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Metric Dimension and Determining Number of Cayley Graphs

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Abstract: A subset W of vertices of a graph G is called a resolving set for G if for every pair of distinct vertices u and v of G, there exists a vertex $w \in W$ such that the distance between u and w is different from the distance between v and w. A resolving set containing a minimum number of vertices is called a metric basis for G and the number of vertices in a metric basis is called the metric dimension of G, denoted by $\beta(G)$. A subset S of vertices of a graph G is called a determining set if whenever two automorphisms agree on the elements of S, they agree on all of G. The minimum cardinality of a determining set of G is called the determining number of G, denoted by Det(G). In this paper, we find the metric dimension of Cayley graphs, $Cay(Z_n: S)$ for all $n \geq 7$ and $S = \{\pm 1, \pm 3\}$. Also we show that, for all prime numbers $n = 2p+1$ with p prime and any subset S of $Z_n \setminus \{0\}$ with $S = -S$, $S \neq \phi$ and $S \neq Z_n \setminus \{0\}$, $Det(Cay(Z_n : S)) = 2$.

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Key words: Cayley graph . resolving set . metric basis . metric dimension . determining number

INTRODUCTION

Let G be a connected graph. The distance between two vertices u and v of G is the length of a shortest path between them, denoted by $d(u,v)$. For a vertex v in a graph G, the eccentricity of v, denoted by e(v), is the distance between v and a vertex farthest from v in G. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in G, the code of v with respect to W is the ordered k-tuple $c_W(v) = (d(v,w_1), d(v,w_2), \ldots, d(v,w_k))$. The set W is called a resolving set [16, [25] for G if every two vertices of G have distinct codes with respect to W. A resolving set W of minimum cardinality is a metric basis [6] and |W| is the metric dimension of G, denoted by $\beta(G)$ [16]. For a set $W \subseteq V(G)$ of a graph G to be resolving, we need only to check that $c_W(u) \neq c_W(v)$ for distinct vertices $u,v \in V(G) - W$, because w_i is the only vertex of W for which the ith coordinate of its code with respect to W is 0.

Resolving sets were firstly defined by Slater [25], in 1975. Independently, Harary and Melter studied resolving sets in 1976 and used the term metric dimension [16], the terminology which we have adopted. Slater used the term locating set for resolving set and location number for metric dimension and initiated the study of this invariant by its application to the placement of minimum number of Sonar/Loran detecting devices in a network so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Khuller *et al*. [19] studied robot navigation in a graph-structured framework, relating this to the notion of metric dimension of a graph and gave a construction which shows that the metric dimension of a graph is NP-hard. It is assumed that a robot navigating a graph can sense its distance to each of the landmarks (represented by nodes in a graph) and hence uniquely determine its location in the graph. These concepts have also been investigated by Johnson [13] of the Pharmaceutical Company while attempting to develop a capability of large data sets of chemical graphs and he also noted that the problem of finding the metric dimension is NP-hard. Resolving sets have been widely studied, arising in several areas including coin weighing problems [3], network discovery and verification [4], strategies for mastermind game [10], robot navigation [19] and connected joins in graphs [24]. Applications to chemistry are given in [7] and in pattern recognition and image processing are discussed in [21].

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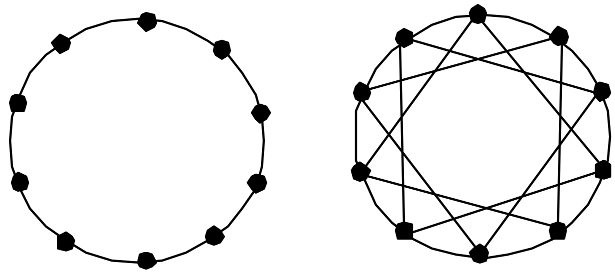

(a) (b)

Fig. 1: (a) Cay(Z_{10}: S) with S = {±1}, (b) Cay(Z_{10}: S) with S = {±1,±3}

The concept of metric dimension helps in studying another notion which can be used to identify the automorphism group of a graph, called the determining number of a graph. A set of vertices $S \subseteq V(G)$ is called a determining set if whenever $g, h \in Aut(G)$ and g(v) = h(v) for all $v \in S$, then g = h. That is, the image of S under an arbitrary automorphism determines the automorphism completely. The determining number is the smallest size of a determining set and is denoted by Det (G). This notion was introduced by Boutin in [5]. Independently, Erwin and Harary studied this notion and used the terms fixing set and fixing number, denoted by fix (G) [11]. Every graph has a determining set, since any set containing all but one vertex is determining. There are graphs, e.g., K_n and $K_{1,n}$, for which such a determining set is minimal [11]. Boutin also proved that, $Det(P_n) = 1; n \geq 2$, $Det(C_n) = 2; n \geq 3$, $Det(P(5,2))=3$ [11]. A basis for a vector space is an analogue of a determining set. Thus, in a vector space the determining number is just the dimension. An early form of a determining set is the concept of a base for a group action. The determining number can be obtained by using its connection with the metric dimension, because the two parameters are intrinsically related. The metric dimension of a graph provides an upper bound for its determining number. [13] For every connected graph G, $Det(G) \leq \beta(G)$. Let H be a finite group and let $S \subseteq H$. The corresponding Cayley (di)graph Cay(H,S) has vertex set H and two vertices $g,h \in H$ are joined by an (arc)edge from g to h if and only if there exists $s \in S$ such that g = sh. A special family of Cayley (di)graphs (also referred to as circulant graphs) with vertex set

$$V(Cay(Z_n : S)) = \{v_i : i \in Z_n\}$$

and edge set

$$E(Cay(Z_n : S)) = \{v_i v_{((i+s) \bmod n)} : 0 \leq i \leq n-1, s \in S\}$$

where $S \subseteq \{1,2,\ldots,\lfloor \frac{n}{2} \rfloor\}$. If S has the property that S = -S, then Cay(Z_n: S) are undirected graphs, otherwise they are directed. Figgure 1 shows undirected Cayley graphs Cay(Z_{10}: S) for S = {±1} and S = {±1, ±3}. We refer to the cycle $v_0 - v_1 - \ldots - v_{n-1} - v_0$ in Cay(Z_n: S) as the principal cycle. The complete and empty graphs are also Cayley graphs, with S = Z_n and S = ϕ, respectively.

The metric dimension of Cayley digraph of the dihedral group D_n of order 2_n with a minimum set of generators is n [12]. Also it was shown that, for positive integers m and n, the metric dimension of Cayley digraph for the group $Z_n \oplus Z_m$ with generating set {(1,0),(1,0)} is min (m,n) [12].

In this paper, we consider the Cayley graphs Cay(Z_n: S). In section 2, for S = {±1,±3}, we show that $\beta(Cay(Z_n : S)) = 3$ when $n \equiv 1$ (mod 6), $\beta(Cay(Z_n : S)) = 4$ when $n \equiv 0,3,4,5$ (mod 6) and $4 \leq \beta(Cay(Z_n : S)) \leq 6$ when $n \equiv 2$ (mod 6). In section 3, we will find the determining number of Cay(Z_n: S), where $S \subseteq Z_n$ has S = -S, $S \neq \phi$ and $S \neq Z_n^*$ and when n, $\frac{n-1}{2}$ both are prime.

METRIC DIMENSION OF CAYLEY GRAPHS

Let $G = (G_n)_{n \geq 1}$ be a family of connected graphs G_n of order $\phi(n)$ for which $\lim_{n \to \infty} \phi(n) = \infty$. If there exists a constant M>0 such that $\beta(G_n) \leq M$ for every n≥1, then we shall say that G has bounded metric dimension, otherwise

G has unbounded metric dimension. If all graphs in G have the same metric dimension (which does not depend upon n), then G is called a family of graphs with constant metric dimension [18]. Paths P_n and cycles $C_n(n\geq 3)$ are families of graphs with constant metric dimension [19]. Also antiprism A_n and generalized Petersen graphs P(n,2) and P(n,3) are families of graphs with constant metric dimension [17, 18]. In [18], Javaid *et al*. considered a family of Cayley graphs $Cay(Z_n: S)$ with $S = \{\pm 1,\pm 2\}$ (also referred to as a 4-regular family of Harary graphs $H_{4,n}$) and proved that this is a family of graphs with constant metric dimension and asked for the characterization of more families of graphs with constant metric dimension. In [23], Salman *et al*. considered a family of Cayley graphs $Cay(Z_n: S)$ with $S=\{1,\frac{n}{2},n-1\}$ and proved that it is a family of graphs with constant metric dimension. In this paper, we consider a family of Cayley graphs, $Cay(Z_n: S)$ with $S = \{\pm 1,\pm 3\}$ and show that this is a family of graphs with constant metric dimension by proving Theorem 2. We give exact value for the metric dimension of $Cay(Z_n: \{\pm 1,\pm 3\}$ when $n\equiv 0,1,3,4,5$ (mod 6) and give upper bound when $n\equiv 2$(mod 6). In what follows, the indices on vertices are taken modulo n.

Lemma 2.1: For all $n\geq 7$ and $n\equiv 1$ (mod 6), $\beta(Cay(Z_n: S)) = 3$

Proof: Let $n = 6k+1$ and $k(\geq 1)\in Z^+$. First, we show that $\beta(Cay(Z_n: S))\leq 3$. For the chosen index i; $0\leq i\leq n-1$, we will show that $W = \{v_i, v_{i+2}, v_{i+3k}\}$ is a resolving set for $Cay(Z_n: S)$. The codes of the vertices of $V(Cay(Z_n: S))\backslash W$ are:
For $1\leq j\leq 3k$,

$$c_W(v_{i+j}) = \frac{1}{3}\begin{cases}(j,j,3k-j) & j\equiv 0(\text{mod}3)\\ (j+2,j+2,3k-j+4) & j\equiv 1(\text{mod } 3)\\ (j+4,j-2,3k-j+2) & j\equiv 2(\text{mod}3)\end{cases}$$

$c_W(v_{i+3k+1}) = (k,k+1,1)$, $c_W(v_{i+3k+2}) = (k+1,k,2)$ and for $0\leq j\leq 3k-3$,

$$c_W(v_{n+i-j-1}) = \frac{1}{3}\begin{cases}(j+3,j+3,3k-j) & j\equiv 0(\text{mod}3)\\ (j+5,j+5,3k-j+4) & j\equiv 1(\text{mod}3)\\ (j+1,j+7,3k-j+2) & j\equiv 2(\text{mod}3)\end{cases}$$

One can see that all the vertices in $V(Cay(Z_n: S))\backslash W$ have distinct codes with respect to W which shows that W is a resolving set for $Cay(Z_n: S)$. Hence, $\beta(Cay(Z_n: S))\leq 3$.

Now for the lower bound, assume that the metric dimension is two and $W=\{v_i, v_{i+j}\}$, for fixed i; $0\leq i\leq n-1$, forms a metric basis for $V(Cay(Z_n: S))$. Then, for j = 1, $c_W(v_{i+3}) = c_W(v_{i+n-3})$ and for $2\leq j\leq n-1$,

$$\begin{cases}c_W(v_{i+n-3}) = c_W(v_{i+n-1}) & j\equiv 0(\text{mod}3)\\ c_W(v_{i+1}) = c_W(v_{i+3}) & j\equiv 1(\text{mod}3)\\ c_W(v_{i+1}) = c_W(v_{i+n-1}) & j\equiv 2(\text{mod } 3)\end{cases}$$

a contradiction since at least two vertices in $Cay(Z_n: S)$ have the same codes with respect to W. Thus, $\beta(Cay(Z_n: S))\leq 3$.

Now we define a gap between two vertices of a graph which will be useful in the next lemmas. Let S be a set of two or more vertices of the principal cycle, let v_i and v_j be two distinct vertices of S, let P and P′ denote the two distinct v_i-v_j paths determined by the principal cycle. If either P or P′, say P, contains only two vertices of S (namely; v_i and v_j), then we refer to v_i and v_j as neighboring vertices of S and a set of vertices of P-$\{v_i, v_j\}$ is called the gap of S (determined by v_i and v_j) and is denoted by γ. The number of vertices in a gap determined by v_i and v_j is called the order of a gap, denoted by $|\gamma|$. Consequently, if $|S| = r$, then S has r gaps, some of which may be empty.

Lemma 2.2: For each even $n\geq 10$, when $n\equiv 4$ (mod 6), $\beta(Cay(Z_n: S)) = 4$.

Proof: For the chosen index i such that $0\le i\le n-1$, we will show that $W = \{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$ is a resolving set for $Cay(Z_n: S)$. Let $n = 6k+4$ and $k(\ge 1)\in Z^+$, then the codes of the vertices of $V(Cay(Z_n: S))\backslash W$ are:
$c_W(v_{i+1})=(1,1,1,3)$, $c_W(v_{i+3})=(1,1,1,1)$ and for $5\le j\le 3k+3$,

$$c_W(v_{i+j}) = \frac{1}{3}\begin{cases}(j,j,j,j-6) & j\equiv 0(\text{mod}3)\\(j+2,j+2,j-4,j-4) & j\equiv 1(\text{mod}3)\\(j+4,j-2,j-2,j-2) & j\equiv 2(\text{mod}3)\end{cases}$$

For $0\le j\le 3k-1$,

$$c_W(v_{n+i-j-1}) = \frac{1}{3}\begin{cases}(j+3,j+3,j+9,j-3l+9) & j\equiv 0(\text{mod}3)\\(j+5,j+5,j+5,j-3l+11) & j\equiv 1(\text{mod}3)\\(j+1,j+7,j-3l+7,j-3l+7) & j\equiv 2(\text{mod}3)\end{cases}$$

where

$$l=0 \text{ for } 0\le j\le 3k-4 \text{ and } l=2 \text{ for } 3k-3\le j\le 3k-1$$

One can see that all the vertices of $Cay(Z_n: S)$ have distinct codes with respect to W which shows that W is a resolving set for $Cay(Z_n: S)$. Hence, $\beta(Cay(Z_n: S))\le 4$.

Now for the lower bound, we contrarily suppose that the metric dimension is two and $W=\{v_i, v_{i+j}\}$, for fixed i; $0\le i\le n-1$, forms a metric basis for $Cay(Z_n: S)$. Then, for $j = 1$, $c_W(v_{i+3}) = c_W(v_{i+n-3})$ and for $2\le j\le n-1$,

$$\begin{cases}c_W(v_{i+n-3}) = c_W(v_{i+n-1}) & j\equiv 0(\text{mod}3)\\c_W(v_{i+1}) = c_W(v_{i+3}) & j\equiv 1(\text{mod}3)\\c_W(v_{i+1}) = c_W(v_{i+n-1}) & j\equiv 2(\text{mod } 3)\end{cases}$$

a contradiction, since at least two vertices in $Cay(Z_n: S)$ have the same codes with respect to W. Thus, $\beta(Cay(Z_n: S))\ne 2$. Now, assume that the metric dimension of $Cay(Z_n: S)$ is three and a resolving set W of three vertices forms a metric basis for $Cay(Z_n: S)$. For each fixed i; $0\le i\le n-1$, we have the following two claims:

Claim 1: No adjacent vertices are in W.

(1) Suppose that W consists of three consecutive adjacent vertices v_i, v_{i+1}, v_{i+2}, then $c_W(v_{i+3}) = c_W(v_{i+n-1})=(1,2,1)$, a contradiction.
(2) Suppose that W consists of two adjacent vertices then
(i) If $W=\{v_i, v_{i+1}, v_{i+j}\}$ then, we have

when $j\equiv 0$ (mod 3) then $c_W(v_{i+n-3}) = c_W(v_{i+n-1})$ for $3\le j\le n-4$; when $j\equiv 1$ (mod 3) then $c_W(v_{i+2}) = c_W(v_{i+n-2})$ for $j = n-1$ and $4\le j\le 3k+1$, $c_W(v_{i+3}) = c_W(v_{i+n-1})$ for $3k+4\le j\le n-3$, when $j\equiv 2$ (mod 3) then $c_W(v_{i+3}) = c_W(v_{i+n-1})$ for $2\le j\le 3k+2$ and $c_W(v_{i+n-3}) = c_W(v_{i+n-1})$ for $3k+5\le j\le n-2$, a contradiction.

(ii) If $W=\{v_i, v_{i+3}, v_{i+j}\}$ then we have

$$\begin{cases}c_W(v_{i+n-3}) = c_W(v_{i+n-1}) & j\equiv 0(\text{mod}3)\\c_W(v_{i+1}) = c_W(v_{i+n-1}) & j\equiv 2(\text{mod } 3)\end{cases}$$

and $c_W(v_{i+1}) = c_W(v_{i+n-3})$ for $j = n-1$. When $j\equiv 1$ (mod 3) then $c_W(v_{i+n-3}) = c_W(v_{i+n-1})$ for $1\le j\le 3k-2$ and $c_W(v_{i+n-4}) = c_W(v_{i+n-2})$ for $3k+1\le j\le n-3$, a contradiction.

(iii) If $W=\{v_i, v_{i-1}, v_{i+j}\}$ then we have $c_W(v_{i+2})=c_W(v_{i+n-2})$ for j = 1, $c_W(v_{i+1})=c_W(v_{i+3})$ for j = 2 and when $j \equiv 0$ (mod 3) then $c_W(v_{i+1})=c_W(v_{i+n-3})$ for 3≤j≤3k and $c_W(v_{i+3k})=c_W(v_{i+3k+4})$ for 3k+3≤j≤n-4 and

$$\begin{cases} c_W(v_{i+1})=c_W(v_{i+3}) & j\equiv 1(\text{mod}3) \\ c_W(v_{i+2})=c_W(v_{i+4}) & j\equiv 2(\text{mod}3) \end{cases}$$

a contradiction.

(iv) If $W=\{v_i, v_{i-3}, v_{i+j}\}$ then we have $c_W(v_{i+2})=c_W(v_{i+4})$ for j = 1 and when $j \equiv 0$ (mod 3) then $c_W(v_{i+3k+4})=c_W(v_{i+3k+6})$ for 3≤j≤3k+3 and $c_W(v_{i+1})=c_W(v_{i+3})$ for 3k+6≤j≤n-1 and

$$\begin{cases} c_W(v_{i+1})=c_W(v_{i+3}) & j\equiv 1(\text{mod}3) \\ c_W(v_{i+1})=c_W(v_{i+n-1}) & j\equiv 2(\text{mod}3) \end{cases}$$

a contradiction.

Claim 2: No non-adjacent vertices are in W.

Suppose that W consists of three non-adjacent vertices. Since |W| = 3 and W consists of three vertices of the principal cycle of Cay(Z_n: S), there are three gaps of W with no empty gap. We call a gap between first and second vertex of W, γ_1; a gap between second and third vertex of W, γ_2 and a gap between third and first vertex of W, γ_3. We have the following two cases:

Case 1: First two gaps of W are of the same order.
Suppose that $\gamma_1=\gamma_2=j$ then, by Claim 1, 1≤j≤3k-1. Let $W=\{v_i, v_{i+j+1}, v_{i+2(j+1)}\}$. Then we note that

$$\begin{cases} c_W(v_{i+j-1})=c_W(v_{i+j+3}) & j\equiv 0(\text{mod}3) \\ c_W(v_{i+1})=c_W(v_{i+3}) & j\equiv 1(\text{mod}3) \\ c_W(v_{i+n-3})=c_W(v_{i+n-1}) & j\equiv 2(\text{mod } 3) \end{cases}$$

a contradiction. The case when all the three gaps are of the same order does not exist, since |W| = 3 and n-3 is not divisible by 3.

Case 2: First two gaps of W are of different order.
Suppose that $\gamma_1=l, \gamma_2=m$ and $l\neq m$.

(i) When $l \equiv 1$ (mod 3), we have $c_W(v_{i+1})=c_W(v_{i+3})$, or $c_W(v_{i+3})=c_W(v_{i+n-1})$ or $c_W(v_{i+n-1})=c_W(v_{i+1})$ for $m \equiv 0$ (mod 3). Also

$$\begin{cases} c_W(v_{i+1})=c_W(v_{i+l+2}) & \text{for} \quad m\equiv 1(\text{mod}3) \\ c_W(v_{i+1})=c_W(v_{i+n-1}) & \text{for} \quad m\equiv 2(\text{mod}3) \end{cases}$$

a contradiction.

(ii) When $l \equiv 2$ (mod 3) and 1≤j≤n-1, we have

$$\begin{cases} c_W(v_{i+l+2})=c_W(v_{i+l+4}) & \text{for} \quad m\equiv 0(\text{mod}3) \\ c_W(v_{i+j-1})=c_W(v_{i+j+1}) & \text{for} \quad m\equiv 1(\text{mod}3) \\ c_W(v_{i+n-3})=c_W(v_{i+n-1}) & \text{for} \quad m\equiv 2(\text{mod}3) \end{cases}$$

a contradiction.

(iii) When $l \equiv 0 \pmod 3$ we have, for $m \equiv 0 \pmod 3$, $c_W(v_{i+l+2}) = c_W(v_{i+l+4})$, where $3\le l\le 3k-3$ and $3\le m\le n-7$, $c_W(v_{i+1}) = c_W(v_{i+n-3})$, where $l = 3k$ and $m = 3k$, $c_W(v_{i+1}) = c_W(v_{i+l-2})$, where $3k\le l\le n-7$ and $3\le m\le 3k-3$.

For $m \equiv 1 \pmod 3$,

$c_W(v_{i+l+2}) = c_W(v_{i+l+4})$, where $3\le l\le 3k-3$ and $1\le m\le 3k+1$,
$c_W(v_{i+1}) = c_W(v_{i+l+4})$, where $l = 3k$ and $1\le m\le 3k-2$,
$c_W(v_{i+1}) = c_W(v_{i+l+2})$, where $3k+3\le l\le n-7$ and $1\le m\le 3k-5$.

For $m \equiv 2 \pmod 3$, $c_W(v_{i+1}) = c_W(v_{i+3})$, a contradiction, since at least two vertices in $Cay(Z_n: S)$ have the same codes with respect to W. Hence we have $\beta(Cay(Z_n: S))\ge 4$.

Lemma 2.3: For all $n\ge 8$,

$$4 \le \beta(Cay(Z_n : S)) \le 6 \text{ when } n \equiv 2 \pmod 6 \text{ and } \beta(Cay(Z_n : S)) = 4 \text{ when } n \equiv 5 \pmod 6$$

Proof: First we prove an upper bound for the metric dimension of $Cay(Z_n: S)$. We have the following two cases:

Case 1: When $n\equiv 2 \pmod 6$, i.e., $n = 6k+2$ and $k(\ge 1) \in Z^+$.

For all $n\ge 8$ and for the chosen index i; $0\le i\le n-1$, we will show that W= $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}\}$ is a resolving set for $Cay(Z_n: S)$. Codes of the vertices of $Cay(Z_n: S)$ are:

For $4\le j\le 3k+3$

$$c_W(v_{i+j}) = \frac{1}{3}\begin{cases} (j, j-3l+3, j, j-3, j, j-3) & j \equiv 0 \pmod 3 \\ (j+2, j-1, j+2, j-1, j-4, j-1) & j \equiv 1 \pmod 3 \\ (j-3l+4, j+1, j-2, j+1, j-2, j-5) & j \equiv 2 \pmod 3 \end{cases}$$

where

$$l = 0 \text{ for } 4 \le j \le 3k+1 \text{ and } l = 2 \text{ for } j = 3k+2, 3k+3$$

For $0\le j\le 3k-3$,

$$c_W(v_{n+i-j-1}) = \frac{1}{3}\begin{cases} (j+3, j+6, j+3, j+6, j-3l+9, j+6) & j \equiv 0 \pmod 3 \\ (j+5, j+2, j+5, j+8, j+5, j+8) & j \equiv 1 \pmod 3 \\ (j+1, j+4, j+7, j+4, j+7, j-3l+10) & j \equiv 2 \pmod 3 \end{cases}$$

where

$$l = 0 \text{ for } 0 \le j \le 3k-5 \text{ and } l = 2 \text{ for } j = 3k-4, 3k-3$$

One can see that all the vertices of $V(Cay(Z_n: S))\backslash W$ have distinct codes with respect to W which implies that W is a resolving set for $Cay(Z_n: S)$. Hence, $\beta(Cay(Z_n: S))\le 6$

Case 2: When $n\equiv 5 \pmod 6$, i.e., $n = 6k+5$ and $k \in Z^+$.

For all $n\ge 11$ and for the chosen index i; $0\le i\le n-1$, we will show that W= $\{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$ is a resolving set for $Cay(Z_n: S)$. Codes of the vertices of $V(Cay(Z_n: S))\backslash W$ are: $c_W(v_{i+1})=(1,1,1,3)$, $c_W(v_{i+3})=(1,1,1,1)$ and for $5\le j\le 3k+5$

$$c_W(v_{i+j}) = \frac{1}{3}\begin{cases} (j, j, j, j-6) & j \equiv 0 \pmod 3 \\ (j-3l+2, j-3l+2, j-4, j-4) & j \equiv 1 \pmod 3 \\ (j-3l+4, j-2, j-2, j-2) & j \equiv 2 \pmod 3 \end{cases}$$

where $l = 0$ for $5 \le j \le 3k+1$

$$l=1 \text{ for } 3k+2\leq j\leq 3k+4 \text{ and } l=3 \text{ for } j=3k+5$$

For $0\leq j\leq 3k-2$

$$c_W(v_{n+i-j-1})=\frac{1}{3}\begin{cases}(j+3,j+3,j-3l+9,j-3l+9) & j\equiv 0(\bmod 3)\\(j+5,j+5,j+5,j-3l+11) & j\equiv 1(\bmod 3)\\(j+1,j+7,j+7,j+7) & j\equiv 2(\bmod 3)\end{cases}$$

where $l=0$ for $0\leq j\leq 3k-6$

$$l=1 \text{ for } 3k-5\leq j\leq 3k-3 \text{ and } l=3 \text{ for } j=3k-2$$

One can see that all the vertices of $V(Cay(Z_n: S))\backslash W$ have distinct codes with respect to W. Therefore W is a resolving set for $Cay(Z_n: S)$. Hence, $\beta(Cay(Z_n: S))\leq 4$.

Now for the lower bound, first note that $\beta(Cay(Z_n: S))\neq 2$ since at least two vertices in $Cay(Z_n: S)$ have the same codes with respect to W as shown below: For each fixed i; $0\leq i\leq n-1$, if $W=\{v_i, v_{i+j}\}$ forms a metric basis for $Cay(Z_n: S)$ then, for $1\leq j\leq n-2$,

$$\begin{cases}c_W(v_{i+1})=c_W(v_{i+n-1}) & j\equiv 0(\bmod 3)\\c_W(v_{i+n-3})=c_W(v_{i+n-1}) & j\equiv 1(\bmod 3)\\c_W(v_{i+1})=c_W(v_{i+3}) & j\equiv 2(\bmod 3)\end{cases}$$

and $c_W(v_{i+2})=c_W(v_{i+n-2})$, for $j=n-1$, a contradiction. Now, we assume that the metric dimension of $Cay(Z_n: S)$ is three and a resolving set W of three vertices forms a metric basis for $Cay(Z_n: S)$. For each fixed i; $0\leq i\leq n-1$, we have the following two claims:

Claim 1: No adjacent vertices are in W.

(1) Suppose that W consists of three consecutive adjacent vertices v_i, v_{i+1}, v_{i+2}, then $c_W(v_{i+3})=c_W(v_{i+n-1})=(1,2,1)$, a contradiction.

(2) Suppose that W consists of two adjacent vertices then

(i) If $W=\{v_i, v_{i+1}, v_{i+j}\}$ then, for $1\leq j\leq n-2$,

$$\begin{cases}c_W(v_{i+n-3})=c_W(v_{i+n-1}) & j\equiv 0,1(\bmod 3)\\c_W(v_{i+3})=c_W(v_{i+n-1}) & j\equiv 2(\bmod 3)\end{cases}$$

and $c_W(v_{i+2})=c_W(v_{i+n-2})$, for $j=n-1$, a contradiction.

(ii) If $W=\{v_i, v_{i+3}, v_{i+j}\}$ then, for $1\leq j\leq n-2$,

$$\begin{cases}c_W(v_{i+n-3})=c_W(v_{i+n-1}) & j\equiv 0,1(\bmod 3)\\c_W(v_{i+1})=c_W(v_{i+n-1}) & j\equiv 2(\bmod 3)\end{cases}$$

and $c_W(v_{i+1})=c_W(v_{i+n-3})$, for $j=n-1$, a contradiction.

(iii) If $W=\{v_i, v_{i-1}, v_{i+j}\}$ then, for $j=1$, $c_W(v_{i+2})=c_W(v_{i+n-2})$ and for $2\leq j\leq n-1$,

$$\begin{cases}c_W(v_{i+1})=c_W(v_{i+n-3}) & j\equiv 0(\bmod 3)\\c_W(v_{i+1})=c_W(v_{i+3}) & j\equiv 1,2(\bmod 3)\end{cases}$$

a contradiction.

(iv) If $W = \{v_i, v_{i-3}, v_{i+j}\}$ then, for j = 1, $c_W(v_{i+2}) = c_W(v_{i+4})$ and for 2≤j≤n-1,

$$\begin{cases} c_W(v_{i+1}) = c_W(v_{i+n-1}) & j \equiv 0 (\bmod 3) \\ c_W(v_{i+1}) = c_W(v_{i+3}) & j \equiv 1,2 (\bmod 3) \end{cases}$$

a contradiction.

Claim 2: No non-adjacent vertices are in W.

Suppose that W consists of three non-adjacent vertices, so there are three gaps with no empty gap. We have the following two cases:

Case 1: First two gaps of W are of same order.

Suppose that $\gamma_1 = \gamma_2 = j$ then, by Claim 1, 1≤j≤3k-1. Let $W = \{v_i, v_{i+j+1}, v_{i+2(j+1)}\}$ then we note that,

$$\begin{cases} c_W(v_{i+1}) = c_W(v_{i+3}) & j \equiv 0,1 (\bmod 3) \\ c_W(v_{i+1}) = c_W(v_{i+n-1}) & j \equiv 2 (\bmod 3) \end{cases}$$

a contradiction. The case when all the three gaps are of the same order does not exist here, since |W| = 3 and n-3 is not divisible by 3 when n ≡ 2,5 (mod 6).

Case 2: First two gaps of W are of different order.

Suppose that $\gamma_1 = l, \gamma_2 = m$ and $l \neq m$, then

(i) When l ≡ 0 (mod 3), we have

$$\begin{cases} c_W(v_{i+l+2}) = c_W(v_{i+l+4}) & m \equiv 0,1 (\bmod 3) \\ c_W(v_{i+l}) = c_W(v_{i+3}) & m \equiv 2 (\bmod 3) \end{cases}$$

a contradiction.

(ii) When l ≡ 1 (mod 3), we have

$$\begin{cases} c_W(v_{i+l}) = c_W(v_{i+l+2}) & m \equiv 1 (\bmod 3) \\ c_W(v_{i+l}) = c_W(v_{i+n-1}) & m \equiv 0,2 (\bmod 3) \end{cases}$$

a contradiction.

(iii) When l ≡ 1 (mod 3) and 0≤j≤n-1, we have

$$\begin{cases} c_W(v_{i+l+2}) = c_W(v_{i+l+4}) & m \equiv 0 (\bmod 3) \\ c_W(v_{i+j-1}) = c_W(v_{i+j+l}) & m \equiv 1 (\bmod 3) \\ c_W(v_{i+n-3}) = c_W(v_{i+n-1}) & m \equiv 2 (\bmod 3) \end{cases}$$

a contradiction. Since at least two vertices in Cay(Z_n: S) have the same codes with respect to W, so we have β(Cay(Z_n: S))≥4. Thus, we conclude that, 4≤β(Cay(Z_n: S))≥6 when n ≡ 2 (mod 6) and β(Cay(Z_n: S))=4 when n ≡ 5 (mod 6).

Lemma 2.4: For all n≥6 and n ≡ 0,3 (mod 6), β(Cay(Z_n: S))=4.

Proof: First we prove an upper bound for the metric dimension of Cay(Z_n: S). We have the following two cases:

Case 1: When n ≡ 0(mod 6), i.e., n = 6k and $k \in Z^+$.

For n≥6 and for the chosen index i; 0≤i≤n-1, we will show that W = $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ is a resolving set for Cay(Z_n: S). The codes of the vertices of V(Cay(Z_n: S))\W are:

For $4\leq j\leq 3k+1$

$$c_W(v_{i+j})=\frac{1}{3}\begin{cases}(j,j+3,j,j-3) & j\equiv 0(\mathrm{mod}3)\\(j+2,j-1,j+2,j-1) & j\equiv 1(\mathrm{mod}3)\\(j+4,j+1,j-2,j+1) & j\equiv 2(\mathrm{mod}3)\end{cases}$$

For $0\leq j\leq 3k-3$

$$c_W(v_{n+i-j-1})=\frac{1}{3}\begin{cases}(j+3,j+6,j+3,j+6) & j\equiv 0(\mathrm{mod}3)\\(j+5,j+2,j+5,j+8) & j\equiv 1(\mathrm{mod}3)\\(j+1,j+4,j+7,j+4) & j\equiv 2(\mathrm{mod}3)\end{cases}$$

It is clear that all the vertices of $V(Cay(Z_n: S))\backslash W$ have distinct codes with respect to W which shows that W is a resolving set for $Cay(Z_n: S)$. Hence $\beta(Cay(Z_n: S))\leq 4$.

Case 2: When $n\equiv 3(\mathrm{mod}\ 6)$, i.e., $n=6k+3$ and $k\in Z^+$.

For $n\geq 9$ and for the chosen index i; $0\leq i\leq n-1$, we will show that $W=\{v_i, v_{i+2}, v_{i+4}, v_{i+6}\}$ is a resolving set for $Cay(Z_n: S)$. The codes of the vertices of $V(Cay(Z_n: S))\backslash W$ are: $c_W(v_{i+1})=(1,1,1,3)$, $c_W(v_{i+3})=(1,1,1,1)$ and for $5\leq j\leq 3k+4$

$$c_W(v_{i+j})=\frac{1}{3}\begin{cases}(j-3l,j,j,j-6) & j\equiv 0(\mathrm{mod}3)\\(j-3l+2,j-3l+2,j-4,j-4) & j\equiv 1(\mathrm{mod}3)\\(j-3l+4,j-2,j-2,j-2) & j\equiv 2(\mathrm{mod}3)\end{cases}$$

where

$$l=0 \text{ for } 5\leq j\leq 3k+1 \text{ and } l=1 \text{ for } 3k+2\leq j\leq 3k+4$$

For $0\leq j\leq 3k-3$

$$c_W(v_{n+i-j-1})=\frac{1}{3}\begin{cases}(j+3,j+3,j-3l+9,j-3l+9) & j\equiv 0(\mathrm{mod}3)\\(j+5,j+5,j+5,j-3l+11) & j\equiv 1(\mathrm{mod}3)\\(j+1,j+7,j+7,j-3l+7) & j\equiv 2(\mathrm{mod}3)\end{cases}$$

where

$$l=0 \text{ for } 0\leq j\leq 3k-6 \text{ and } l=1 \text{ for } 3k-5\leq j\leq 3k-3$$

One can see that all the vertices of $V(Cay(Z_n: S))\backslash W$ have distinct codes with respect to W which shows that W is a resolving set for $Cay(Z_n: S)$. Hence, $\beta(Cay(Z_n: S))\leq 4$.

Now for the lower bound, first note that $\beta(Cay(Z_n: S))\neq 4$ since, for each fixed i; $0\leq i\leq n-1$, if $W=\{v_i, v_{i+j}\}$ is a metric basis for $Cay(Z_n: S)$ then, for $1\leq j\leq n-1$,

$$\begin{cases}c_W(v_{i+1})=c_W(v_{i+n-1}) & j\equiv 0(\mathrm{mod}3)\\c_W(v_{i+n-3})=c_W(v_{i+n-1}) & j\equiv 1(\mathrm{mod}3)\\c_W(v_{i+1})=c_W(v_{i+3}) & j\equiv 2(\mathrm{mod}\ 3)\end{cases}$$

a contradiction, since at least two vertices in $Cay(Z_n: S)$ have the same codes with respect to W. Now, assume that the metric dimension of $Cay(Z_n: S)$ is three and a resolving set W of three vertices forms a metric basis for $Cay(Z_n: S)$. For each fixed i; $0\leq i\leq n-1$, we have the following two claims:

Claim 1: No adjacent vertices are in W.

(1) Suppose that W consists of three consecutive adjacent vertices v_i, v_{i+1}, v_{i+2}, then $c_W(v_{i+3})=c_W(v_{i+n-1})=(1,2,1)$, a contradiction.

(2) Suppose that W consists of two adjacent vertices then, we have

(i) If $W=\{v_i, v_{i+1}, v_{i+j}\}$ then we have

$$\begin{cases} c_W(v_{i+3k-1}) = c_W(v_{i+3k+1}) & j \equiv 0 (\bmod 3) \\ c_W(v_{i+n-3}) = c_W(v_{i+n-1}) & j \equiv 1 (\bmod 3) \end{cases}$$

When $j \equiv 2 \pmod 3$ then $c_W(v_{i+3}) = c_W(v_{i+n-1})$ for $2 \le j \le 3k-1$, $c_W(v_{i+n-3}) = c_W(v_{i+n-1})$ for $3k+2 \le j \le n-4$, $c_W(v_{i+2}) = c_W(v_{i+n-2})$ for $j = n-1$, a contradiction.

(ii) If $W = \{v_i, v_{i+3}, v_{i+j}\}$ then we have $c_W(v_{i+3k-1}) = c_W(v_{i+3k+1})$ when $j \equiv 0 \pmod 3$ and for $k \ge 2$,

$$\begin{cases} c_W(v_{i+n-3}) = c_W(v_{i+n-1}) & j \equiv 1 (\bmod 3) \\ c_W(v_{i+3k-2}) = c_W(v_{i+3k}) & j \equiv 2 (\bmod 3) \end{cases}$$

a contradiction.

(iii) If $W = \{v_i, v_{i+n-3}, v_{i+j}\}$ then we have

$$\begin{cases} c_W(v_{i+3k-1}) = c_W(v_{i+3k+1}) & j \equiv 0 (\bmod\ 3) \\ c_W(v_{i+1}) = c_W(v_{i+3}) & j \equiv 2 (\bmod 3) \text{ and } k \ge 2 \end{cases}$$

When $j \equiv 1 \pmod 3$ then $c_W(v_{i+3}) = c_W(v_{i+n-1})$ for $j = 1$, $c_W(v_{i+1}) = c_W(v_{i+3})$ for $4 \le j \le 3k+1$, $c_W(v_{i+1}) = c_W(v_{i+n-1})$ for $3k+4 \le j \le n-2$, a contradiction.

(iv) If $W = \{v_i, v_{i+n-1}, v_{i+j}\}$ then we have

$$\begin{cases} c_W(v_{i+n-4}) = c_W(v_{i+n-2}) & j \equiv 0 (\bmod 3) \\ c_W(v_{i+1}) = c_W(v_{i+3}) & j \equiv 2 (\bmod 3) \text{ and } k \ge 2 \end{cases}$$

When $j \equiv 1 \pmod 3$ then $c_W(v_{i+3}) = c_W(v_{i+n-3})$ for $j = 1$, $c_W(v_{i+1}) = c_W(v_{i+3})$ for $4 \le j \le 3k+1$, $c_W(v_{i+1}) = c_W(v_{i+n-3})$ for $3k+4 \le j \le n-2$, a contradiction.

Claim 2: No non-adjacent vertices are in W.

Suppose that W consists of three non-adjacent vertices, so there are three gaps with no empty gap. We have the following two cases:

Case 1: First two gaps of W are of the same order.

Suppose that $\gamma_1 = \gamma_2 = j$ then, by Claim 1, $1 \le j \le 3k-1$. Let $W = \{v_i, v_{i+j+1}, v_{i+2(j+1)}\}$ then we note that

$$\begin{cases} c_W(v_{i+j-1}) = c_W(v_{i+j+3}) & j \equiv 0 (\bmod 3) \\ c_W(v_{i+j}) = c_W(v_{i+j+2}) & j \equiv 1,2 (\bmod 3) \end{cases}$$

The case when all the three gaps are of the same order occurs only for j = 2k-1 or j = 2k.

Case 2: First two gaps of W are of different order.

Suppose that $\gamma_1 = l, \gamma_2 = m$ and $l \ne m$.

(i) When $l \equiv 0 \pmod 3$ and $3 \le l \le n-6$, then $c_W(v_{i+l+2}) = c_W(v_{i+l+4})$ for $m \equiv 1 \pmod 3$ and $1 \le m \le n-8$, $c_W(v_{i+n-4}) = c_W(v_{i+n-6})$ for $m \equiv 2 \pmod 3$ and $2 \le m \le n-7$, where $k \ge 2$ and for $m \equiv 0 \pmod 3$ and $3 \le l \le 3k$, $c_W(v_{i+1}) = c_W(v_{i+3})$ where $3 \le m \le n-9$ and for $3k+3 \le l \le n-6$, $c_W(v_{i+l+2}) = c_W(v_{i+l+4})$, where $3 \le m \le n-9-3k$.

(ii) When $l \equiv 1 \pmod 3$ and $1 \le l \le n-8$, then for $m \equiv 1 \pmod 3$ when $1 \le l \le 3k-2$, $c_W(v_{i+1}) = c_W(v_{i+l+2})$ where $1 \le m \le 3k-2$ and $c_W(v_{i+1}) = c_W(v_{i+n-1})$ where $3k+1 \le m \le n-5$, and for $3k+1 \le l \le n-5$, $c_W(v_{i+n-3}) = c_W(v_{i+n-1})$ where $1 \le m \le 3k-2$. Now for $m \equiv 2 \pmod 3$ and $1 \le l \le n-8$, $c_W(v_{i+1}) = c_W(v_{i+3})$

where $2 \le m \le n-7$, and for $m \equiv 0 \pmod 3$ when $1 \le l \le 3k-2$, $c_W(v_{i+1}) = c_W(v_{i+n-1})$ where $3 \le m \le n-6$ and when $3k+1 \le l \le n-8$, $c_W(v_{i+1}) = c_W(v_{i+3})$ where $3 \le m \le 3k-6$.

(iii) When $l \equiv 2 \pmod 3$ and $0 \le l \le n-7$, then for $m \equiv 1 \pmod 3$ when $2 \le l \le 3k-4$, $c_W(v_{i+l+2}) = c_W(v_{i+l+4})$ where $0 \le m \le n-8$ and when $3k-1 \le l \le n-7$, $c_W(v_{i+1}) = c_W(v_{i+l+2})$ where $1 \le m \le 3k-5$. Now for $m \equiv 2 \pmod 3$ and $2 \le l \le n-7$, $c_W(v_{i+2}) = c_W(v_{i+4})$ where $2 \le m \le n-7$. Also for $m \equiv 0 \pmod 3$ when $2 \le l \le 3k-4$, $c_W(v_{i+l+2}) = c_W(v_{i+l+4})$ where $3 \le m \le 3k$ and when $2 \le l \le 3k-4$, $c_W(v_{i+1}) = c_W(v_{i+n-1})$ where $3k+3 \le m \le 3k+6$ and when $3k-1 \le l \le n-7$, $c_W(v_{i+1}) = c_W(v_{i+n-1})$ where $3 \le m \le 3k$, a contradiction, since at least two vertices in Cay(Z_n: S) have the same codes with respect to W. Hence we have $\beta(\text{Cay}(Z_n: S)) \ge 4$, for $n \equiv 0,3 \pmod 6$. Finally, we conclude, $\beta(\text{Cay}(Z_n: S)) = 4$ when $n \equiv 0,3 \pmod 6$.

Hence, by Lemmas 2.1 -- 2.4, we have the following main result:

$$\beta(\text{Cay}(Z_n : S)) = \begin{cases} 3 & ; \ n \equiv 1 (\text{mod} 3) \\ 4 & ; \ n \equiv 0,3,4,5 (\text{mod} 3) \\ & \end{cases}$$

and

$$4 \le \beta(\text{Cay}(Z_n : S)) \le 6 \ \text{ for } \ n \equiv 2 (\text{mod} 3)$$

DETERMINING NUMBER OF Cay(Z_n: S)

In this section, we will find the determining number of Cay(Z_n: S), where $S \subseteq Z_n \setminus \{0\}$ has the property that S = -S, $S \ne \phi$ and $S \ne Z_n^*(= Z_n \setminus \{0\})$ and when n, $\frac{n-1}{2}$ both are prime. We use an approach called the symmetry breaking, which was formalized by Albertson and Collins [1] and independently by Harary [14, 15], in which a subset of the vertex set is determined(fixed) in such a way that the automorphism group of the graph is destroyed. Symmetry breaking has been applied to the problem of programming a robot to manipulate objects [20]. In [5], Boutin raised the following question:

Question: [5] Can the difference between the metric dimension and the determining number of a graph of order n be arbitrarily large?

This question turns out to be interesting since an automorphism preserves distances and resolving sets are also determining sets [5, 11]. It arises first as an open problem in [5] and its answer has led to a number of results on the determining number of some families of graphs in which the metric dimension is known. In 1973, Alspach [22] proved the following result on automorphism group of Cayley graphs Cay(Z_n: S) on prime number of vertices.

Theorem 3.1: [22] Let p be prime. If $S = \phi$ or $S = Z_p^*$, then the automorphism group of Cayley graph Cay(Z_p: S) is given by, $\text{Aut}(\text{Cay}(Z_p : S)) = S_p$; otherwise,

$$\text{Aut}(\text{Cay}(Z_p : S)) = \{T_{a,b} \ ; \ a \in E(S), b \in Z_p\}$$

where

$$T_{a,b}(v_i) = v_{ai+b}$$

and E(S) is the largest even order subgroup of Z_p^* such that S is a union of cosets of E(S). The following result gives the determining number of Cay(Z_n: S).

Theorem 3.2: For all prime numbers n = 2p+1 with p is prime, let Cay(Z_n: S) be a Cayley graph, where $S \subseteq Z_n \setminus \{0\}$ has S = -S, $S \ne \phi$ and $S \ne Z_n^*$, then $\text{Det}(\text{Cay}(Z_n : S)) = 2$.

Proof: First, we will find the automorphism group of Cay(Z_n: S). Since $S \neq \Phi$ and $S \neq Z_n^*$, we note that E(S) = {1,-1} is the largest even order subgroup of Z_n^* such that S is the union of cosets of E(S) = {1,-1}. Hence, by Theorem 3, we have

$$\mathrm{Aut}(\mathrm{Cay}(Z_n : S) = \{T_{a,b} \; ; \; a \in \{1,-1\}, b \in Z_n\}$$

where

$$T_{a,b}(v_i) = v_{ai+b}$$

Since |E(S)| = 2 and $|Z_n|$ = n, so $|\mathrm{Aut}(\mathrm{Cay}(Z_n : S)| = 2n$. Now, we will show that all these automorphisms become trivial by determining(fixing) only two vertices in Cay(Z_n: S). Let v_i be any arbitrary vertex of Cay(Z_n: S) such that

$$v_i = T_{a,b}(v_i) = v_{ai+b}$$

Then b = 0 when a = 1 and b = 2i when a = -1.
In the first case, for all 0≤j≤n-1, we have

$$T_{a,b}(v_j) = v_j$$

In the second case, for all 0≤j≤n-1, we have

$$T_{a,b}(v_j) = \begin{cases} v_j & \text{if } j = i \\ v_{2i-j} & \text{otherwise} \end{cases}$$

One can see that 2n-1 automorphisms become trivial in the first case and the remaining non-trivial automorphism of Cay(Z_n: S) we have in the second case, which maps each v_j to v_{2i-j} for i+1≤j≤i+n-1. This non-trivial automorphism become trivial by determining(fixing) the vertex v_{i+1} of Cay(Z_n: S). Thus the automorphism group of Cay(Z_n: S) is destroyed by determining(fixing) the subset $\{v_i, v_{i+1}\}$ of vertices of Cay(Z_n: S). Hence, Det (Cay(Z_n: S)) = 2.

CONCLUDING REMARKS

The aim of this work is to find the difference between metric dimension and determining number. We notice that for the graphs in Cay(Z_n; {±1,±3}) with prime order n = 2p+1 where p is a prime, the difference between metric dimension and determining number is either 1 or 2.

Question: Does there exist a regular graph G such that β(G) is not bounded?

Conjecture: For all regular families of graphs with finite eccentricities, the metric dimension is independent of the choice of graph in the family.

REFERENCES

1. Albertson, M.O. and K. Collins, 1996. Symmetry breaking in graphs. The Electron. J. Combin., 3 (1): R18.
2. Albertson, M.O. and K. Collins, 1999. A note on breaking the symmetries of tournaments. Congr. Numer., 136: 129-131.
3. Alon, N., D.N. Kozlov and V.H. Vu, 1996. The geometry of coin-weighing problems. Proc. 37th Annual Symposium on Foundations of Computer Science, FOCS, 96: 524-532.
4. Beerliova, Z., F. Eberhard, T. Erlebach, A. Hall, M. Hoffmann, M. Mihalak and L.S. Ram, Network discovery and verification. IEEE Journal on Selected Areas in Communications, 24 (12): 2168-2181.
5. Boutin, D.L., 2006. Identifying graphs automorphisms using determining sets. The Electron. J. Combin., 13: R78.
6. Buczkowski, P., G. Chartrand, C. Poisson and P. Zhang, 2003. On k-dimensional graphs and their basis. Period. Math. Hungar., 46 (1): 9-15.

7. Chartrand, G., L. Eroh, M.A. Johnson and O.R. Oellermann, 2000. Resolvability in graphs and the metric dimension of a graph. Disc. Appl. Math., 105 (1-3): 99-113.
8. Chartrand, G., D. Erwin, M.A. Henning, P.J. Slater and P. Zhang, 2003. Graphs of order n with locating-chromatic number n-1. Disc. Math., 269 (1-3): 65-79.
9. Chartrand, G. and P. Zhang, 2003. The theory and applications of resolvability in graphs. A survey. Congr. Numer., 160: 47-68.
10. Chv´atal, V., 1983. Mastermind, Combinatorica, 3 (3-4): 325-329.
11. Erwin, D. and F. Harary, 2006. Destroying automorphisms by fixing nodes. Disc. Math., 24: 3244-3252.
12. Fehr, M., S. Gosselin and O.R. Oellermann, 2006. The metric dimension of Cayley digraph. Disc. Math., 306: 31-41.
13. Garey, M.R. and D.S. Johnson, 1979. Computers and Intractability. A Guide to the Theory of NP-Completeness. A Series of Books in the Mathematical Sciences, W.H. Freeman, California.
14. Harary, F., 1996. Survey of methods of automorphism destruction in graphs, Invited address, Eighth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications, Kalamazoo, Michigan.
15. Harary, F., 2001. Methods of destroying the symmetries of a graph, Bull. Malaysian Math. Sci. Soc., 24 (2): 183-191.
16. Harary, F. and R.A. Melter, 1976. On the metric dimension of a graph, Ars Combin., 2: 191-195.
17. Imran, M., A.Q. Baig and M.K. Shafiq, On metric dimension of generalized Petersen graphs P(n,3), to appear.
18. Javaid, I., M.T. Rahim and K. Ali, 2008. Families of regular graphs with constant metric dimension. Util. Math., 75: 21-33.
19. Khuller, S., B. Raghavachari and A. Rosenfeld, 1996. Landmarks in graphs. Disc. Appl. Math., 70: 217-229.
20. Lynch, K., Determining the orientation of a painted sphere from a single image: A graph coloring problem, URL:http://citeseer.nj.nec.com/469-475.html.
21. Melter, R.A. and I. Tomescu, 1984. Metric basis in digital geometry. Computer Vision, Graphics and Image Processing, 25: 113-121.
22. Morris, J., 2006. Automorphism groups of circulant graphs. A survey, Graph Theory Trends in Mathematics, 1: 311-325.
23. Salman, M., I. Javaid and M.A. Chaudhry, Resolvability in Circulant Graphs. Acta. Math. Sinica, English Series, DOI: 10.1007/s10114-012-0417-4.
24. Sebö, A. and E. Tannier, 2004. On metric generators of graph. Math. Oper. Res., 29 (2): 383-393.
25. Slater, P.J., 1975. Leaves of trees. Congr. Numer., 14: 549-559.