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On Generalized Recurrent Kenmotsu Manifolds

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Abstract: We study on generalized recurrent, generalized Ricci-recurrent and generalized concircular recurrent Kenmotsu manifolds.

Key words: Kenmotsu manifold . generalized recurrent . generalized Ricci-recurrent manifold

INTRODUCTION

A Riemannian manifold (M^n, g) is called generalized recurrent [1] if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y,Z)W = \alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{1}$$

where, α and β are two 1-forms, β is non-zero and these are defined by:

$$\alpha(X) = g(X,A),\ \beta(X) = g(X,B) \tag{2}$$

and A, B are vector fields associated with 1-forms α and β, respectively.

A Riemannian manifold (M^n, g) is called a generalized Ricci-recurrent [1] if its Ricci tensor S satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + (n-1)\beta(X)g(Y,Z), \tag{3}$$

where, α and β are defined as in (2).

A Riemannian manifold (M^n, g) is called generalized concircular recurrent if its concircular curvature tensor $\bar{C}$ [2]

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y) \tag{4}$$

satisfies the condition

$$(\nabla_X \bar{C})(Y,Z)W = \alpha(X)\bar{C}(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z], \tag{5}$$

[3], where α and β are defined as in (2) and r is the scalar curvature of (M^n, g).

In [4], Q. Khan studied on generalized recurrent and generalized Ricci-recurrent Sasakian manifolds. In this study, we consider generalized recurrent and generalized Ricci-recurrent Kenmotsu manifolds. We also consider generalized concircular recurrent Kenmotsu manifolds. The paper is organized as follows: In Section 2, we give a brief account of Kenmotsu manifolds. In Section 3, we find the characterizations of generalized recurrent, generalized Ricci-recurrent and generalized concircular recurrent Kenmotsu manifolds.

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KENMOTSU MANIFOLDS

Let M be an almost contact manifold [5] equipped with an almost contact metric structure (φ, ξ, η, g) consisting of a (1, 1) tensor field φ, a vector field ξ, a 1-form η and a compatible Riemannian metric g satisfying

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \tag{6}$$

$$g(X,Y) = g(\varphi X, \varphi Y) + \eta(X)\eta(Y), \tag{7}$$

$$g(X,\varphi Y) = -g(\varphi X, Y), \quad g(X,\xi) = \eta(X) \tag{8}$$

for all X, Y$\in\chi$(M). An almost contact metric manifold M is called a Kenmotsu manifold if it satisfies [6]

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X, \quad X, Y \in \chi(M), \tag{9}$$

where, ∇ is Levi-Civita connection of the Riemannian metric g. From the above equation it follows that

$$\nabla_X \xi = X - \eta(X)\xi, \tag{10}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y). \tag{11}$$

Moreover, the curvature tensor R and the Ricci tensor S satisfy [6]

$$S(X,\xi) = (1-n)\eta(X), \tag{12}$$

$$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi, \tag{13}$$

where, n = 2m + 1. Kenmotsu manifolds have been studied various authors. For example see [7, 8, 9, 10, 11, 12].

RESULTS

In this section, our aim is to find the characterizations of Kenmotsu manifolds which are generalized recurrent, generalized Ricci-recurrent and generalized concircular recurrent.

Theorem 3.1: Let (M^n, g) be a generalized recurrent Kenmotsu manifold. Then $\alpha = \beta$.

Proof: Assume that (M^n, g) be a generalized recurrent Kenmotsu manifold. Then the curvature tensor of (M^n, g) satisfies the condition (1) for all vector fields X, Y, Z, W. Putting X = Y = ξ in (1) we have:

$$(\nabla_X R)(\xi,Z)\xi = \alpha(X)R(\xi, Z)\xi + \beta(X)[\eta(Z)\xi - Z]. \tag{14}$$

It is also well-known that:

$$(\nabla_X R)(\xi,Z)\xi = \nabla_X R(\xi, Z)\xi - R(\nabla_X \xi, Z)\xi - R(\xi, \nabla_X Z)\xi - R(\xi,Z)\nabla_X \xi. \tag{15}$$

Then in view of (10) the equation (15) can be written as:

$$(\nabla_X R)(\xi,Z)\xi = \nabla_X R(\xi, Z)\xi - R(X,Z)\xi + \eta(X)R(\xi,Z)\xi - R(\xi,\nabla_X Z)\xi - R(\xi, Z)X + \eta(X)R(\xi,Z)\xi. \tag{16}$$

Hence by the use of (11) and (13) in (16) it can be easily seen that:

$$(\nabla_X R)(\xi,Z)\xi = 0. \quad (17)$$

Therefore from the equality of left hand sides of the equations (14) and (17) we have:

$$\alpha(X)R(\xi,Z)\xi + \beta(X)[\eta(Z)\xi - Z] = 0. \quad (18)$$

So by the use of (13) in (18) we get:

$$[\alpha(X) - \beta(X)][\eta(Z)\xi - Z] = 0,$$

which implies that $\alpha(X) = \beta(X)$ for any vector field X. This proves the theorem.

Theorem 3.2: Let (M^n, g) be a generalized recurrent Kenmotsu manifold. Then the scalar curvature r of (M^n, g) is related to

$$\eta(A)r = (1-n)[(n-2)\eta(B) + 2\eta(A)]. \quad (19)$$

Proof: Suppose that (M^n, g) is a generalized recurrent Kenmotsu manifold. Then by the use of second Bianchi identity and (1) we have:

$$\begin{aligned} &\alpha(X)R(Y,Z)W + \beta(X)[g(Z,W)Y - g(Y,W)Z] \\ &+\alpha(Y)R(Z,X)W + \beta(Y)[g(X,W)Z - g(Z,W)X] \\ &+\alpha(Z)R(X,Y)W + \beta(Z)[g(Y,W)X - g(X,W)Y] = 0. \end{aligned} \quad (20)$$

So by a suitable contraction from (20) we get:

$$\alpha(X)S(Z,W) + (n-1)\beta(X)g(Z,W) + R(Z,X,W,A) + \beta(Z)g(X,W) - \beta(X)g(Z,W) - \alpha(Z)S(X,W) + (1-n)\beta(Z)g(X,W) = 0. \quad (21)$$

Contracting (21) with respect to Z, W we find:

$$r\alpha(X) + (n-1)(n-2)\beta(X) - 2S(X,A) = 0. \quad (22)$$

Hence putting $X = \xi$ in (22) we obtain (19). Our theorem is thus proved.

Theorem 3.3: Let (M^n, g) be a generalized Ricci-recurrent Kenmotsu manifold. Then $\alpha = \beta$.

Proof: Assume that (M^n, g) is a generalized recurrent Kenmotsu manifold. Then the Ricci tensor S of (M^n, g) satisfies the condition (3) for all vector fields X, Y, Z. Taking $Z = \xi$ in (3) we have:

$$(\nabla_X S)(Y,\xi) = (1-n)[\alpha(X) - \beta(X)]\eta(Y). \quad (23)$$

On the other hand, by the definition of covariant derivative of S it is well-known that:

$$(\nabla_X S)(Y,\xi) = \nabla_X S(Y,\xi) - S(\nabla_X Y,\xi) - S(Y,\nabla_X \xi). \quad (24)$$

Then in view of (10) and (12) the equation (24) can be written as:

$$(\nabla_X S)(Y,\xi) = (1-n)\nabla_X \eta(Y) - (1-n)\eta(\nabla_X Y) - S(X,Y) + (1-n)\eta(X)\eta(Y). \quad (25)$$

So using (11) we get:

$$(\nabla_X S)(Y,\xi) = (1-n)g(X,Y) - S(X,Y). \tag{26}$$

From the equality of the left hand sides of the equations (23) and (26) we can write:

$$(1-n)[\alpha(X) - \beta(X)]\eta(Y) = (1-n)g(X,Y) - S(X,Y). \tag{27}$$

Hence taking $Y = \xi$ in (27) and using (12) we obtain $\alpha(X) = \beta(X)$ for any vector field X. This proves the theorem.

Theorem 3.4: Let (M^n, g) be a generalized concircular recurrent Kenmotsu manifold. Then the condition

$$\alpha(X)\left(1 + \frac{r}{n(n-1)}\right) - \beta(X) - \frac{1}{n(n-1)}X[r] = 0 \tag{28}$$

holds for any vector field X, where X[r] denotes the covariant derivative of the scalar curvature r with respect to the vector field X.

Proof: Suppose that (M^n, g) is a generalized concircular recurrent Kenmotsu manifold. Then the concircular curvature tensor $\bar{C}$ of (M^n, g) satisfies the condition (5) for all vector fields X, Y, Z, W. Taking $Y = W = \xi$ in (5) we have:

$$(\nabla_X \bar{C})(\xi,Z)\xi = \alpha(X)\bar{C}(\xi,Z)\xi + \beta(X)[\eta(Z)\xi - Z].$$

So using (4) and (13) this gives us:

$$(\nabla_X \bar{C})(\xi,Z)\xi = [\alpha(X)(1 + \frac{r}{n(n-1)}) - \beta(X)][Z - \eta(Z)\xi]. \tag{29}$$

On the other hand, from the definition of covariant derivative, it is well-known that:

$$(\ {}_X\bar{C})(\ ,Z)\quad {}_X\bar{C}(\ ,Z)\quad \bar{C}(\ {}_X\ ,Z)\quad \bar{C}(\ ,\ {}_XZ)\quad \bar{C}(\ ,Z)\ {}_X\ .$$