$$\frac{du^*}{dy^*} = 0 \text{ at } y^* = 1 \tag{3.9}$$

For simplicity, we drop asterisks. Integrating Eq. (3.7) once, we get

$$\frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3 + my + m = 0 \tag{3.10}$$

at $y = 0$, Eq. (3.10) reads as

$$\left[\frac{\partial u}{\partial y} + 2\beta\mu\left(\frac{\partial u}{\partial y}\right)^3\right]_{y=0} = -m \tag{3.11}$$

Comparison of Eq. (3.8) and (3.11) yields

$$u(0) = -\frac{\mu\gamma m}{\delta} \tag{3.12}$$

Solution by OHAM: OHAM formulation of our problem as presented in Section 2 is

$$L(u(y,p)) = \frac{\partial^2 u}{\partial y^2} - \lambda u + m \tag{4.1}$$

$$N(u(y,p)) = 6\beta\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} \tag{4.2}$$

$$g(y) = 0 \tag{4.3}$$

with boundary conditions

$$u(0) = -\frac{\mu\gamma m}{\delta},\ u'(1) = 0 \tag{4.4}$$

which satisfies

$$\begin{aligned}&(1-p)\left[\left(u_0''(y) + pu_1''(y) + p^2u_2''(y) + \cdots\right) - \lambda\left(u_0(y) + pu_1(y) + p^2u_2(y) + \cdots\right) + m\right]\\ &= \left(pc_1 + p^2c_2 + p^3c_3 + \cdots\right)\left[\left(u_0''(y) + pu_1''(y) + p^2u_2''(y) + \cdots\right) - \lambda\left(u_0(y) + pu_1(y) + p^2u_2(y) + \cdots\right)\right.\\ &\left. + m + 6\beta\left(u_0'(y) + pu_1'(y) + p^2u_2'(y) + \cdots\right)^2\left(u_0''(y) + pu_1''(y) + p^2u_2''(y) + \cdots\right)\right]\end{aligned} \tag{4.5}$$

With boundary conditions

$$u_0(0) + pu_1(0) + p^2u_2(0) + \cdots = -\frac{\mu\gamma m}{\delta},$$

$$u_0'(1) + pu_1'(1) + p^2u_2'(1) + \cdots = 0.$$

By equating the coefficients of like powers of p, we get a series of problems.

The zeroth order problems defined as

$$O\left(p^0\right):\ u_0''(y) = -m,\ u_0(0) = -\frac{\mu\gamma m}{\delta},\ u_0'(1) = 0 \tag{4.6}$$

which has solution

$$u_0(y) = \frac{1}{2}\left(-2m\frac{\gamma\mu}{\delta} + 2my - my^2\right) \tag{4.7}$$

The first order problem is defined as

$$O\left(p^1\right):\ u_1''(y) = 6\beta c_1 u_0'^2 u_0'',\ u_1(0) = 0,\ u_1'(1) = 0$$

which has the solution

$$u_1(y) = -m\left[20\gamma\delta\mu + c_1 m^2(-2+y)y\beta\left(\left(8+4y+2y^2-4y^3+y^4\right)\delta^2 + 10\left(-4-2y+y^2\right)\gamma\delta\mu + 60\gamma^2\mu^2\right)\right]\Big/20\delta^2 \tag{4.9}$$

The second order problem is defined

$$O\left(p^2\right):\ u_2''(y) = (1+c_1)u_1''(y) + 6\beta\left(c_1u_0^2u_1'' + 2u_0u_1u_0'' + c_1u_0^2u_0''\right),\ u_2(0)=0,\ u_2'(1)=0 \tag{4.10}$$

$$u_2(0)=0, u_2'(1)=0$$

which has the solution

$$\begin{aligned}u_2(y) = -m\Big(&210c_1m^2(-2+y)y\beta\delta^2\left(\left(8+4y+2y^2-4y^3+y^4\right)\delta^2 + 20\left(-4-2y+y^2\right)\gamma\delta\mu + 180\gamma^2\mu^2\right) + \\ &c_1^2m^2(-2+y)y\beta\Big(\Big(-2518m^2y^5\beta + 2641m^2y^6\beta - 952m^2y^7\beta + 119m^2y^8\beta + y^4\left(210+4m^2\beta\right) - \\ &40y^3\left(21+25m^2\beta\right) + 80\left(21+68m^2\beta\right) + 40y\left(21+68m^2\beta\right) + 20y^2\left(21+68m^2\beta\right)\Big)\delta^4 + 60\big(80m^2y^3\beta + \\ &250m^2y^4\beta - 162m^2y^5\beta + 27m^2y^6\beta - 5y^2\left(-7+24m^2\beta\right) - 4\left(35+176m^2\beta\right) - 2y\left(35+176m^2\beta\right)\big)\gamma\delta^3\mu + \\ &840\left(15+m^2\left(144+72y+16y^2-52y^3+13y^4\right)\beta\right)\gamma^2\delta^2\mu^2 + 37800m^2\left(-4-2y+y^2\right)\beta\gamma^3\delta\mu^3 + \\ &756000m^2\beta\gamma^4\mu^4\Big) + 210\delta^2\big(20\gamma\delta\mu + c_2m^2(-2+y)y\beta\left(\left(8+4y+2y^2-4y^3+y^4\right)\delta^2 + \right. \\ &\left. 10\left(-4-2y+y^2\right)\gamma\delta\mu + 60\gamma^2\mu^2\right)\big)\Big)\Big/4200\delta^4\end{aligned} \tag{4.11}$$

By substituting the zeroth, first and second order solution in Eq. (2.11), we get second order approximate analytic solution of our problem. For the constants c_1,c_2 we use least square method as described in Section 2. Hence

$$c_1 = 0.9756921619499443,\ c_2 = -0.25443722127835977$$

Using c_1,c_2 in Eq. (2.11), we have the approximate solution of the problem.

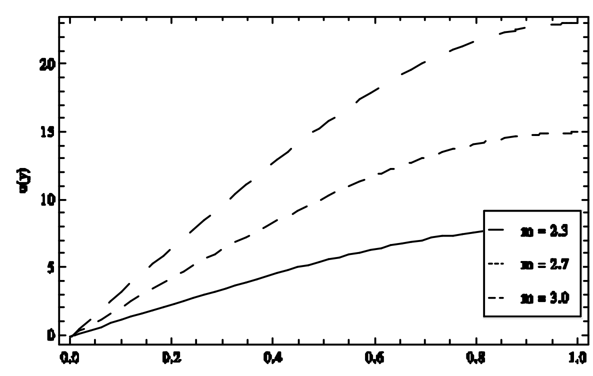

Fig. 1: Dimensionless velocity profiles with fixed values of parameters $\delta = 1.0$, $\gamma = 0.1$ $\beta = 0.2$, $\mu = 0.2$ and different values of the parameter

RESULTS AND DISCUSSION

Figure 1-3 exhibits the effect of parameters m, γ, β velocity field. In Figure 1, velocity function $u(y)$ is plotted against y for different values of m. Clearly, increasing values of m cause to increase in the velocity. This is because the reason that increasing value of m correspond to the increasing angle of inclination, which shows that by increasing the angle of inclination of inclined plane, the velocity increases. In Figure 2, graph is plotted against different values of slip parameter γ, as the values

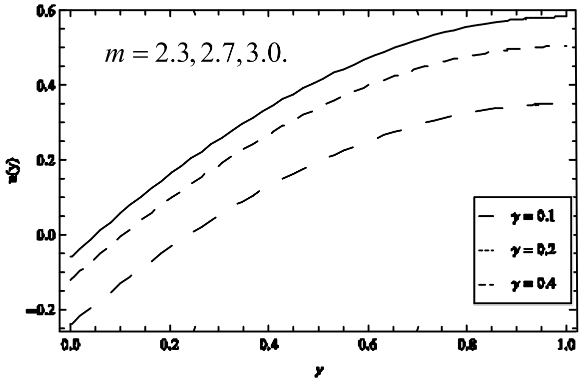

Fig. 2: Dimensionless velocity profiles with fixed values of parameters $m = 1.0$, $\beta = 0.1$ $\mu = 0.2$, $\delta = 1$ and different values of the parameter $\gamma = 0.1, 0.2, 0.4$.

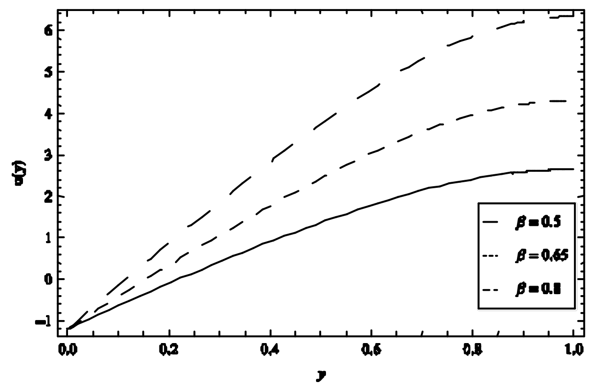

Fig. 3: Dimensionless velocity profiles with fixed values of parameters $m = 2.0$, $\mu = 0.2$, $\delta = 1$, $\gamma = 1$ and different values of the parameter

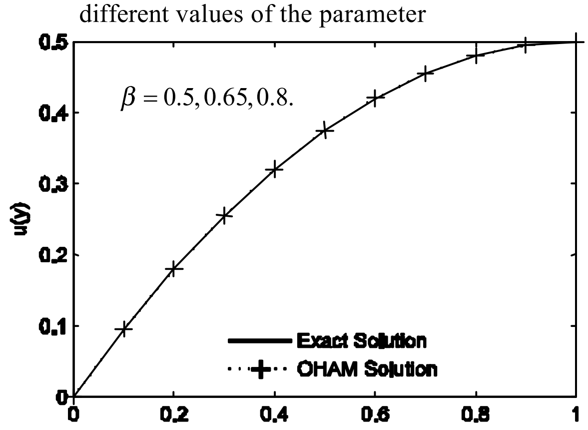

Fig. 4: Comparison of velocity profiles of exact and approximate solution using OHAM

of γ increases the velocity decreases. Figure 3 is plotted for different values of non-Newtonian parameter β, we see that with increase in β, the velocity increases in this region and as a consequence, the velocity gradient also increases, due to which skin friction also increases.

In Figure 4, a comparisons is presented for the exact solution obtained in [15] for third grade fluid and solution obtained by OHAM by setting $\gamma = 0$. It is found that the solution obtained by OHAM has an excellent agreement with exact solution.

Concluding Remarks: A thin film flow of a non-Newtonian third grade fluid over an inclined plane with partial slip has been studied analytically. The effect of non-Newtonian parameter is to increase the flow velocity and hence the skin friction at the plate. An increase in slip parameter also causes the velocity to decrease. It is explicitly shown that an agreement between the derived solutions and exact solution is excellent. This confirms our belief that the efficiency of the OHAM gives it much wider applicability.

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