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Shape Preserving Rational Bi-cubic Function for Positive Data

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Abstract: In this paper, an attempt has been made to construct a shape preserving rational bi-cubic interpolant (cubic/quadratic) with twelve free parameters to depict a more pleasant and smooth display of positive surface through positive data. Simple data dependent constraints are derived for four free parameters to preserve the positivity of data while the remaining eight are left free for designer's choice to refine the positive surface as desired. The developed scheme is C^1, simple, local, computationally economical and time saving as compared to existing schemes. Numerical examples are provided to demonstrate that the proposed scheme is interactive and smooth.

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Key words: Shape preserving interpolation . rational bi-cubic function . positive data . free parameters

INTRODUCTION

Positivity is a prevailing shape property of curves and surfaces. Positivity preserving problems occur in visualizing a physical quantity that cannot be negative which may arise if the data is taken from some scientific, social or business environments. There are some physical quantities which are always positive: In the modelling of earth surface, measurement of altitude above sea level at different positions on the surface of earth, carbon dating used to measure the age of mummies and fossils, terrain modelling, formation of geological crust movement to forecast earth quake and volcanic eruptions, the rate of dissemination of drugs in the blood, depreciation of the price of computers, probability distributions and resistance offered by an electric circuit.

The ordinary spline surfaces schemes are failed to preserve the inherent shape features (positivity, monotonicity, convexity) of regular data as shown in Fig. 1, 4 and 8. These schemes merely depend on the data points, so just a change in data points can cause a modification or an alteration in the shape of curves and surfaces. Although cubic and bi-cubic Hermite splines are smooth and visually pleasing but they do not preserve the inherited shape features of data due to unwanted oscillations.

A number of authors and references have contributed to the shape preserving interpolations, for brevity, the reader is referred to [1-13]. Abbas *et al.* [2, 3] solved the problem of positivity preserving surfaces using rational bi-cubic and bi-cubic partially blended rational functions (cubic/cubic) with shape parameters. The authors derived simple data dependent constraints for shape parameters to preserve the positivity of data. Abbas *et al.* [4] developed a rational bi-cubic function (cubic/cubic) with shape parameters. Simple dependent conditions for shape parameters were derived to maintain the shape of constrained surface data that lies above the plane. Asim and Brodlie [5] developed a piecewise cubic Hermite interpolant to preserve the positivity of positive data. The interpolant preserved the shape of data by inserting extra knots where it lost positivity of positive data. Brodlie *et al.* [6] developed a piecewise bi-cubic Hermite function for the positivity of data. Sufficient conditions in the term of the first and mixed partial derivatives were derived at the rectangular grid points to preserve the required shape of surface.

Butt and Brodlie [7] developed a piecewise cubic Hermite interpolant to preserve the shape of curve through positive data by interval subdivision technique. In [7], the authors inserted extra knots in the interval where the function lost the positivity. Goodman [8] surveyed the shape preserving interpolating algorithms for 2D data. Srafraz *et al*. [10] developed a rational cubic function with two free parameters to preserve the shape of curve through positive data. The authors extended the rational cubic function to partially bi-cubic blended rational function. They derived data dependent constraints for shape parameters to preserve the shape of surface through positive surface data. But the proposed scheme did not involve any free parameters thus was not able to provide any freedom to the designer in modification of the shape of curves and surfaces.

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This paper is related to the solution of problem of positivity preserving surface through positive data using rational bi-cubic function. In this scheme, we extend the rational cubic function [1] to rational bi-cubic function with twelve free parameters which guarantee to preserve the shape of surface. Simple data dependent constraints are derived for four parameters and the remaining eight are left free for the user to refine the surfaces. The technique used in this paper has some salient features.

- In [9], the smoothness of interpolant is C^0 while in this work it is C^1.
- This scheme works well for both equally and unequally space data.
- The proposed scheme is equally applicable for the data with derivative or without derivatives while in [13], scheme works only if partial derivatives at the knots are known.
- The proposed surface scheme is unique in its representation.
- The proposed visual model of positive surface is pleasant, smooth and C^1 while in [10], the authors claimed that positive surfaces generated by their scheme is C^1 and smooth but unfortunately the visual models did not depict the smooth and C^1 surfaces.
- The developed scheme has been tested through different numerical examples and it is observed that the scheme is not only local, computationally economical, easy to compute, time saving but also visually pleasant as compared to existing schemes [9-12] due to less number of constraints for free parameters and flexibility bestowed to designer for the refinement of surfaces.
- In [5-7], the authors achieved the desired shape of data by inserting extra knots in the interval where the interpolant lost the positivity while the proposed interpolant preserves the shape of surface through positive data without any extra knots.

REVIEW OF RATIONAL CUBIC SPLINE FUNCTION

Let $\{(x_i, f_i), i = 0,1,2,...,n\}$ be the given set of data points such as $x_0 < x_1 < x_2 < ... < x_n$. The rational cubic function with three free parameters [1], in each subinterval $I_i = [x_i, x_{i+1}], i = 0,1,2,....,n-1$ is defined as:

$$S(x) \equiv S_i(x) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i\xi_i}{q_i(\theta)} \qquad (1)$$

Let $S_i'(x)$ denotes the derivative with respect to $'x'$ and d_i denotes the derivative values at given knots. The following interpolatory conditions are imposed on piecewise rational cubic function (1) for C^1 continuity as:

$$\begin{cases} S_i(x_i) = f_i, & S_i(x_{i+1}) = f_{i+1} \\ S_i'(x_i) = d_i, & S_i'(x_{i+1}) = d_{i+1} \end{cases} \qquad (2)$$

From equation (2), the following unknown values ξ_i, i=0,1,2,3 are

$$\begin{aligned} \xi_0 &= u_i f_i \\ \xi_1 &= (2u_i + w_i + v_i) f_i + u_i h_i d_i \\ \xi_2 &= (u_i + w_i + 2v_i) f_{i+1} - v_i h_i d_{i+1} \\ \xi_3 &= v_i f_{i+1} \end{aligned} \qquad (3)$$

where $\theta = x - x_i / h_i$, $h_i = x_{i+1} - x_i$, u_i, $v_i > 0$ and $w_i \geq 0$ are free parameters.

The C^1 piecewise rational cubic function (1) is reformulated after using equation (3) as:

$$S(x) \equiv S_i(x) = \frac{p_i(\theta)}{q_i(\theta)} \qquad (4)$$

with

$$p_i(\theta) = \begin{cases} u_i f_i (1-\theta)^3 + \left((2u_i + w_i + v_i) f_i + u_i h_i d_i\right)\theta(1-\theta)^2 \\ + \left((u_i + w_i + 2v_i) f_{i+1} - v_i h_i d_{i+1}\right)\theta^2(1-\theta) + v_i f_{i+1}\theta^3 \end{cases}$$

$$q_i(\theta) = u_i(1-\theta)^2 + (u_i + w_i + v_i)\theta(1-\theta) + v_i\theta^2$$

Remark 1: For the values of free parameters $u_i = 1$, $v_i = 1$ and $w_i = 0$ in each subinterval, the C^1 piecewise rational cubic function (4) reduces to standard cubic Hermite spline.

Abbas *et al*. [1] developed the following result as:

Theorem 1: The piecewise rational cubic function (4) preserves the positivity preserving curve through positive data, if in each subinterval $I_i = [x_i \quad x_{i+1}], i = 0,1,2,...,n$, the free parameter u_i, v_i and w_i satisfy the following sufficient conditions,

$$\begin{cases} u_i > 0, v_i > 0 \\ w_i > \max\left\{0, \dfrac{-u_i(2f_i + h_i d_i)}{f_i} - v_i, \dfrac{v_i(h_i d_{i+1} - 2f_{i+1})}{f_{i+1}} - u_i\right\} \end{cases}$$

Table 1: Positive wind velocity data set

i	1	2	3	4	5	6	7	8	9
x_i	0	0.25	0.5	1.0	1.5	2.0	2.5	3.0	4.0
f_i	2.0	0.6	0.10	0.13	1.0	0.5	1.1	0.25	0.2

The above result can be rearranged as:

$$\begin{cases} u_i > 0, v_i > 0 \\ w_i = l_i + \max\left\{0, \dfrac{-u_i(2f_i + h_i d_i)}{f_i} - v_i, \dfrac{v_i(h_i d_{i+1} - 2f_{i+1})}{f_{i+1}} - u_i\right\}, l_i > 0 \end{cases}$$

Note: The d_i ,i=0,1,2,...n are derivative values (Tangents) which are calculated at data points by using arithmetic mean method [11] so the smoothness of positive curve in Fig. 2 and 3 is C^1. On the other hand, if these derivative values are calculated by solving the tri-diagonal system of linear equations [1] then the smoothness of interpolant is C^2.

DETERMINATION OF DERIVATIVES

In this paper, the partial derivatives $F^x_{i,j}$ and $F^y_{i,j}$ are calculated at given data points by using arithmetic mean method which was proposed in [11]. It is the three-point difference approximation method based on arithmetic calculation for the positive surface manipulation. It is considered to be computationally economical and suitable for shape preserving schemes. This method can be oriented and extended for the 3D data visualization as follows:

Arithmetic mean method for 3D data [11]

$$F^x_{0,j} = \Delta_{0,j} + \frac{(\Delta_{0,j} - \Delta_{1,j})h_0}{(h_0 + h_1)}, F^x_{n,j} = \Delta_{n-1,j} + \frac{(\Delta_{n-1,j} - \Delta_{n-2,j})h_{n-1}}{(h_{n-1} + h_{n-2})}$$

$$F^x_{i,j} = 0.5(\Delta_{i,j} + \Delta_{i-1,j}) \quad ,i = 1,2,3,\ldots,n-1; \; j = 0,1,2,\ldots,m$$

$$F^y_{i,0} = \hat{\Delta}_{i,0} + \frac{(\hat{\Delta}_{i,0} - \hat{\Delta}_{i,1})\hat{h}_0}{(\hat{h}_0 + \hat{h}_1)}, F^y_{i,m} = \hat{\Delta}_{i,m-1} + \frac{(\hat{\Delta}_{i,m-1} - \hat{\Delta}_{i,m-2})\hat{h}_{m-1}}{(\hat{h}_{m-1} + \hat{h}_{m-2})}$$

$$F^y_{i,j} = 0.5(\hat{\Delta}_{i,j} + \hat{\Delta}_{i,j-1}), i = 0,1,2,\ldots,n; j = 1,2,3,\ldots,m-1$$

where

$$\Delta_{i,j} = \frac{F_{i+1,j} - F_{i,j}}{h_i}, \hat{\Delta}_{i,j} = \frac{F_{i,j+1} - F_{i,j}}{\hat{h}_j}$$

Example 1: A positive data set [1] in Table 1 shows the velocity of wind which is noted at different time interval. One can observe that the velocity of wind inherently positive. Figure 1 is drawn by cubic Hermite spline that does not preserve the positivity of positive data which is the drawback of this spline. Figure 2 and 3 are generated by developed rational cubic function with three shape parameter [1] to preserve the positivity of wind velocity data.

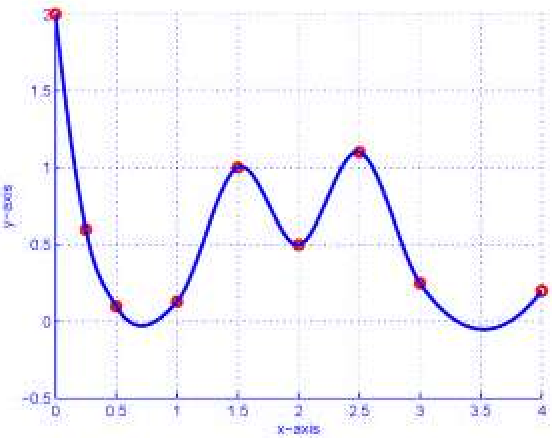


Fig. 1: Cubic Hermite curve

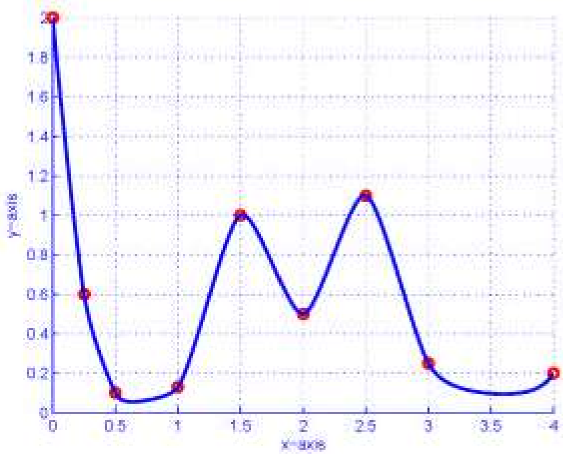


Fig. 2: Positive rational cubic function with $u_i = 0.1$ and $v_i = 0.1$

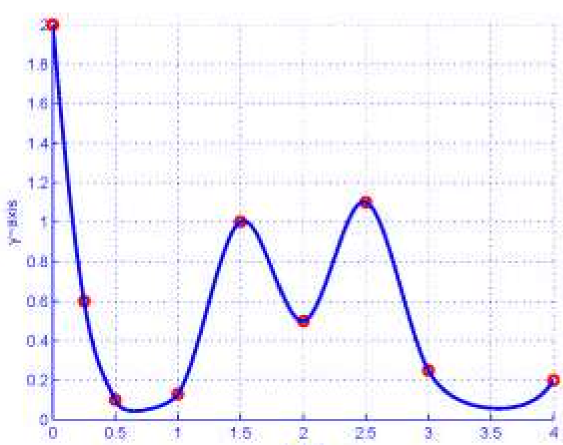


Fig. 3: Positive rational cubic function with $u_i = 0.5$ and $v_i = 0.5$

RATIONAL BI-CUBIC FUNCTION

The C^1 piecewise rational cubic function (4) is extended to rational bi-cubic function S(x, y) over the rectangular Domain Ω = [a,b]×[c,d] for the visualization of 3D regular data. The partition of arbitrary intervals [a,b] and [c,d] is defined as: $\pi: a = x_0 < x_1 < x_2 < \ldots < x_n = b$, $\hat{\pi}: c = y_0 < y_1 < y_2 < \ldots < y_m = d$ respectively. The partially blended rational bi-cubic function is defined over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0,1,2,\ldots,n-1; j = 0,1,2,\ldots,m-1$ as:

$$S(x,y) = \begin{bmatrix} a_0(\theta) & a_1(\theta) \end{bmatrix} \begin{bmatrix} S(x_i,y) \\ S(x_{i+1},y) \end{bmatrix} + \begin{bmatrix} b_0(\varphi) & b_1(\varphi) \end{bmatrix} \begin{bmatrix} S(x,y_j) \\ S(x,y_{j+1}) \end{bmatrix} - \begin{bmatrix} a_0(\theta) & a_1(\theta) \end{bmatrix} \begin{bmatrix} S(x_i,y_j) & S(x_i,y_{j+1}) \\ S(x_{i+1},y_j) & S(x_{i+1},y_{j+1}) \end{bmatrix} \begin{bmatrix} b_0(\varphi) \\ b_1(\varphi) \end{bmatrix} \tag{5}$$

with

$$a_0(\theta) = (1-\theta)^2(1+2\theta), a_1(\theta) = \theta^2(3-2\theta)$$

$$b_0(\varphi) = (1-\varphi)^2(1+2\varphi), b_1(\varphi) = \varphi^2(3-2\varphi)$$

$$\theta = (x - x_i)/h_i, \varphi = (y - y_j)/\hat{h}_j$$

and

$$h_i = x_{i+1} - x_i, \hat{h}_j = y_{j+1} - y_j$$

$S(x,y_j), S(x,y_{j+1}), S(x_i,y)$ and $S(x_{i+1},y)$ are the rational cubic functions defined on boundary of each rectangular patch $[x_i \quad x_{i+1}] \times [y_j \quad y_{j+1}]$ as:

$$S(x,y_j) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i L_i}{q_1(\theta)} \tag{6}$$

with

$$L_0 = u_{i,j}F_{i,j}$$
$$L_1 = (2u_{i,j} + w_{i,j} + v_{i,j})F_{i,j} + u_{i,j}h_iF^x_{i,j}$$
$$L_2 = (u_{i,j} + w_{i,j} + 2v_{i,j})F_{i+1,j} - v_{i,j}h_iF^x_{i+1,j}$$
$$L_3 = v_{i,j}F_{i+1,j}$$
$$q_1(\theta) = u_{i,j}(1-\theta)^3 + (u_{i,j} + w_{i,j} + v_{i,j})\theta(1-\theta) + v_{i,j}\theta^3$$

$$S(x,y_{j+1}) = \frac{\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i M_i}{q_2(\theta)} \tag{7}$$

with

$$M_0 = u_{i,j+1}F_{i,j+1}$$
$$M_1 = (2u_{i,j+1} + w_{i,j+1} + v_{i,j+1})F_{i,j+1} + u_{i,j+1}h_iF^x_{i,j+1}$$
$$M_2 = (u_{i,j+1} + w_{i,j+1} + 2v_{i,j+1})F_{i+1,j+1} - v_{i,j+1}h_iF^x_{i+1,j+1}$$
$$M_3 = v_{i,j+1}F_{i+1,j+1}$$
$$q_2(\theta) = u_{i,j+1}(1-\theta)^3 + (u_{i,j+1} + w_{i,j+1} + v_{i,j+1})\theta(1-\theta) + v_{i,j+1}\theta^3$$

$$S(x_i,y) = \frac{\sum_{i=0}^{3}(1-\varphi)^{3-i}\varphi^i N_i}{q_3(\varphi)} \tag{8}$$

with

$$N_0 = \hat{u}_{i,j}F_{i,j}$$
$$N_1 = (2\hat{u}_{i,j} + \hat{w}_{i,j} + \hat{v}_{i,j})F_{i,j} + \hat{u}_{i,j}\hat{h}_jF^y_{i,j}$$
$$N_2 = (\hat{u}_{i,j} + \hat{w}_{i,j} + 2\hat{v}_{i,j})F_{i,j+1} - \hat{v}_{i,j}\hat{h}_jF^y_{i,j+1}$$
$$N_3 = \hat{v}_{i,j}F_{i,j+1}$$
$$q_3(\theta) = \hat{u}_{i,j}(1-\varphi)^3 + (\hat{u}_{i,j} + \hat{w}_{i,j} + \hat{v}_{i,j})\varphi(1-\varphi) + \hat{v}_{i,j}\varphi^3$$

$$S(x_{i+1},y) = \frac{\sum_{i=0}^{3}(1-\varphi)^{3-i}\varphi^i O_i}{q_4(\varphi)} \tag{9}$$

with

$$O_0 = \hat{u}_{i+1,j}F_{i+1,j}$$
$$O_1 = (2\hat{u}_{i+1,j} + \hat{w}_{i+1,j} + \hat{v}_{i+1,j})F_{i+1,j} + \hat{u}_{i+1,j}\hat{h}_jF^y_{i+1,j}$$
$$O_2 = (\hat{u}_{i+1,j} + \hat{w}_{i+1,j} + 2\hat{v}_{i+1,j})F_{i+1,j+1} - \hat{v}_{i+1,j}\hat{h}_jF^y_{i+1,j+1}$$
$$O_3 = \hat{v}_{i+1,j}F_{i+1,j+1}$$
$$q_4(\theta) = \hat{u}_{i+1,j}(1-\varphi)^3 + (\hat{u}_{i+1,j} + \hat{w}_{i+1,j} + \hat{v}_{i+1,j})\varphi(1-\varphi) + \hat{v}_{i+1,j}\varphi^3$$

SHAPE PRESERVING RATIONAL BI-CUBIC POSITIVE INTERPOLATION

In this section, we deal with the problem of positivity preserving rational bi-cubic function (5) that does not preserve the shape of surface through positive data. So, it is required to assign suitable constraints on free parameters by some mathematical treatment which guarantees to preserve inherited shape feature (positivity) of data.

Let

$$\{(x_i, y_j, F_{i,j}), i = 0,1,2,\ldots,n-1; j = 0,1,2,\ldots,m-1\}$$

be the positive surface data arranged over each rectangular patch

$$[x_i, x_{i+1}] \times [y_j, y_{j+1}], i = 0,1,2,\ldots,n-1; j = 0,1,2,\ldots,m-1$$

Since the data is positive so

$$F_{i,j}>0,\ \forall i, j$$

The necessary conditions for free parameters for positivity of positive data are:

$$\begin{cases} u_{i,j}>0, v_{i,j}>0, u_{i,j+1}>0, v_{i,j+1}>0 \\ \hat{u}_{i,j}>0, \hat{v}_{i,j}>0, \hat{u}_{i+1,j}>0, \hat{v}_{i+1,j}>0 \end{cases} \tag{10}$$

The surface patch (5) preserves the positivity of positive data if the four boundary curves $S(x, y_j)$, $S(x, y_{j+1})$, $S(x_i, y)$ and $S(x_{i+1}, y)$ are defined in Equation (6)-(9) are positive.

$$S(x, y_j)>0 \text{ if } \sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i L_i>0 \text{ and } q_1(\theta)>0.$$

We have $q_1(\theta)>0$ if equation (10) is satisfied and $\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i L_i>0$ if $L_i>0$, $i = 0,1,2,3$.

$$L_i>0,\ \forall i \text{ if}$$

$$w_{i,j} > \max\left\{ \begin{array}{l} 0, \dfrac{-u_{i,j}(2F_{i,j}+h_iF^x_{i,j})}{F_{i,j}} - v_{i,j}, \\ \dfrac{v_{i,j}(h_iF^x_{i+1,j}-2F_{i+1,j})}{F_{i+1,j}} - u_{i,j} \end{array} \right\} \tag{11}$$

Likewise, $S(x, y_{j+1})>0$ if $\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i M_i>0$ and $q_2(\theta)>0$.

We have $q_2(\theta)>0$ if equation (10) is satisfied and

$$\sum_{i=0}^{3}(1-\theta)^{3-i}\theta^i M_i>0 \text{ if } M_i>0,\ i = 0,1,2,3$$

$$M_i>0,\ i = 0,1,2,3 \text{ if}$$

$$w_{i,j+1} > \max\left\{ \begin{array}{l} 0, \dfrac{-u_{i,j+1}(2F_{i,j+1}+h_iF^x_{i,j+1})}{F_{i,j+1}} - v_{i,j+1}, \\ \dfrac{v_{i,j+1}(h_iF^x_{i+1,j+1}-2F_{i+1,j+1})}{F_{i+1,j+1}} - u_{i,j+1} \end{array} \right\} \tag{12}$$

$S(x_i, y)>0$ if $\sum_{i=0}^{3}(1-\varphi)^{3-i}\varphi^i N_i>0$ and $q_3(\theta)>0$.

We have $q_3(\varphi)>0$ if equation (10) is satisfied and $\sum_{i=0}^{3}(1-\varphi)^{3-i}\varphi^i N_i>0$ if $N_i>0$, $i = 0,1,2,3$.

$$N_i>0,\ i = 0,1,2,3 \text{ if}$$

$$\hat{w}_{i,j} > \max\left\{ \begin{array}{l} 0, \dfrac{-\hat{u}_{i,j}(2F_{i,j}+\hat{h}_jF^y_{i,j})}{F_{i,j}} - \hat{v}_{i,j}, \\ \dfrac{\hat{v}_{i,j}(\hat{h}_jF^y_{i,j+1}-2F_{i,j+1})}{F_{i,j+1}} - \hat{u}_{i,j} \end{array} \right\} \tag{13}$$

Lastly, $S(x_{i+1}, y)>0$ if $\sum_{i=0}^{3}(1-\varphi)^{3-i}\varphi^i O_i>0$ and $q_4(\varphi)>0$.

We have $q_4(\varphi)>0$ if equation (10) is satisfied and

$$\sum_{i=0}^{3}(1-\varphi)^{3-i}\varphi^i O_i>0 \text{ if } O_i>0,\ i = 0,1,2,3.$$

$$O_i>0,\ \forall i \text{ if}$$

$$\hat{w}_{i+1,j} > \max\left\{ \begin{array}{l} 0, \dfrac{-\hat{u}_{i+1,j}(2F_{i+1,j}+\hat{h}_jF^y_{i+1,j})}{F_{i+1,j}} - \hat{v}_{i+1,j}, \\ \dfrac{\hat{v}_{i+1,j}(\hat{h}_jF^y_{i+1,j+1}-2F_{i+1,j+1})}{F_{i+1,j+1}} - \hat{u}_{i+1,j} \end{array} \right\} \tag{14}$$

Theorem.2: The rational bi-cubic function (5) preserves the positivity preserving surface through positive surface data, if in each rectangular patch $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$, $i= 0,1,2,...,n-1; j= 0,1,2,...,m-1$, the free parameters are satisfying the following sufficient conditions as:

$$\begin{cases} u_{i,j}>0, u_{i,j+1}>0, \hat{u}_{i,j}>0, \hat{u}_{i+1,j}>0 \\ v_{i,j}>0, v_{i,j+1}>0, \hat{v}_{i,j}>0, \hat{v}_{i+1,j}>0 \end{cases}$$

$$w_{i,j} > \max\left\{0, \frac{-u_{i,j}(2F_{i,j}+h_iF^x_{i,j})}{F_{i,j}} - v_{i,j}, \frac{v_{i,j}(h_iF^x_{i+1,j}-2F_{i+1,j})}{F_{i+1,j}} - u_{i,j}\right\}$$

$$w_{i,j+1} > \max\left\{0, \frac{-u_{i,j+1}(2F_{i,j+1}+h_iF^x_{i,j+1})}{F_{i,j+1}} - v_{i,j+1}, \frac{v_{i,j+1}(h_iF^x_{i+1,j+1}-2F_{i+1,j+1})}{F_{i+1,j+1}} - u_{i,j+1}\right\}$$

$$\hat{w}_{i,j} > \max\left\{0, \frac{-\hat{u}_{i,j}(2F_{i,j}+\hat{h}_j F^y_{i,j})}{F_{i,j}} - \hat{v}_{i,j}, \frac{\hat{v}_{i,j}(\hat{h}_j F^y_{i,j+1} - 2F_{i,j+1})}{F_{i,j+1}} - \hat{u}_{i,j}\right\}$$

$$\hat{w}_{i+1,j} > \max\left\{0, \frac{-\hat{u}_{i+1,j}(2F_{i+1,j}+\hat{h}_j F^y_{i+1,j})}{F_{i+1,j}} - \hat{v}_{i+1,j}, \frac{\hat{v}_{i+1,j}(\hat{h}_j F^y_{i+1,j+1} - 2F_{i+1,j+1})}{F_{i+1,j+1}} - \hat{u}_{i+1,j}\right\}$$

The above constraints can be rearranged as:

$$w_{i,j} = p_{i,j} + \max\left\{0, \frac{-u_{i,j}(2F_{i,j}+h_i F^x_{i,j})}{F_{i,j}} - v_{i,j}, \frac{v_{i,j}(h_i F^x_{i+1,j} - 2F_{i+1,j})}{F_{i+1,j}} - u_{i,j}\right\}, p_{i,j} > 0$$

$$w_{i,j+1} = r_{i,j} + \max\left\{0, \frac{-u_{i,j+1}(2F_{i,j+1}+h_i F^x_{i,j+1})}{F_{i,j+1}} - v_{i,j+1}, \frac{v_{i,j+1}(h_i F^x_{i+1,j+1} - 2F_{i+1,j+1})}{F_{i+1,j+1}} - u_{i,j+1}\right\}, r_{i,j} > 0$$

$$\hat{w}_{i,j} = s_{i,j} + \max\left\{0, \frac{-\hat{u}_{i,j}(2F_{i,j}+\hat{h}_j F^y_{i,j})}{F_{i,j}} - \hat{v}_{i,j}, \frac{\hat{v}_{i,j}(\hat{h}_j F^y_{i,j+1} - 2F_{i,j+1})}{F_{i,j+1}} - \hat{u}_{i,j}\right\}, s_{i,j} > 0$$

$$\hat{w}_{i+1,j} = t_{i,j} + \max\left\{0, \frac{-\hat{u}_{i+1,j}(2F_{i+1,j}+\hat{h}_j F^y_{i+1,j})}{F_{i+1,j}} - \hat{v}_{i+1,j}, \frac{\hat{v}_{i+1,j}(\hat{h}_j F^y_{i+1,j+1} - 2F_{i+1,j+1})}{F_{i+1,j+1}} - \hat{u}_{i+1,j}\right\}, t_{i,j} > 0.$$

Proof: The result follows immediately from the above equations (10)-(14).

NUMERICAL EXAMPLES

In this section, a numerical demonstration of positivity preserving rational bi-cubic scheme given in previous section is presented.

Example 2: A positive data set up to 4 decimal places taken in Table 2 is generated by following function,

$$F_1(x,y) = e^{-(x^2+y^2)/15}(\sin x + \cos y) + 0.33 \quad 0 \le x, y \le 6$$

Figure 4 (with xz-view in Fig. 5(a) and yz-view in Fig. 5 (b)) is produced by using bi-cubic Hermite spline through positive data that does not preserve the positivity. To overcome this flaw, Fig. 6 (with xz-view in Fig. 7 (a) and yz-view in Fig. 7 (b)) is drawn by using positivity preserving rational bi-cubic function with the values of free parameters as: $u_{i,j} = 0.5, \hat{u}_{i,j} = 0.5$, $v_{i,j} = 0.5$ and $\hat{v}_{i,j} = 0.5$ to preserve the positivity through same positive data.

Example 3: A positive data set up to 4 decimal places taken in Table 3 is produced by following function,

$$F_2(x,y) = \sin y e^{-x} + 1.0 \qquad -3 \le x, y \le 3$$

Figure 8 (with xz-view in Fig. 9 (a) and yz-view in Fig. 9 (b)) is drawn using bi-cubic Hermite spline through positive data. It shows that the scheme does not preserve the positivity. To remove this flaw, Fig. 10

Table 2: Positive surface data

y/x	0	2	4	6
0	1.33000	0.011261	0.10505	0.41710
2	1.79240	0.619300	0.39739	0.45990
4	0.41370	0.020814	0.16294	0.33635
6	0.39537	0.281670	0.30087	0.33560

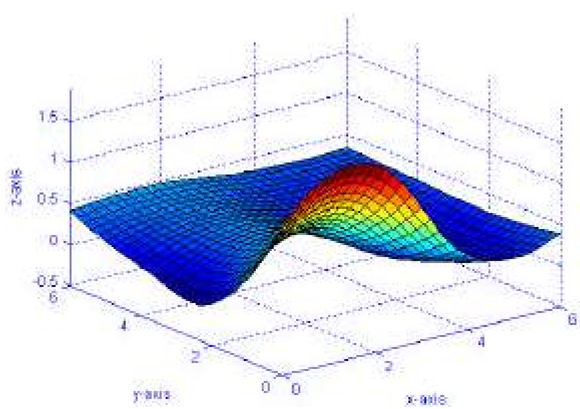
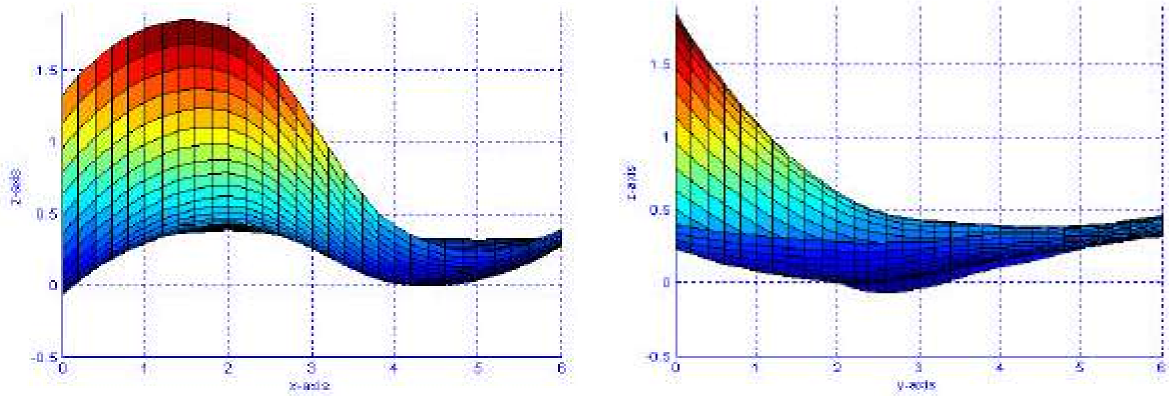


Fig. 4: Bi-cubic Hermite surface



(a) (b)

Fig. 5: Bi-cubic Hermite surface a) xz-view of Fig. 4; b) yz-view of Fig. 4

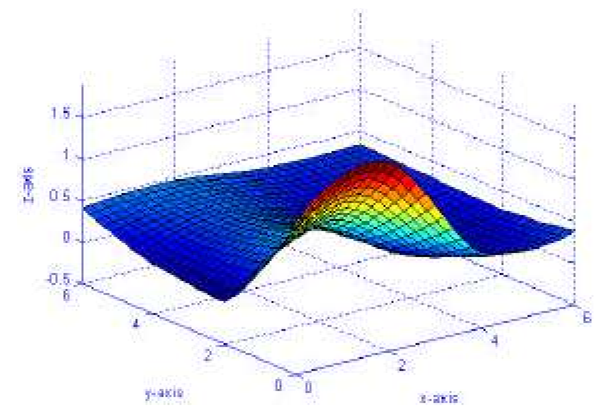
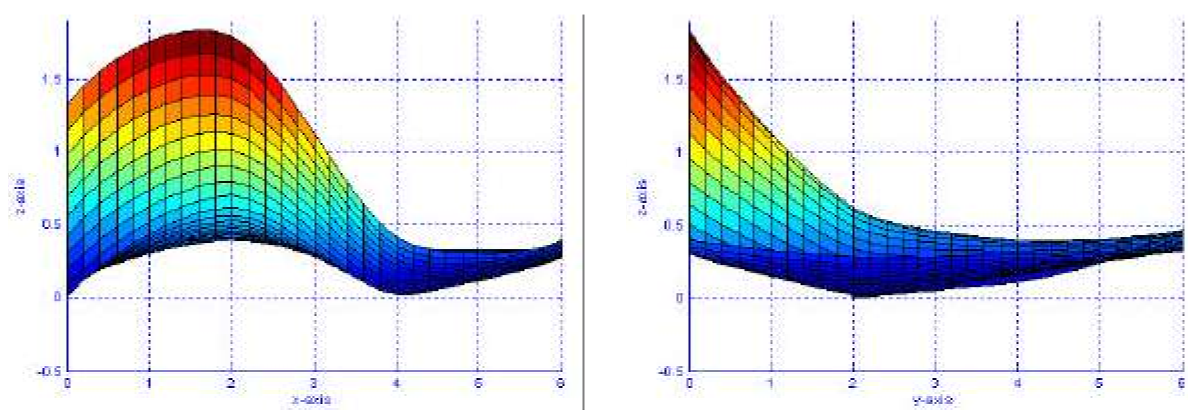


Fig. 6: Positivity preserving surface using developed rational bi-cubic function



(a) (b)

Fig. 7: Positivity preserving surface a) xz-view of Fig. 6; b) yz-view of Fig. 6

Table 3: Positive surface data

y/x	-3	-2	-1	1	2	3
-3	1.536600	0.37930	0.055529	1.9445	1.62070	0.46344
-2	1.175100	0.19847	0.106150	1.8939	1.80150	0.82489
-1	0.044919	1.74900	0.589220	1.4108	0.25095	1.95510
1	0.107150	0.32885	0.640360	1.3596	1.67110	1.89290
2	0.605060	0.73262	0.865080	1.1349	1.26740	1.39490
3	0.851190	0.90059	0.950230	1.0498	1.09940	1.14880

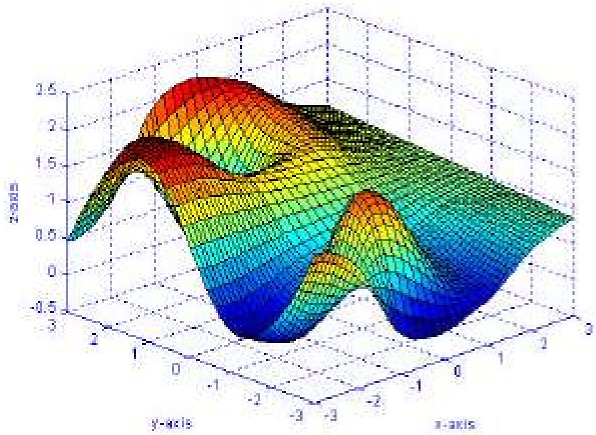
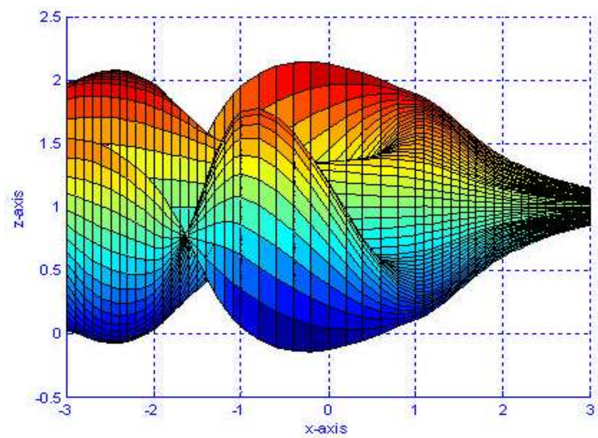
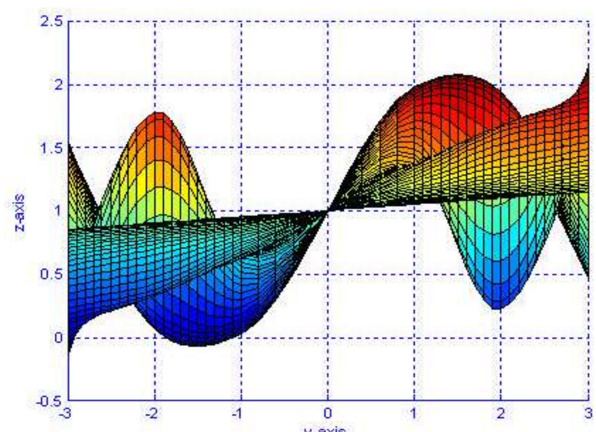


Fig. 8: Bi-cubic Hermite surface



(a)



(b)

Fig. 9: Bi-cubic Hermite surface a) xz-view of Fig. 8; b) yz-view of Fig. 8

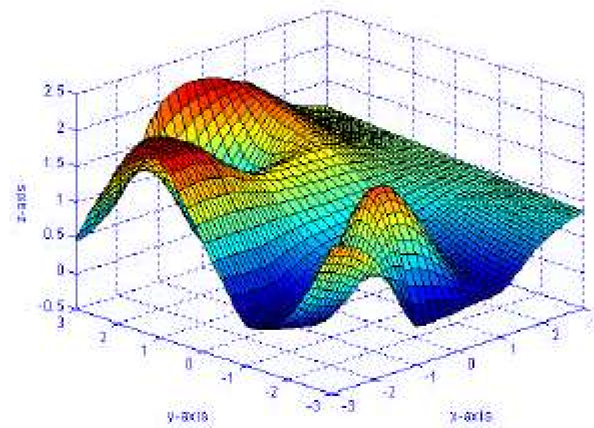
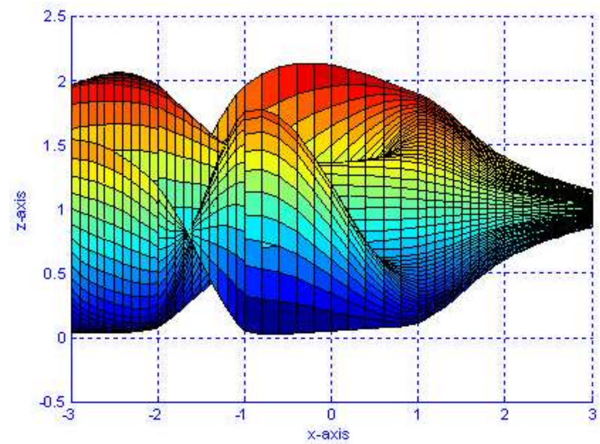
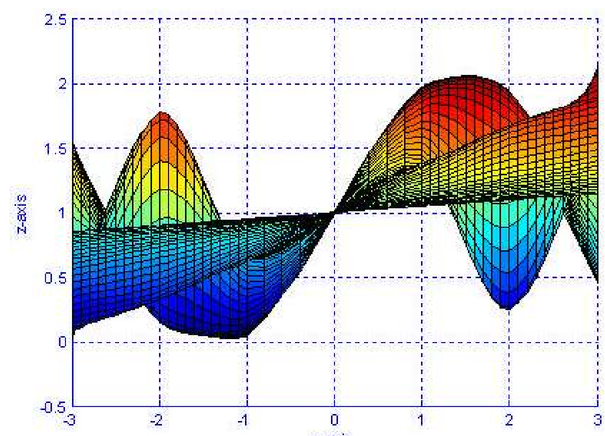


Fig. 10: Positivity preserving surface by proposed rational bi-cubic function



(a)



(b)

Fig. 11: a) Positivity preserving surface a) xz-view of Fig. 10; b) yz-view of Fig. 10

(with xz-view in Fig. 11 (a) and yz-view in Fig. 11 (b)) is generated by positive rational bi-cubic interpolant with the values of free parameters

$$u_{i,j} = 0.5, \hat{u}_{i,j} = 0.5, v_{i,j} = 0.5 \text{ and } \hat{v}_{i,j} = 0.5$$

to preserve the positivity preserving surface through same positive surface data.

CONCLUSION

In this paper, we have extended the rational cubic function [1] to rational bi-cubic function with twelve free parameters in each rectangular patch to preserve the shape of surface through positive surface data. Four of these free parameters are constrained parameters to preserve desired the shape of surface while the remaining are left free for the user to modify the surface as desired. No extra knots are inserted in the interval when the interpolant loses the positivity. The developed surface scheme has been demonstrated through different numerical examples and it is concluded that the scheme is not only local and computationally economical but also visually pleasant. The proposed scheme is equally applicable for the data with derivative or without derivatives. It also works well for both equally and unequally space data.

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