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Numerical Solution of the Nonlinear Fredholm Integral Equation and the Fredholm Integro-differential Equation of Second Kind using Chebyshev Wavelets

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Abstract: In this paper, a numerical method to solve nonlinear Fredholm integral equations of second kind is proposed and some numerical notes about this method are addressed. The method utilizes Chebyshev wavelets constructed on the unit interval as a basis in the Galerkin method. This approach reduces this type of integral equation to solve a nonlinear system of algebraic equation. The method is also used to solve Fredholm integro-differential equation of second kind. Several numerical examples are presented to compare accuracy of Chebyshev wavelet Galerkin method with methods using polynomial Chebyshev basis.

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Key words: Fredholm integral equation of second kind . Hammerstein integral equation . Integro-differential equation . Chebyshev wavelet . Sparse matrix

INTRODUCTION

Integral equations have significant applications in various fields of science and engineering [1-5]. These equations also occur as reformulations of other mathematical problems such as partial differential equations and ordinary differential equations. A large class of integral equations is nonlinear Fredholm integral equation of second kind, namely

$$\lambda u(s) - \lambda\int_0^1 K(s,t)[u(t)]^m dt = f(s), \ \ 0 \le s \le 1, \lambda \ne 0 \qquad (1)$$

where f and K are known functions, u is unknown function to be determined and m>1 is an integer. Several methods have been proposed for numerical solution of these types of integral equation. In [6], the Petrov-Galerkin method and the iterated Petrov-Galerkin method have been applied to solve a class of nonlinear integral equations. Lardy [7] described a variation of the Nystrom method for nonlinear integral equations of the second kind. In [8], a method for numerical solution of nonlinear Fredholom integral equations utilizing positive definite functions is proposed by Alipanah and Dehghan. Authors of [9] introduced a numerical method for solving nonlinear Fredholm integral equations of the second kind using Sinc-collocation method.

A different problem take place when we have derivative of unknown function u in the equation, like

$$u'(s) + q(s)u(s) = \int_0^1 K(s,t)[\alpha u(t) + \beta u'(t)]dt + f(s), \ \ 0 \le s \le 1 \quad (2)$$

where α and β are constant numbers, function q is known, with the initial condition u(0) = r. This is Fredholm integro-differential equation.

Here, we would like to review some computational method for solving these equations. In [10], a Chebyshev finite difference method has been proposed in order to solve linear and nonlinear second-order Fredholm integro-differential equations. The variational iteration method (VIM) is considered to solve integral and integro-differential equations in [11]. Golbabai and Seifollahi [12] have applied RBF networks for solving the linear integro-differential equations. In [13] investigated the numerical solution of the nonlinear integro-differential equations based on the meshless method. In recent years, orthogonal functions receive a lot of attentions to solve these types of problems. The main advantage of using orthogonal basis is that it

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reduces the problem into solving a system of linear algebraic equations, by truncated approximation series

$$y(t) \simeq y_N(t) = \sum_{i=0}^{N-1} c_i \phi_i(t) \tag{3}$$

There are many different families of orthogonal functions which can be used. Orthogonal wavelet basis are one of them which are considerably useful to solve integral equations.

An efficient algorithm based on Haar wavelet -as simplest wavelet-approach for numerical solution of integral equations is given in [14]. Other wavelet families like CAS wavelet [15-17], Coifman wavelet [18],... are used to solve Fredholm integral or integro-differential equation of the second kind, directly. But among them polynomial wavelets have useful properties like high vanishing moments, easy structure and closed formula. These characteristics cause utilizing polynomial wavelets to be very striking. Polynomial Legendre wavelets are used in solving many kinds of problems [19, 20]. Legendre wavelet is also used to solve integro-differential equation [21]. Legendre wavelets are also used to approximate the solution of linear and nonlinear integral equations [22] and Fred-holm integro-differential equations [23] with weakly singular kernels. Z. Abbas *et al*. [24] improved this method by using Legendre multi-wavelet. B-spline wavelets are applied to solve Fredholm-Hammerstein integral equations of the second kind in [25].

First time, Chebyshev wavelet as another polynomial wavelet was discussed in [26, 27]. Using of chebyshev wavelet to solve integral equations continued in some other papers [28, 29]. The main purpose of this article is using properties of Chebyshev wavelet for solving Eq. (1) and Eq. (2). Note that the proposed method is not a pure Galerkin method, but it is a version of it. The properties of Chebyshev wavelets are used to convert Eq. (1) and Eq. (2) into a system of algebraic equations. This system may be solved by using an appropriate numerical method, such as Newtons iteration method. We will notice that, these wavelets make the wavelet coefficient matrices sparse and accordingly leads to the sparsity of the coefficient matrix of the final system and provide accurate solutions.

The outline of the paper is as follows: In Section 2, wavelets and Chebyshev wavelets and their properties are described. Then Chebyshev wavelet is used to approximate one and two dimensional functions. Sections 3 and 4 present computational methods for solving (1) and (2) utilizing Chebyshev wavelets and numerical examples are presented in last part of both sections. Finally, we conclude the article in Section 5.

CHEBYSHEV WAVELETS

Wavelets and Chebyshev wavelets: Wavelets consist of a family of functions constructed from dilation and translation of a single function called the mother wavelet $\psi(x)$. We have the following family of continuous wavelets as

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}} \psi\left(\frac{t-b}{a}\right), a,b \in \mathbb{R}, a \neq 0 \tag{4}$$

Chebyshev wavelets, $\psi_{n,m}(t) = \psi(k,n,m,t)$ have four arguments; $n = 1,2,\ldots, 2^{k-1}$, k can assume any non-negative integer, m is the degree of Chebyshev polynomial of the first kind and t denotes independent variable in [0,1] [5]:

$$\psi_{n,m}(t) = \begin{cases} \alpha_m \sqrt{\dfrac{2^k}{\pi}} T_m(2^k t - 2n + 1), & \dfrac{n-1}{2^{k-1}} \le t < \dfrac{n}{2^{k-1}} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

where

$$\alpha_m = \begin{cases} 1, & m = 0 \\ 2, & m > 0 \end{cases} \tag{6}$$

and $m = 0,1,\ldots,M-1$, $n = 1,2,\ldots,2^{k-1}$. Here T_m, $m = 0,1,\ldots$, are Chebyshev polynomials of the first kind which are orthogonal with respect to the weight function

$$w(t) = \frac{1}{\sqrt{1-t^2}}$$

on the interval [-1,1] and satisfy the following recursive formula:

$$T_0(t) = 1,\ T_1(t) = t,\ T_m(t) = 2tT_{m-1}(t) - T_{m-2}(t),\ m = 2,3,\ldots.$$

For these functions we also have the following useful formula:

$$T_m(\cos\theta) = \cos m\theta,\ m = 0,1,2,\ldots.$$

We should note that Chebyshev wavelets are orthonormal set with respect to the weight function

$$\overline{w}_k(t) = \begin{cases} w_{1,k}(t), & 0 \le t < \dfrac{1}{2^{k-1}} \\ w_{2,k}(t), & \dfrac{1}{2^{k-1}} \le t < \dfrac{2}{2^{k-1}} \\ \vdots & \vdots \\ w_{2^{k-1},k}(t), & \dfrac{2^{k-1}-1}{2^{k-1}} \le t < 1 \end{cases} \tag{7}$$

where

$$w_{n,k}(t) = w(2^{k-1}t - n + 1)$$

Function approximation: A function $f(x) \in L^2_{\overline{w}_k}[0,1]$, may be expanded as

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{n,m}\psi_{n,m}(x) \tag{8}$$

where

$$c_{n,m} = \langle f(x), \psi_{n,m}(x) \rangle_{\overline{w}_k} \tag{9}$$

in which $\langle .,. \rangle_{\overline{w}_k}$ denotes the inner product in $L^2_{\overline{w}_k}[0,1]$. The series (8) is truncated as

$$f(x) \simeq \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{n,m}\psi_{n,m}(x) = C^T\Psi(x) \tag{10}$$

where C and Ψ are $2^{k-1}M$ vectors given by

$$C = [c_{1,0}, c_{1,1}, \ldots, c_{1,M-1}, c_{2,0}, c_{2,0}, \ldots, c_{2,M-1}, \ldots, c_{2^{k-1},0}, \ldots, c_{2^{k-1},M-1}]^T$$
$$= [c_1, c_2 \ldots, c_{2^{k-1}M}]^T$$

and

$$\Psi = [\psi_{1,0}, \psi_{1,1}, \ldots, \psi_{1,M-1}, \psi_{2,0}, \psi_{2,1}, \ldots,$$
$$\psi_{2,M-1}, \ldots, \psi_{2^{k-1},0}, \ldots, \psi_{2^{k-1},M-1}]^T \tag{11}$$
$$= [\psi_1, \psi_2, \ldots, \psi_{2^{k-1}M}]^T$$

It means if $\psi_i = \psi_{n,m}$ or $c_i = c_{n,m}$ then we have $i = M(n-1) + m + 1$. Similarly, by considering $i = M(n-1) + m + 1$ and $j = M(n'-1)m' + 1$, we approximate $K(x,y) \in L^2_{\overline{w}_k}([0,1]\times[0,1])$ as

$$K(x,y) \simeq \sum_{i=1}^{2^{k-1}M}\sum_{j=1}^{2^{k-1}M} K_{ij}\psi_i(x)\psi_j(y) = \Psi^T(x)\mathbf{K}\Psi(y)$$

where $\mathbf{K} = [K_{ij}]_{1\le i,\, j\le 2^{k-1}M}$ with the entries

$$K_{ij} = \langle \psi_i(x), \langle K(x,y), \psi_j(y) \rangle_{\overline{w}_k} \rangle_{\overline{w}_k} \tag{12}$$

Ordinary calculation of c_i's and K_{ij}'s to approximate one or two dimensional functions take considerable times. But using Gauss-Chebyshev quadrature rule [30]

$$\int_{-1}^{1} f(s)w(s)ds \simeq \frac{\pi}{N}\sum_{p=1}^{N} f\left(\cos\xi_p\right) \tag{13}$$

where

$$\xi_p = \pi(2p-1)/2N,\ p = 1,2,\ldots,N$$

can reduce number of operations and make commodious methods to calculate inner product in (9) and (12).

$$\begin{aligned} c_i = c_{n,m} &= \int_0^1 f(x)\psi_{n,m}(x)\overline{w}_k(x)dx \\ &\simeq \frac{\sqrt{\pi}\alpha_m}{2^{k/2}N}\sum_{p=1}^{N} f(2^{-k}(\cos\xi_p + 2n - 1))\cos(m\xi_p) \end{aligned} \tag{14}$$

and

$$\begin{aligned} K_{ij} &= \int_0^1\int_0^1 \psi_{n,m(x)}K(x,y)\psi_{n',m'}(y)\overline{w}_k(x)\overline{w}_k(y)dxdy \\ &\simeq \frac{\pi\alpha_m\alpha_{m'}}{2^k N^2}\sum_{q=1}^{N}\sum_{p=1}^{N} K\left(\frac{\cos\xi_p + 2n - 1}{2^k}, \frac{\cos\xi_q + 2n' - 1}{2^k}\right) \\ &\cos(m\xi_p)\cos(m'\xi_q) \end{aligned} \tag{15}$$

SOLUTION OF FREDHOLM INTEGRAL EQUATIONS

In this section, the Chebyshev wavelet method is used to solve (1) by approximating functions *f*(x), u(x) and K(x,y) in the matrix forms

$$f(x) \simeq F^t\Psi(x)$$

$$[u(x)]^m \simeq \overline{U}^t\Psi(x) \tag{16}$$

$$K(x,y) \simeq \Psi^t(x)\mathbf{K}\Psi(y)$$

where $\overline{U}$ a column vector function whose elements are nonlinear combinations of elements of the vector U, which is computed next. By substituting (16) into (1), we obtain

$$\lambda\Psi^T(x)U - \Psi^T(x)\mathbf{K}\left(\int_0^1 \Psi(y)\Psi^T(y)dy\right)\overline{U} \simeq \Psi^T(x)F \tag{17}$$

Let L be the $2^{k-1}M\times 2^{k-1}M$ matrix with the following definition:

$$\mathbf{L} = \int_0^1 \Psi(y)\Psi^T(y)dy \tag{18}$$

Therefore, replacing $\simeq$ with \$=\$ in (17) and using (18), we have

$$\Psi^T(x)\lambda U - \Psi^T(x)\mathbf{KL}\overline{U} = \Psi^T(x)F \tag{19}$$

Now by taking inner product $\langle ., \Psi(x) \rangle_{\overline{w}_k}$ upon both sides of (19) and using orthonormality of Chebyshev wavelets, we can reduce (1) to the system

$$\lambda U - \mathbf{KL}\overline{U} = F \tag{20}$$

Notice that in some earlier works [27, 29], the matrix L has been considered as identity matrix, mistakenly, while

$$\int_0^1 \Psi(y)\Psi^T(y)\overline{w}_k(y)dy = I \tag{21}$$

but structure of L is different from (21).

Now we want to calculate $L_{i,j}, 1 \le i, j \le 2^{k-1}M$ as an entire of matrix L. Similar to (11) let $i = M(n-1) + m + 1$ and $j = M(n'-1)m' + 1$. By definition of L

$$\mathbf{L}_{i,j} = \int_0^1 \psi_i(y)\psi_j(y)dy \tag{22}$$

Pending $n \neq n'$, supports of ψ_i and ψ_j have no intersection and $\psi_i(y)\psi_j(y) = 0$ hence, $L_{i,j} = 0$. Therefore, suppose that $n=n'$. By substituting $2^k x - 2n + 1 = \cos\theta$ in (22) we have

$$L_{i,j} = \frac{-C_{mm'}}{\pi}\int_0^{\pi} \cos m\theta \cos m'\theta \sin\theta d\theta \tag{23}$$

where

$$C_{mm'} = \begin{cases} 1, & m = m' = 0 \\ 2, & m \neq 0 \neq m' \\ \sqrt{2}, & \text{otherwise} \end{cases}$$

By calculating integral in (23), matrix L has the following form

$$\mathbf{L} = \operatorname{diag}(\underbrace{A, A, \ldots, A}_{2^{k-1}\text{ times}}) \tag{24}$$

where

$$A = [A_{mm'}],\ m, m' = 0, 1, \ldots, M-1$$

is an $M \times M$ matrix with the following entries

$$A_{mm'} = \begin{cases} C_{mm'}\dfrac{-2(m^2 + m'^2 - 1)}{\pi(1 + m^4 - 2m'^2m^2 - 2m'^2 - 2m^2 + m'^4)}, & m + m' \text{ is even} \\ 0, & m + m' \text{ is odd} \end{cases}$$

Clearly L is a sparse matrix. On the other hand choosing a threshold parameter $\varepsilon_0 > 0$ [27] and replacing $\overline{\mathbf{K}} = [\overline{K}_{i,j}]$ with

$$\overline{K}_{ij} = \begin{cases} K_{ij}, & |K_{ij}| \ge \varepsilon_0 \\ 0, & \text{otherwise} \end{cases}$$

by K in (20), we get the following system of nonlinear equations whose coefficients matrix is sparse:

$$\lambda U - \overline{\mathbf{K}}\mathbf{L}\overline{U} = F \tag{25}$$

Computation of $\overline{\mathbf{U}}$: For this purpose, we introduce the matrix

$$\mathbf{P} = [P_{ij}]_{M2^{k-1} \times M2^{k-1}},\ i, j = 1, 2, \ldots, M2^{k-1} \tag{26}$$

where $P_{ij} = \psi_i(x'_j)$ and x'_j's are the collocation points

$$x'_j = \frac{(j - \frac{1}{2})}{M2^{k-1}}\ j = 1, 2, \ldots, M2^{k-1} \tag{27}$$

Consequently, P has the following form

$$\mathbf{P} = \operatorname{diag}\left(A_0, A_1, \ldots, A_{2^{k-1}-1}\right) \tag{28}$$

in which $A_s, s = 0, 1, \ldots, 2^{k-1} - 1$, is the $M \times M$ matrix

$$A_s = [P_{ij}]_{1+sM \le i \le (s+1)M,\ 1+sM \le j \le (s+1)M} \tag{29}$$

Now, we get:

$$\overline{U}^t\Psi(x) = [U^t\Psi(x)]^m \tag{30}$$

By evaluating (30) at collocation points $\{x'_j\}_{j=1}^{M2^{k-1}}$, we obtain the following system of equations

$$\overline{U}^tP_r = (U^tP_r)^m,\ r = 1, 2, \ldots, M2^{k-1} \tag{31}$$

where P_r is the r-th column of P. Finally, by using the representation of P in (28), (31) is reduced to 2^{k-1} system of M equations in M unknowns.

Example 3.1: Consider the Fredholm integral equation of the second kind:

$$u(x) + \int_0^1 e^{2x - \frac{5}{3}y}[u(y)]^2 dy = (4/7)e^{2x} + (3/7)e^{2x-7/3}, \quad 0 \le x \le 1 \tag{32}$$

The exact solution of this problem is $u(x) = e^{2x}$. Table 1 shows the numerical results for this example with k = 3, M = 4 and k = 4, M = 4, with $\varepsilon_0 = 10^{-6}$. Also, the approximate solution for k = 2, M = 3 with $\varepsilon_0 = 10^{-4}$ are compared with exact solution graphically in Fig. 1.

Table 1: Some numerical results for example 3.1

x	Exact solution	Approximate solution k = 3, M = 4 with $e_0 = 10^{-6}$	Approximate solution k = 4, M = 4 with $e_0 = 10^{6}$
0.1	1.2214028	1.2213838	1.2214045
0.2	1.4918247	1.4918468	1.4918249
0.3	1.8221188	1.8221535	1.8221184
0.4	2.2255409	2.2255090	2.2255435
0.5	2.7182818	2.7182108	2.7182767
0.6	3.3201169	3.3200670	3.3201206
0.7	4.0552000	4.0552577	4.0551970
0.8	4.9530324	4.9531283	4.9530274
0.9	6.0496475	6.0495601	6.0496537
1.0	7.3890561	7.3890930	7.3890524

Table 2: Some numerical results for example 3.2

x	Exact solution	Approximate solution k = 4, M = 4	Relative error	Absolute error
0.0	0.0000000000	6.787e-7	---	6.787e-7
0.1	0.1847732567	0.1847736927	2.3596e-6	4.360e-7
0.2	0.3677003201	0.3677026918	6.4500e-6	2.3717e-6
0.3	0.5469534435	0.5469568199	6.1731e-6	3.3764e-6
0.4	0.7207415888	0.7207420707	6.6861e-7	4.819e-7
0.5	0.8873283225	0.8873188634	1.0660e-5	9.4591e-6
0.6	1.0450491650	1.0450503780	1.1607e-6	1.213e-6
0.7	1.1923282210	1.1923356910	6.2650e-6	7.470e-6
0.8	1.3276939280	1.3277021230	6.1723e-6	8.195e-5
0.9	1.4497937570	1.4497949030	7.9045e-7	1.146e-6
1.0	1.5574077250	1.5574072890	2.79952e-7	4.36e-7

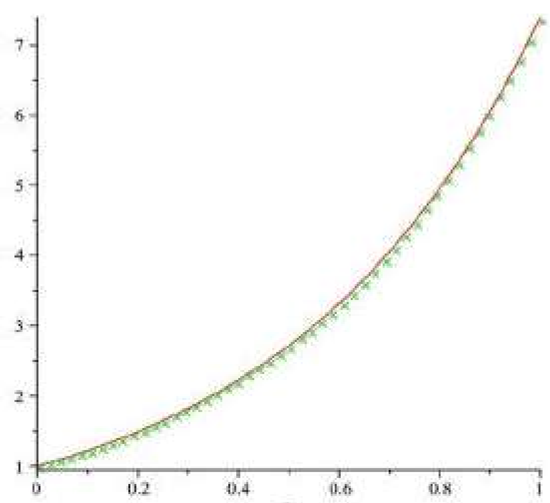

Fig. 1: Approximate solution for example 3.1 with k = 2, M = 3 with $\varepsilon_0 = 10^{-4}$

Example 3.2: Consider integral equation

$$u(s) = (1/3)\sin(s) - (1/9)\sin(s)^3 + (1/3)s\cos(1)\sin(1)^2 + (2/3)s\cos(1) + \int_0^1 K(s,t)[u(t)]^3\,dt$$

where

$$K(s,t) = \begin{cases} s, & s < t \\ t, & s \geq t \end{cases}$$

with exact solution u(s) = sin (s). Table 2 shows the numerical results for this example with k = 4, M = 4 and $\varepsilon_0 = 10^{-5}$. Absolute and relative errors in sample points are also reported.

INTEGRO-DIFFERENTIAL EQUATION

In this section, we use Chebyshev wavelet method to solve integro-differential equation (2). The integration of the vector $\Psi(t)$ can be written as:

$$\int_0^t \Psi(x)dx \simeq P\Psi(t) \qquad (33)$$

where P is a $2^{k-1}M \times 2^{k-1}M$ matrix called operational matrix for integral (OMI) and is given in [27]. The following property of the product of two Chebyshev wavelet function vectors will be used:

$$\int_0^t \Psi(x)dx \simeq P\Psi(t) \qquad (34)$$

where X is an arbitrary $2^{k-1}M$ vector and $\tilde{X}$ is a $2^{k-1}M \times 2^{k-1}M$ matrix, which called Product Operation Matrix (POM) of Chebyshev wavelet that is given in [27].

In this case we approximate with $u'(t)$ Chebyshev wavelet series

$$u'(t) = \sum_{i=1}^{2^{k-1}M} c_i \Psi_i(t) = C^T \Psi(t) \tag{35}$$

Direct conclusion of (35) is

$$u(t) = \sum_{i=1}^{2^{k-1}M} c_i s_i(t) + u(0) \tag{36}$$

where $s_i(t) = \int_0^t \psi_i(x)dx$. Then scalar $u(0)$ can be written as:

$$u(0) \simeq [\underbrace{\alpha\mathbf{e}_1, \alpha\mathbf{e}_1, \ldots, \alpha\mathbf{e}_1}_{2^{k-1}\text{ times}}]\Psi(t) = U_0^T\Psi(t) \tag{37}$$

where $\alpha = \sqrt{\frac{\pi}{2^k}}u(0)$ and $\mathbf{e}_1 = [1,0,0,\ldots,0]_{M\times 1}$.

Therefore, we can approximate u by

$$u(t) \simeq C^T P\Psi(t) + U_0^T\Psi(t) \tag{38}$$

and

$$q(s) \simeq Q^T\Psi(s), \quad f(s) \simeq F^T\Psi(s), \quad K(s,t) \simeq \Psi^T(s)\mathbf{K}\Psi(t) \tag{39}$$

Substituting (35),(38) and(39) into (2) and replacing $\simeq$ with =, gives

$$\begin{aligned} &\Psi^T(s)C + Q^T\Psi(s)\Psi^T(s)(P^T C + U_0) \\ &= \int_0^1 \Psi^T(s)\mathbf{K}\Psi(t)\Psi^T(t)[\alpha(P^T C + U_0) + \beta C]dt + \Psi^T(s)F \end{aligned} \tag{40}$$

Utilizing (33), (34) and L, definition (24), we obtain

$$\Psi^T(s)[C + \tilde{Q}(P^T C + U_0)] = \Psi^T(s)[\mathbf{KL}(\alpha(P^T C + U_0) + \beta C) + F] \tag{41}$$

Now, by taking inner product $<\Psi(s),.>_{\varpi_k}$ from both sides of (41) lead to the following linear syste

$$[I + \tilde{Q}P^T - \mathbf{KL}(\alpha P^T + \beta I)]C = -\tilde{Q}U_0 + \mathbf{KL}U_0 + F \tag{42}$$

Solving (42) gives unknown vector C, so

$$u(t) \simeq (P^T C + U_0)\Psi(t) \tag{43}$$

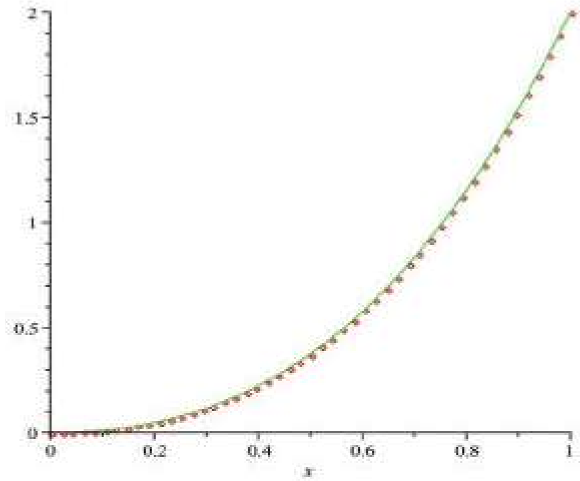

Fig. 2: Approximate solution for example 4.1 with $k = 2$, $M = 3$, $\varepsilon_0 = 10^{-4}$

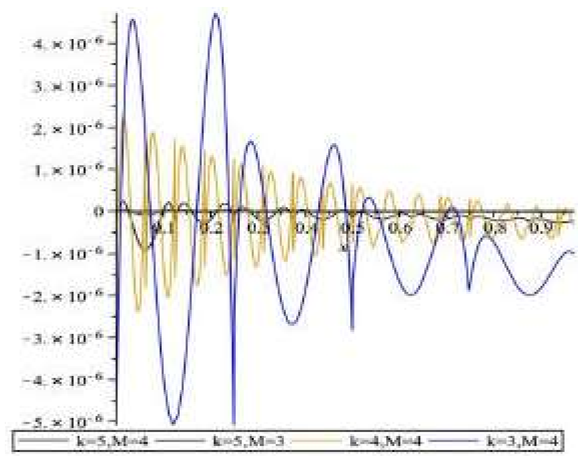

Fig. 3: Distribution of errors for Example 4.2

Example 4.1: Consider equation (2) with $q(s) = 2s$, $K(s,t) = s + t, \alpha = 0, \beta = 1, \gamma = 0$ and

$$f(x) = 2x^4 + 2x^3 + 3x^2 - \frac{17}{12}$$

In Table 3, the results of Chebyshev wavelet method are compared with exact solution, which is $u_{ex}(s) = s^2(1+x)$ [14]. Table 4 reports error of approximation solution in L^2-norm. This is

$$\| u - u_{ex} \|_2 = \left(\int_0^1 |u(s) - u_{ex}(s)|^2 \, ds\right)^{\frac{1}{2}}$$

Also, the approximate solution for k = 3, M = 3 with $\varepsilon_0 = 10^{-4}$ is compared with exact solution graphically in Fig. 2.

Table 3: Some numerical results for example 4.1

x	Exact solution	Approximate solution k = 3, M = 3, $\varepsilon_0 = 10^{-4}$	Approximate solution k = 4, M = 4, $\varepsilon_0 = 10^{4}$
0.1	0.01100000	0.010719840	0.01099999
0.2	0.04800000	0.048450063	0.04799999
0.3	0.11700000	0.116534354	0.11699999
0.4	0.22400000	0.224264710	0.22399999
0.5	0.37500000	0.375474480	0.37500000
0.6	0.57600000	0.575705322	0.57600000
0.7	0.83300000	0.833436680	0.83300000
0.8	1.15200000	1.151522503	1.15199999
0.9	1.53900000	1.539254613	1.53899999
1.0	2.00000000	1.999487303	1.99999999

Table 4: Error for example 4.1

k	M	$\|u-u_{ex}\|_2$
2	3	7.269439334e-6
3	3	1.152331560e-7
4	3	1.807189679e-9
3	4	5.854994865e-19
4	4	9.393085527e-20

Table 5: Some numerical results for example 4.2

x	Exact solution	Approximate solution k = 3, M = 3, $\varepsilon_0 = 10^{-4}$	Approximate solution k = 4, M = 4, $\varepsilon_0 = 10^{4}$
0.1	0.0953101798	0.0952889907	0.0953101493
0.2	0.1823215568	0.1825449463	0.1823217648
0.3	0.2623642645	0.2624945635	0.2623644469
0.4	0.3364722366	0.3367692254	0.3364722266
0.5	0.4054651081	0.4058829389	0.4054652876
0.6	0.4700036292	0.4704350184	0.4700036824
0.7	0.5306282511	0.5312459776	0.5306284418
0.8	0.5877866649	0.5884913528	0.5877868810
0.9	0.6418538862	0.6427604341	0.6427604341
1.0	0.6931471806	0.6939446730	0.6931473158

Table 6: Error for example 4.2

k	M	$\|u-u_{ex}\|_2$
3	3	2.763993163e-7
4	3	3.688787084e-11
5	3	5.811129897e-13
3	4	5.854994865e-19
4	4	9.393085527e-20

Example 4.2: Consider

$$u'(t)-u(t)-\frac{1}{(\ln 2)^2}\int_0^1 \frac{t}{s+1}u(s)ds$$
$$=\frac{-1}{2}t+\frac{1}{t+1}-\ln(1+t), u(0)=0$$

with exact solution u(t) = ln(t+1) [21].

In Table 5, the results of chebyshev wavelet method compared with exact solution. Table 6 reports error value of approximation solution in L^2-norm. Distribution of errors for different k and M are graphically shown in Fig. 3. It reveals that little increase in k and M, improve approximation solution considerably.

CONCLUSION

The Chebyshev wavelets have been applied for solving integral and integro-differential equations by reducing each equation into a system of algebraic equations. Because of some good properties of Chebyshev wavelets like having vanish moments and local support,... using Chebyshev wavelets give high accuracy approximation of solution. Numerical examples show accuracy of this method compared with some other methods.

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