

## The Homotopy Analysis Method for Solving the Kuramoto-Tsuzuki Equation

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**Abstract:** The homotopy analysis method (HAM) is used to find a family of solitary solutions of the homogenous Kuramoto-Tsuzuki equation. This approximate solution, which is obtained as a series of exponentials has a reasonable residual error. An example is given and the numerical results are compared with those of finite difference scheme and exact solution which the comparison shows the accuracy of the HAM method. The HAM contains the auxiliary parameter  $\hbar$ , which provides us with a simple way to adjust and control the convergence region of solution series.

**Key words:** Partial differential equations • Homogeneous boundry conditions • Kuramoto-Tsuzuki equation  
 • Semi-analytical solution • Homotopy analysis method

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### INTRODUCTION

The Kuramoto-Tsuzuki equation describes the behavior of many two-component systems in a neighborhood of the bifurcation point [1]. Reaction-diffusion type equations have been applied in the study of broad class of nonlinear processes, including a well-known synergetic model [1-13]. The problem of constructing and validating difference schemes for these classes of problems has been in detail taken up in [8, 9]. A finite element Galerkin method had been discussed in [15, 16]. Tsertsvadze studied in [16] the convergence of difference schemes for the Kuramoto-Tsuzuki equation and for systems of reaction-diffusion type. In this paper, we consider the homogenous Kuramoto-Tsuzuki equation [17].

$$\frac{\partial \omega}{\partial t} = (1 + ic_1) \frac{\partial^2 \omega}{\partial x^2} + \omega - (1 + ic_2) |\omega|^2 \omega, \quad (x, t) \in (0, 1) \times (0, T), \quad (1)$$

with the initial condition

$$\omega(x, 0) = \omega_0(x), \quad x \in [0, 1], \quad (2)$$

and homogeneous boundry conditions

$$\frac{\partial \omega}{\partial x}(0, t) = 0, \quad \frac{\partial \omega}{\partial x}(1, t) = 0, \quad t \in (0, T] \quad (3)$$

where  $c_1$  and  $c_2$  are two real constants,  $\omega(x, t)$  is an unknown complex function and  $\omega(x)$  is a given complex function.

In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely homotopy analysis method (HAM) [10, 11] and then modified it, step by step [4, 7, 12]. This method has been successfully applied to solve many types of nonlinear problems by others [3, 5, 6, 18-21]. This method doesn't depend upon any small or large parameters and is valid for most nonlinear models. In this article, we shall apply HAM to find the approximate analytical solution of the Kuramoto-Tsuzuki equation and compare it with the exact solution. The remainder of this paper is arranged as follows: In section 2, we give a brief description for the homotopy analysis method. In section 3, we give one numerical example to assess the efficiency and convenience of the HAM.

**Homotopy Analysis Method:** To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[\omega(x,t)] = 0 \quad (4)$$

where  $N$  is a nonlinear operator,  $\omega(x, t)$  is an unknown function and  $x$  and  $t$  denote spatial and temporal independent variables, respectively.

By means of generalizing the traditional concept of homotopy [12] we constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(x,t;p) - \omega_0(x,t)] = p\hbar N[\phi(x,t;p)] \quad (5)$$

where  $p \in [0, 1]$  is an embedding parameter,  $\hbar$  is a nonzero auxiliary parameter,  $L$  is an auxiliary linear operator,  $\omega_0(x, t)$  is an initial guess of  $\omega(x, t)$  and  $\phi(x, t; p)$  is an unknown function. It should be emphasized that one has great freedom to choose the initial guess, the auxiliary linear operator, the auxiliary parameter  $\hbar$  in HAM. Obviously, when  $p = 0$  and  $p = 1$ , it holds

$$\phi(x,t;0) = \omega_0(x,t), \quad \phi(x,t;1) = \omega(x,t) \quad (6)$$

respectively. Thus, as  $p$  increases from 0 to 1, the solution  $\phi(x, t; p)$  varies from the initial guess  $\omega_0(x, t)$  to the solution  $\omega(x, t)$ . Expanding  $\phi(x, t; p)$  in Taylor series with respect to  $p$ , one has

$$\phi(x,t;p) = \omega_0(x,t) + \sum_{m=1}^{\infty} \omega_m(x,t) p^m, \quad (7)$$

where

$$\omega_m(x,t) = \frac{1}{m!} \frac{\partial^m \phi(x,t;p)}{\partial p^m} \Big|_{p=0}. \quad (8)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter  $\hbar$  and the auxiliary function are so properly chosen, then, as proved by Liao [12], the series (7) converges at  $p = 1$  and one has

$$\omega(x,t) = \omega_0(x,t) + \sum_{m=1}^{\infty} \omega_m(x,t), \quad (9)$$

which must be one of solutions of the original nonlinear equation, as proved by Liao. As  $\hbar = -1$ , Eq. (5) becomes

$$(1-p)L[\phi(x,t;p) - \omega_0(x,t)] + pN[\phi(x,t;p)] = 0, \quad (10)$$

which is used in the homotopy perturbation method [6].

According to the definition (8), the governing equation can be deduced from the zero-order deformation equation (5). Let us define the vector

$$\overline{\omega_n} = \{\omega_0(x,t), \omega_1(x,t), \dots, \omega_n(x,t)\}. \quad (11)$$

Differentiating Eq. (5)  $m$  times with respect to the embedding parameter  $p$  and then setting  $p = 0$  and finally dividing them by  $m!$ , we have the so-called  $m$  th-order deformation equation,

$$L[\omega_m(x,t) - \chi_m \omega_{m-1}(x,t)] = \hbar \Re_m[\overline{\omega_{m-1}}(x,t)], \quad (12)$$

where

$$\Re(\overline{\omega_{m-1}}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}} \Big|_{p=0}, \quad (13)$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases} \quad (14)$$

It should be emphasized that  $\omega_m(x,t)$  for  $m \geq 1$  is governed by the linear equation (12) with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

**Numerical Result:** In this section, we present an example of Kuramoto-Tsuzuki equation. We compare the results with the exact solution. In our work, we use the Maple Package to calculate the numerical solutions obtained by this method.

Consider the homogenous equation ([17])

$$\frac{\partial \omega}{\partial t} = (1+i) \frac{\partial^2 \omega}{\partial x^2} + \omega - (1+i) |\omega|^2 \omega, \quad (15)$$

with the initial condition

$$\omega(x,0) = \frac{\sqrt{3}}{2} \exp\left(i \frac{x}{2}\right), \quad (16)$$

whose exact solution is

$$\omega(x,t) = \frac{\sqrt{3}}{2} \exp\left(i \left(\frac{x}{2} - t\right)\right). \quad (17)$$

In [17], the authors have been used the finite difference method for solving Eq. (15) with the condition (16). The finite difference method has long computations. Also the consistent and stability of the proposed formulas is not easy.

To solve Eq. (15) by means HAM, we choose the initial approximation

$$\omega_0(x, t) = \omega(x, 0) = \frac{\sqrt{3}}{2} \exp\left(i \frac{x}{2}\right), \quad (18)$$

Eq. (15) suggests the nonlinear operator as

$$N[\phi(x, t; p)] = \frac{\partial \phi(x, t; p)}{\partial t} - (1+i) \frac{\partial^2 \phi(x, t; p)}{\partial x^2} - \phi(x, t; p) + (1+i) |\phi(x, t; p)|^2 \phi(x, t; p) \quad (19)$$

and the linear operator

$$L[\phi(x, t; p)] = \frac{\partial \phi(x, t; p)}{\partial t}, \quad (20)$$

with the property

$$L(c_1) = 0,$$

where  $c_1$  is the integration constant.

Using the above definition, we construct the zeroth-order deformation equation

$$(1-p)L[\phi(x, t; p) - \omega_0(x, t)] = p\hbar N[\phi(x, t; p)]. \quad (21)$$

Obviously, when  $p = 0$  and  $p = 1$ ,

$$\phi(x, t; 0) = \omega_0(x, t), \quad \phi(x, t; 1) = \omega(x, t).$$

Therefore, as the embedding parameter  $p$  increases from 0 to 1,  $\phi(x, t; p)$  varies from the initial guess  $\omega_0(x, t)$  to the solution  $\omega(x, t)$ . Then, we obtain the  $m$  th-order deformation equation

$$L[\omega_m(x, t) - \chi_m \omega_{m-1}(x, t)] = \hbar \Re_m[\bar{\omega}_{m-1}(x, t)], \quad (22)$$

subject to initial condition

$$\omega_m(x, 0) = 0, \quad m \geq 1$$

where

$$\Re_m(\bar{\omega}_{m-1}) = \frac{\partial \omega_{m-1}(x, t)}{\partial t} - (1+i) \frac{\partial^2 \omega_{m-1}(x, t)}{\partial x^2} - \omega_{m-1}(x, t) + (1+i) \left[ \sum_{s=0}^{m-1} \left( \sum_{i=0}^s \omega_i(x, t) \omega_{s-i}(x, t) \right) \bar{\omega}_{m-1-s}(x, t) \right]. \quad (23)$$

Now, the solution of the  $m$  th-order deformation equation (22) for  $m \geq 1$  becomes

$$\omega_m(x, t) = \chi_m \omega_{m-1}(x, t) + \hbar L^{-1}[\Re_m(\bar{\omega}_{m-1})], \quad (24)$$

From (18) and (24) we now successively obtain

$$\omega_0(x, t) = \frac{\sqrt{3}}{2} \exp\left(i \frac{x}{2}\right),$$

$$\omega_1(x, t) = \frac{\sqrt{3}}{2} i \hbar \exp\left(\frac{1}{2} i x\right) t,$$

$$\omega_2(x, t) = \frac{\sqrt{3}}{4} \hbar \exp\left(\frac{1}{2} i x\right) t(2i + 2i\hbar - \hbar t),$$

$$\omega_3(x, t) = -\frac{\sqrt{3}}{12} \hbar \exp\left(\frac{1}{2} i x\right) t(-6i - 12i\hbar + 6\hbar t + i t^2 \hbar^2 + 6t\hbar^2 - 6i\hbar^2),$$

$$\omega_4(x, t) = -\frac{\sqrt{3}}{48} \hbar \exp\left(\frac{1}{2} i x\right) t(-24i + 12i t^2 \hbar^2 + 36\hbar t + 12i\hbar^3 t^2 + 72t\hbar^2 - 72i\hbar^2 - \hbar^3 t^3 - 24i\hbar^3 - 72i\hbar + 36t\hbar^3)$$

and so on. Therefore, we use five terms in evaluating the approximate solution

$$\omega_{app} = \sum_{i=0}^4 \omega_i. \quad (25)$$

Then,

$$\omega_{app} = \frac{\sqrt{3}}{2} \exp\left(\frac{1}{2} i x\right) \left[ 1 + i\hbar t + \frac{1}{2} \hbar t(2i + 2i\hbar - \hbar t) - \frac{1}{6} \hbar t(-6i - 12i\hbar + 6\hbar t + i t^2 \hbar^2 + 6t\hbar^2 - 6i\hbar^2) - \frac{1}{24} \hbar t(-24i + 12i t^2 \hbar^2 + 36\hbar t + 12i\hbar^3 t^2 + 72t\hbar^2 - 72i\hbar^2 - \hbar^3 t^3 - 24i\hbar^3 - 72i\hbar + 36t\hbar^3) \right] \quad (26)$$

Table 1:  $\|\omega_{exact} - \omega_{app}\|_2^2$  for different values of  $h$  and  $n$

$h$	$n=5$	$n=10$	$n=15$
-2	2.4158E+00	2.4889E+01	1.8907E+02
-1.9	9.9171E-01	3.9190E+00	1.1187E+01
-1.8	3.7327E-01	5.1009E-01	4.9111E-01
-1.7	1.2643E-01	5.2509E-02	1.4893E-02
-1.6	3.7565E-02	4.0187E-03	2.8201E-04
-1.5	9.4476E-03	2.0892E-04	2.8716E-06
-1.4	1.9087E-03	6.4168E-06	1.2423E-08
-1.3	2.8596E-04	9.2262E-08	1.5231E-11
-1.2	2.8033E-05	4.0344E-10	2.4013E-15
-1.1	1.5374E-06	2.1630E-13	1.3589E-20
-1	1.0529E-07	1.9509E-17	2.1890E-21
-0.9	4.4141E-07	2.6241E-14	1.3762E-20
-0.8	4.5664E-06	2.0407E-11	4.8572E-17
-0.7	3.3134E-05	2.6262E-09	1.4292E-13
-0.6	1.7573E-04	1.1905E-07	6.4287E-11
-0.5	7.4415E-04	2.7712E-06	9.1254E-09
-0.4	2.7179E-03	4.0797E-05	5.9433E-07
-0.3	9.0506E-03	4.3776E-04	2.2182E-05
-0.2	2.8161E-02	3.8693E-03	5.6380E-04
-0.1	8.2095E-02	3.0728E-02	1.1720E-02
0	2.2371E-01	2.2371E-01	2.2371E-01

Table 2: The absolute error given by HAM for  $\hbar = -1$  at  $x = 1$ .

$n/t_i$	1	2	3	4	5
10	2.16322E-08	4.39168E-05	3.74434E-03	8.69144E-02	9.86945E-01
15	4.13279E-14	2.69606E-09	1.75748E-06	1.73518E-04	6.08307E-03
20	1.69347E-20	3.54146E-14	1.75820E-10	7.34446E-08	7.89733E-06
25	2.14716E-27	1.43743E-19	5.42732E-15	9.57368E-12	3.14991E-09

Table 3:  $\|e^N\|$  computed by the finite difference scheme at  $t = 5$  ( $h$  and  $\tau$  are mesh lengths)

$h$	$\tau$	$\lambda = \tau/h^2$	$\ e^N\ $
$\pi/10$	0.1000	1.0132	2.9852E-03
$\pi/20$	0.0500	2.0264	7.5220E-04
$\pi/40$	0.0250	4.0528	1.8854E-04
$\pi/80$	0.0125	8.1057	4.7182E-05

To investigate the influence of  $\hbar$  on the solution series, we plot the so-called  $\hbar$ -curve  $\omega_t(0, 0)$  obtained from the 4 th-order, 5 th-order and 6 th-order HAM approximation solution as shown in Figure 1. According to this  $\hbar$ -curve, it is easy to discover the valid region of  $\hbar$  which corresponds to the line segment nearly parallel to the horizontal axis. From Figure 1 it is clear that the series of  $\omega_t(0, 0)$  is convergent when  $-1.6 < \hbar < -0.2$ . hcur 3tae. The  $\hbar$ -curve of  $\omega_t(0, 0)$  given by the 4th-order, 5th-order and 6th-order HAM approximate solution. Table 1 shows  $\|\omega_{exact} - \omega_{ap}\|_2^2$  for  $x \in [0, 1]$  and  $t \in [0, 1]$ . This table which was computed for  $n = 5, 10$  and  $15$ , shows that the valid region of  $\hbar$  is  $[-1.6, -0.2]$ , as we have obtained from Figure 1 early. Also Table 1 shows that the best choice is  $\hbar = -1$ . Figure 2 shows the behavior of the absolute error for 5th-order HAM with  $\hbar = -1$ .

In Table 2, we have computed the absolute error for some values of  $t$  at  $x = 1$ . It is resulted from Figure 2 that the absolute error does not vary for fixed values of

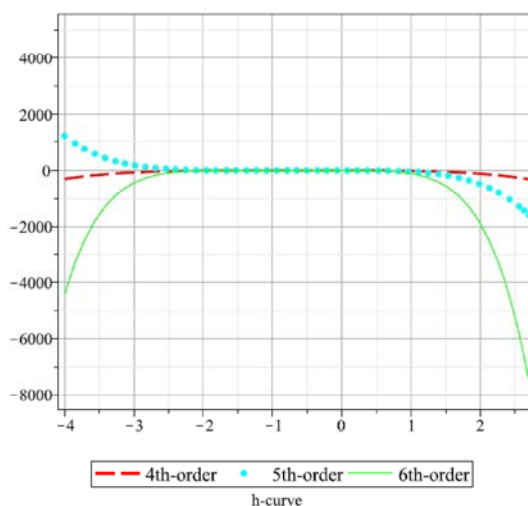


Fig. 1: The  $h$ -curve of  $\omega(0, 0)$  given by the 4th-order, 5th-order and 6th-order HAM approximate solution.

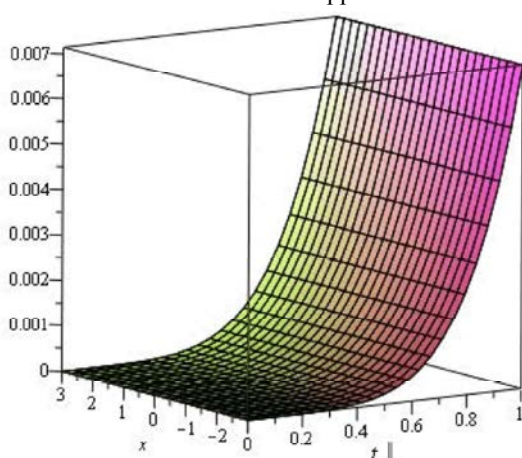


Fig. 2: The behavior of the absolute error for 5th-order HAM with  $h = -1$ .

$t$ . Table 3, shows the second-order convergence in  $L_\infty$ -norm of the finite difference scheme for solving periodic boundary-initial value problem (15)-(17) [19]. Comparing the results in Table 2 and Table 3 shows that HAM produces more accurate results with respect to finite differences scheme.

## CONCLUSIONS

In this paper, HAM has successfully developed for solving Kuramoto-Tsuzuki equation. It is obvious to see that the HAM is very powerful and efficient technique in finding analytical solutions for wide classes of nonlinear problems. HAM provides accurate numerical solution for Kuramoto-Tsuzuki equation in comparison with finite difference scheme.

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