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On Positive Solution of Nonlinear Fractional Differential Equation

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Abstract: In this paper, the method of upper and lower solutions and the Schauder fixed point theorem are used to investigate the existence and uniqueness of a positive solution for a class of nonlinear fractional differential equations with non-monotone term. An example is also given to illuminate our results.

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INTRODUCTION

We consider the following nonlinear fractional differential equation

$$D^{\alpha}u = f(t,u), 0 < t < 1, u(0) = 0 \qquad (1)$$

where $0<\alpha<1$, D^{α} is the Riemann-Liouville fractional derivative defined as follows

$$D^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}u(s)ds \qquad (2)$$

where Γ denotes the Gamma function and $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ is a given continuous function. Fractional differential equation (1) can be extensively applied to the various physics, mechanics, chemistry and engineering etc. [1-5]. Hence, in recent years, fractional differential equations have been of great interest and there have been many results on the existence and uniqueness of solutions of FDE. D. Delbosco and L.Rodino proved the existence of a solution for nonlinear fractional equation (1) using the Banach contraction principle and the Schauder fixed point theorem respectively [6]; Shuqin Zhang obtained the existence and uniqueness of a positive solution utilizing the method of upper and lower solutions and the cone fixed-point theorem [7]; Qingliu Yao considered the existence of a positive solution for fractional order differential equation controlled by the power function employing the Krasnosel'skii fixed-point theorem of cone expansion-compression type [8]. Recently, V. Lakshmikantham obtained the existence of a local and global solution for equation (1) using the classical differential equation theorem [9]. However, in the previous works, the nonlinear term has to satisfy the monotone or other control conditions. In fact, the nonlinear fractional differential equation with non-monotone term can respond better to impersonal law, so it is very important to study this kind of problems. In this paper we investigate the nonlinear fractional differential equations with non-monotone term by constructing a pair of upper and lower control functions and exploiting the method of upper and lower solutions as well as the Schauder fixed-point theorem. The existence and uniqueness of a positive solution for equation (1) is obtained under as few possible as condition. This work is motivated from the references [7, 10]. Other related results on the fractional differential equations can be found in refs [11-22].

This paper is organized as follow. In section 2 we consider the existence of positive solution for equation (1) utilizing the method of upper and lower solutions and the Schauder fixed-point theorem. Section 3 deals with the uniqueness of positive solution and gives an example to illuminate our results.

EXISTENCE OF POSITIVE SOLUTION

Let X = C[0,1] be the Banach space endowed with the maximum norm

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$$||u(t)||= \max_{0\le t\le 1} |u(t)|$$

and define one positive cone

$$K = \{u \in X : u(t) \ge 0, 0 \le t \le 1\}$$

The positive solution which we consider in this paper is such that

$$u(0) = 0, u(t) > 0, 0 < t \le 1, u(t) \in X$$

According to Proposition 2.4 in [4], equation (1) is equivalent to the integral equation

$$u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds, 0 \le t \le 1 \quad (3)$$

where Γ denotes the Gamma function. It is easy to verify that to solve the integral equation (3) is also equivalent to solve the following fixed point equation

$$Tu(t) = u(t), u(t) \in C[0,1]$$

where operator T: K$\to$K is defined as

$$(Tu)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds, 0 \le t \le 1 \quad (4)$$

First, we have the following compactness Lemma.

Lemma 2.1: [7] The operator T: K$\to$K is completely continuous.

Proof: The operator T: K$\to$K is continuous in view of the assumption of nonnegativeness and continuity of f(t,u).

Let M$\subset$K be bounded, i.e., there exists a positive constant l such that $||u||\le l$ for any u$\in$M Since f(t,u) is a given continuous function, we have

$$\max_{0\le t\le 1} f(t,u(t)) \le \max_{(t,u)\in D} f(t,u)$$

for any u$\in$M, where

$$D = \{(t,u) | 0 \le t \le 1, 0 \le u \le l\}$$

Let

$$L = \max_{(t,u)\in D} f(t,u)$$

then for any u$\in$M, we have

$$\begin{aligned} |Tu(t)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,u(s))ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} |f(s,u(s))|ds \\ &\le \frac{L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \le \frac{L}{\Gamma(1+\alpha)} t^{\alpha} \end{aligned}$$

Thus

$$\|Tu\| \le \frac{L}{\Gamma(1+\alpha)}$$

Hence the operator T: K$\to$K is uniformly bounded.

Now, we will prove that the operator T is equicontinuous. For each u$\in$M, any $\varepsilon>0$, t_1, $t_2\in[0,1]$, $t_1<t_2$. Let $\delta = (\varepsilon\,\Gamma(1+\alpha)/2L)^{1/\alpha}$, then when $|t_2-t_1|<\delta$, we have

$$\begin{aligned} |Tu(t_1) - Tu(t_2)| &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1} f(s,u(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1} f(s,u(s))ds\right| \\ &= \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1} f(s,u(s))ds - \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_2-s)^{\alpha-1} f(s,u(s))ds - \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_1-s)^{\alpha-1} f(s,u(s))ds\right| \\ &\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1} |(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}||f(s,u(s))|ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} |f(s,u(s))|ds \\ &\le \frac{L}{\Gamma(\alpha)}\int_0^{t_1} ((t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1})ds + \frac{L}{\Gamma(\alpha)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \\ &= \frac{L}{\Gamma(\alpha)}\left(\int_0^{t_1} (t_1-s)^{\alpha-1} ds - \int_0^{t_1} (t_2-s)^{\alpha-1} ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds\right) \\ &= \frac{L}{\Gamma(1+\alpha)}[t_1^{\alpha} + (t_2-t_1)^{\alpha} - t_2^{\alpha} + (t_2-t_1)^{\alpha}] \le \frac{2L}{\Gamma(1+\alpha)}(t_2-t_1)^{\alpha} < \frac{2L}{\Gamma(1+\alpha)}\delta^{\alpha} = \varepsilon \end{aligned}$$

The Arzela-Ascoli Theorem implies that T is completely continuous. The proof is therefore completed.

Let $f(t,u):[0,1]\times[0,+\infty) \to [0,+\infty)$ is a given continuous function. For any u$\in$[0,b] we define the upper-control function

$$H(t,u)=\sup_{0\le\eta\le u} f(t,\eta)$$

and the lower-control function

$$h(t,u)=\inf_{u\le\eta\le b} f(t,\eta)$$

It is obvious that H(t, u) and h(t, u) are monotonous non-decreasing with respect to u. And we have

$$h(t,u)\le f(t,u)\le H(t,u)$$

Definition 2.1: If $\tilde{u}(t),\hat{u}(t)\in K,\quad 0\le\hat{u}(t)\le\tilde{u}(t)\le b$ satisfy

$$D^\alpha\tilde{u}(t)\ge H(t,\tilde{u}(t))\ \text{(or}\ \tilde{u}(t)\ge\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H(s,\tilde{u}(s))ds,\quad 0\le t\le 1\text{)}$$

and

$$D^\alpha\hat{u}(t)\le h(t,\hat{u}(t))\ \text{(or}\ \hat{u}(t)\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s,\hat{u}(s))ds,\quad 0\le t\le 1\text{)}$$

then the functions $\tilde{u}(t)$ and $\hat{u}(t)$ are called a pair of upper and lower solutions for equation (1).
Now, we give the main result of this paper.

Theorem 2.1: Assume $f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ is a continuous function and $\tilde{u}(t)$, $\hat{u}(t)$ is a pair of upper and lower solution of equation (1), then the initial value problem (1) exists at least one solution u(t)∈C[0,1] satisfying

$$\tilde{u}(t)\ge u(t)\ge\hat{u}(t)\ .\ t\in[0,1]$$

Proof Let

$$S=\{v\in K\mid \hat{u}(t)\le v(t)\le\tilde{u}(t), t\in[0,1]\ \}$$

then we have ||v||≤b. Hence S is a convex, bounded and closed subset of the Banach space X. According to Lemma 2.1, the operator T: K→K is completely continuous. Thus the rest is to prove that T: S→S.
For any v(t)∈S, we have $\tilde{u}(t)\ge v(t)\ge\hat{u}(t)$, then

$$\begin{aligned}Tv(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,v(s))ds\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H(s,v(s))ds\\&\le\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}H(s,\tilde{u}(s))ds\le\tilde{u}(t)\end{aligned}$$

and

$$\begin{aligned}Tv(t)&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,v(s))ds\\&\ge\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s,v(s))ds\\&\ge\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}h(s,\hat{u}(s))ds\ge\hat{u}(t)\end{aligned}$$

Hence $\tilde{u}(t)\ge Tv(t)\ge\hat{u}(t), 1\ge t\ge 0$, namely, T: S→S. According to the Schauder fixed point theorem, the operator T has at least one fixed point u(t)∈S, 0≤t≤1. Therefore the initial value problem (1) exists at least one solution u(t)∈C[0,1] and $\tilde{u}(t)\ge u(t)\ge\hat{u}(t)\ .t\in[0,1]$.

Corollary 2.1: Assume $f(t,u):[0,1]\times[0,+\infty)\to[0,+\infty)$ is a continuous function and there exists $k_2\ge k_1>0$, such that

$$k_1\le f(t,l)\le k_2\quad (t,l)\in[0,1]\times[0,+\infty)\tag{5}$$

then the initial value problem (1) has at least a positive solution u(t)∈C[0,1] satisfying

$$\frac{k_1}{\Gamma(1+\alpha)}t^\alpha\le u(t)\le\frac{k_2}{\Gamma(1+\alpha)}t^\alpha$$

Proof: By the assumption (5) and the definition of control function, we have

$$k_1\le h(t,l)\le H(t,l)\le k_2\,(t,l)\in[0,1]\times[a,b]$$

Now, we consider the equation

$$D^\alpha w(t)=k_2,\ w(0)=0\tag{6}$$

Obviously, the equation (6) has a positive solution

$$w(t)=I^\alpha(k_2)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}k_2ds=\frac{k_2}{\Gamma(1+\alpha)}t^\alpha$$

and

$$w(t)=I^\alpha(k_2)\ge I^\alpha(H(t,w(t))$$

namely, w(t) is a upper solution of equation (1). In the similar way, we obtain $v(t)=I^\alpha(k_1)$ is the lower solution of equation (1). By using Theorem 2.1, one gets that the initial value problem (1) has at least one solution u(t)∈C[0,1] satisfying

$$\frac{k_1}{\Gamma(1+\alpha)}t^\alpha\le u(t)\le\frac{k_2}{\Gamma(1+\alpha)}t^\alpha$$

Corollary 2.2: Assume $f(t,u):[0,1]\times[0,+\infty)\to[a,+\infty)$ is a continuous function, where a is a positive constant and also assume that

$$a<\lim_{u\to+\infty} f(t,u(t))<+\infty,\quad t\in[0,1] \tag{7}$$

then the initial value problem (1) has at least one positive solution $u(t)\in C[0,1]$.

Proof: By assumption (7), there exist positive constants N, R such that for any u>R one has $f(t,u)\le N$. Let

$$C=\max_{0\le t\le 1, 0\le u\le R} f(t,u)$$

Then we have $a\le f(t,u)\le N+C$, $0<u<+\infty$. By corollary 2.1, the initial problem (1) has at least one positive solution $u(t)\in C[0,1]$ satisfying

$$\frac{a}{\Gamma(1+\alpha)}t^{\alpha}\le u(t)\le\frac{N+C}{\Gamma(1+\alpha)}t^{\alpha} \tag{8}$$

Corollary 2.3: If $f(t,u):[0,1]\times[0,+\infty)\to[a,+\infty)$ is a continuous function, where a is a positive constant and

$$a<\lim_{u\to+\infty}\max_{0\le t\le 1}\frac{f(t,u(t))}{u(t)}<+\infty \tag{9}$$

then the initial value problem (1) has at least one positive solution $u(t)\in C[0,\delta]$, where $0<\delta<1$.

Proof: According to

$$a<\lim_{u\to+\infty}\max_{0\le t\le 1}\frac{f(t,u(t))}{u(t)}<+\infty$$

there exists M>0, c>0 such that for any $u(t)\in X$ one has

$$f(t,u(t))\le Mu(t)+c$$

By the definition of control function, we have

$$H(t,u(t))\le Mu(t)+c \tag{10}$$

Next, we consider the equation

$$D^{\alpha}u(t)=Mu(t)+c,\quad 0<\alpha<1, 0<t<1 \tag{11}$$

According to proposition 4.2 in [4], the equation (11) is equivalent to the integral equation

$$u(t)=I^{\alpha}(Mu(t)+c)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(Mu(s)+c)ds,\quad 0\le t\le 1$$

Let $A: K\to K$ is an operator defined as follow

$$Au(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}(Mu(s)+c)ds,\quad 0\le t\le 1.$$

By Lemma 2.1, the operator A is completely continuous.

Let

$$B_R=\left\{u(t)\in K \,\middle|\, \left\|u-\frac{c}{\Gamma(1+s)}t^{s}\right\|\le R<+\infty\right\}$$

then B_R is convex, bounded and closed subset of the Banach space $C[0,\delta]$, where $0<\delta<1$.

For any $u\in B_R$, we have

$$\|u\|\le\frac{c}{\Gamma(1+\alpha)}t^{\alpha}+R\le\frac{c}{\Gamma(1+\alpha)}\delta^{\alpha}+R\le\frac{c}{\Gamma(1+\alpha)}+R$$

then

$$\begin{aligned}\left\|Au(t)-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}\right\|&\le\frac{M}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)ds\\&\le\frac{M}{\Gamma(1+\alpha)}\|u(t)\|t^{\alpha}\\&\le\frac{M}{\Gamma(1+\alpha)}\left(\frac{c}{\Gamma(1+\alpha)}+R\right)\delta^{\alpha}\end{aligned}$$

Taking

$$\delta<\min\left\{\left[\frac{\Gamma(1+\alpha)}{2M}\right]^{\frac{1}{\alpha}},\left[\frac{R(\Gamma(1+\alpha))^2}{2Mc}\right]^{\frac{1}{\alpha}},1\right\}$$

then

$$\left\|Au(t)-\frac{c}{\Gamma(1+\alpha)}t^{\alpha}\right\|\le R$$

Hence, by the Schauder fixed theorem, the operator A has at least one fixed point and then the equation (11) has at least one positive solution $w^*(t)$ where $0<t<\delta$. Hence

$$\begin{aligned}w^*(t)&=I^{\alpha}(Mw^*(t)+c)\\&=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}(Mw^*(s)+c)ds, 0<t<1\end{aligned}$$

which combining with (10), yields that

$$w^*(t)\ge\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}H(s,w^*(s))ds, 0<t<1$$

Thus, $w^*(t)$ is the upper solution of the initial value problem (1) and $v^*(t)=I^{\alpha}(a)>0$ is the lower solution of the equation (1). By Theorem 2.1, the system (1) has at least one positive solution $u(t)\in C[0,\delta]$, where $0<\delta<1$ and $v^*(t)\le u(t)\le w^*(t)$.

Corollary 2.4: Assume $f(t,u):[0,1]\times[0,+\infty)\to[a,+\infty)$ is a continuous function, where a is a positive constant and there exists two constants d>0, c>0 such that

$$\max\{f(t,l):(t,l)\in[0,1]\times[0,d]\}\le c\Gamma(1+\alpha) \tag{12}$$

then the initial value problem (1) has at least one positive solution $u(t)\in C[0,1]$ satisfying $0<\|u\|\le c$.

Proof: According to the definition of control function, one has

$$a\le H(t,l)\le c\Gamma(1+\alpha),(t,l)\in[0,1]\times[0,d].$$

By corollary 2.1, the initial problem (1) has at least a positive solution $u(t)\in C[0,1]$ satisfying $0<u(t)\le c$. Hence $0<\|u\|\le c$.

UNIQUENESS OF POSITIVE SOLUTION

In this section, we shall prove the uniqueness of the positive solution by using the Banach contraction mapping principle:

Lemma 3.1: If the operator A: $X\to X$ is the contraction mapping, where X is the Banach space, then A has a unique fixed point in X.

Theorem 3.1: If there exists a pair of positive upper and lower solutions of the equation.(1) and for any $u(t),v(t)\in X, 0<t<1$, there exists l>0 such that

$$\|f(t,u)-f(t,v)\|\le l\|u-v\| \tag{13}$$

then the initial problem (1) has a unique positive solution $u(t)\in C[0,1]$ when

$$\frac{1}{\Gamma(s+1)}<1$$

Proof: It follows from Theorem 2.1 that the initial value problem (1) has at least one positive solution in S. Hence it suffices to prove that the operator T defined in (4) is contract in X. In fact, for any $u_1(t),u_2(t)\in X$� by assumption (13), we have

$$\begin{aligned}\|T(u_1)(t)-T(u_2)(t)\| &\le \frac{1}{\Gamma(s)}\int_0^t (t-\tau)^{s-1}\|f(\tau,u_1(\tau))-f(\tau,u_2(\tau))\|d\tau\\ &\le \frac{1}{\Gamma(s)}\|u_1(\tau)-u_2(\tau)\|\int_0^t (t-\tau)^{s-1}d\tau\\ &\le \frac{1}{\Gamma(1+s)}\|u_1(\tau)-u_2(\tau)\|\end{aligned}$$

Thus, when $\frac{1}{\Gamma(s+1)}<1$, the operator T is the contraction mapping. Therefore, the initial value problem (1) has a unique positive solution $u(t)\in C[0,1]$.

Finally, we give an example to illuminate our results.

Example 3.1: We consider the fractional order differential equation

$$D^{\frac{1}{2}}u(t)=1+\frac{u(t)}{u(t)+\sin[u(t)+1]},\ 0\le t\le 1,\ ,u(0)=0 \tag{14}$$

where

$$f(t,u)=1+\frac{u}{u+\sin(u+1)}$$

Due to $\lim_{u\to+\infty} f(t,u)=2$ and $f(t,u)\ge 1, u\in[0,+\infty)$, by corollary 2.2, then the equation (14) has a positive solution. Nevertheless the function $f(t,u)$ is not monotonous and $f(t,u)$ is neither contract nor controlled by two power functions. Hence the conclusions of [5-7] can not be applied to the above example.

CONCLUSIONS

This paper presents the use of upper and lower solutions method for systems of nonlinear fractional differential equations. This method is a powerful tool for solving nonlinear differential equations in mathematical physics, chemistry and engineering etc. The technique constructing a pair of upper and lower control functions with respect to nonlinear term without monotone demand provides a new efficient method to handle the nonlinear structure.

We have dealt with the problem of positive solution for a class of nonlinear fractional differential equation. The general sufficient conditions have been obtained to ensure the existence and uniqueness of the positive solution for the nonlinear fractional differential equation. These criteria generalize and improve some known results [5-7]. In particular, an example is given to show the effectiveness of the obtained results. In addition, the sufficient conditions that we obtained are very simple, which provide flexibility for the application and analysis of nonlinear fractional differential equation.

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