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Complex Torsions and Holomorphic Helices

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Abstract: Recently, properties of holomorphic helix of Kähler Frenet curves on n-dimensional M Kähler manifold studied by S. Maeda, H. Tanabe and T. Adachi. In this paper we give some characterizations for complex torsions by $\tau_{i,j}$ in the Kähler manifold to be general helix. And by considering κ_1, κ_2 curvatures of order 3. Curvatures of Frenet curve on M Kähler manifold are not constant but their ratios are constant. We investigate relationship between $\tau_{1,2}$ and $\tau_{2,3}$ complex torsions which are not seperately constant but their ratios are constant.

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INTRODUCTION

Let M be a n-dimensional Kähler manifold, with complex structure J and Riemannian metric g. For a helix γ on M of order $d(\leq 2n)$ with the associated Frenet frame $\{V_1,...,V_d\}$ and we define $\tau_{i,j}$ called complex torsions by $\tau_{i,j}=g(V_i(s),JV_j(s))$ for $1\leq i<j\leq d$, γ is a holomorphic helix if all the complex torsions are constant [1]. They are used curvatures κ_i and complex torsions $\tau_{i,j}$ which are constant. A classical result stated by M. A. Lancert in 1802 and first proved by B. De Saint Venant in 1845 is a necessary and sufficient condition that a curve be a general helix is the ratio of curvature of torsion to be constant [2, 3]. In a Kähler manifold, a Frenet curve is called a general helix if $\frac{\tau_{1,2}}{\tau_{2,3}}$ is constant and its first and second curvatures are not constant.

If its first and second curvatures are constant and its third curvature is zero then the Frenet curve is called a helix. We obtained the relations between the complex torsions and their own derivations.

PRELIMINARIES

Complex torsions: A smooth curve $\gamma = \gamma(s)$ parametrized by its arc-length s is called a helix of proper order d if there exist an orthonormal system $\{V_1=\dot{\gamma}, V_2,\ldots,V_d\}$ of vector fields along γ and positive constants $\kappa_1(s),\kappa_2(s),\ldots,\kappa_{d-1}(s)$ which satisfy the system of ordinary differential equations

$$D_{\dot{\gamma}}V_j(s)=-\kappa_{j-1}(s)V_{j-1}(s)+\kappa_j(s)V_{j+1}(s), \qquad j=1,2,\ldots,d$$

where $V_0\equiv V_{d+1}\equiv 0$ and $\kappa_0=\kappa_d=0$ [4].

Let M be a complex n-dimensional Kähler manifold (K-manifold) with complex structure J. $\{V_1,\ldots,V_d,JV_1,\ldots,JV_d\}$ system is a basis of tangent space of M. A smooth curve $\gamma = \gamma(s)$ on M parametrized by its arc-length s is called a Kähler Frenet curve, if it satisfies the following diferential equation

$$D_{\dot{\gamma}}\dot{\gamma}=\kappa(s)J\dot{\gamma} \text{ or } D_{\dot{\gamma}}\dot{\gamma}=-\kappa(s)J\dot{\gamma}$$

for some positive C^∞ function $\kappa=\kappa(s)$, where $D_{\dot{\gamma}}$ denotes the covariant differentiation along γ with respect to the Riemannian connection D of M [5].

For a Frenet curve γ in a K-manifold M of order d with associated Frenet frame $\{V_1,\ldots,V_d,JV_1,\ldots,JV_d\}$, we define functions $\tau_{i,j}$ called complex torsions by [5].

$$\tau_{i,j}(s)=\begin{cases}0 & ,i=j,\ i=0,\ j>d\\ \langle V_i(s),JV_j(s)\rangle & ,1\leq i<j\leq d\end{cases} \quad ,\|\tau_{i,j}(s)\|\leq 1$$

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Definition 1: For a curve γ on a K-manifold M of order d we call a holomophic helix (H-helix) if all its complex torsions are constant functions.

Let a curve γ on a K-manifold M of order d. In this stuation for [6].

$$D_{\dot\gamma}V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j=1,2,\ldots,d$$

and

$$\tau_{i,j}(s) = \langle V_i(s), JV_j(s)\rangle$$

$$D_{\dot\gamma}\tau_{i,j}(s) = -\kappa_{i-1}\tau_{i-1,j}(s) + \kappa_i\tau_{i+1,j}(s) - \kappa_{j-1}\tau_{i,j-1}(s) + \kappa_j\tau_{i,j+1}(s) \quad (2.1)$$

For complex torsions of helix on K-manifold of order 3 from d = 3, 1≤i<j≤3, i = j = 0, i = 1,2, j = 1,2,3 and from (2.1) we obtain

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3}, \quad D_{\dot\gamma}\tau_{1,3} = -\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}, \quad D_{\dot\gamma}\tau_{2,3} = -\kappa_1\tau_{1,3}$$

or

$$\begin{bmatrix} D_{\dot\gamma}\tau_{1,2} \\ D_{\dot\gamma}\tau_{1,3} \\ D_{\dot\gamma}\tau_{2,3} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 \\ -\kappa_2 & 0 & \kappa_1 \\ 0 & -\kappa_1 & 0 \end{bmatrix} \begin{bmatrix} \tau_{1,2} \\ \tau_{1,3} \\ \tau_{2,3} \end{bmatrix}$$

When γ a Frenet curve on K-manifold M of order 2 and $\tau_{1,2}$ is constant. Really for $\tau_{1,2} = \langle V_1, JV_2\rangle$

$$\begin{aligned} D_{\dot\gamma}\langle V_1, JV_2\rangle &= \langle D_{\dot\gamma}V_1, JV_2\rangle + \langle V_1, JD_{\dot\gamma}V_2\rangle \\ &= \kappa\langle V_2, JV_2\rangle - \kappa\langle V_1, JV_1\rangle = 0 \end{aligned}$$

Then a Frenet curve of order 2 is a H-helix.

HOLOMORPHIC HELICES

If we give theorems and results which they known related to holomorphic helices of order 3 and 4.

Theorem 1: The complex torsions of a H-helix of proper order on a K-manifold satisfy

$$\sum_{j=1}^{i-1}\tau_{j,i}^2 + \sum_{j=i+1}^{d}\tau_{i,j}^2 \le 1$$

For every i [1].

We take H-helices of order 3 and choose orthonormal vectors $\{V_1, V_2, V_3\}$ which satisfy

$$\begin{array}{ccccccc} \langle V_1, V_1\rangle & = & \langle V_2, V_2\rangle & = & \langle V_3, V_3\rangle & = & 1 \\ \langle V_1, V_2\rangle & = & \langle V_1, V_3\rangle & = & \langle V_2, V_3\rangle & = & 0 \\ \langle JV_1, JV_1\rangle & = & \langle JV_2, JV_2\rangle & = & \langle JV_3, JV_3\rangle & = & 1 \\ \langle JV_1, JV_2\rangle & = & \langle JV_1, JV_3\rangle & = & \langle JV_2, JV_3\rangle & = & 0 \end{array}$$

And then we set V_1, V_2 and V_3 as:

$$\begin{aligned} V_1 &= (1,0,\ldots,0) \\ V_2 &= (-i\tau, \sqrt{1-\tau^2}, 0,\ldots 0) \\ V_3 &= \left(0, \frac{-i\rho}{\sqrt{1-\tau^2}}, \frac{\sqrt{1-\tau^2-\rho^2}}{\sqrt{1-\tau^2}}, 0,\ldots,0\right) \end{aligned}$$

For positive constants $\tau = \tau_{1,2}$ and $\rho = \tau_{2,3}$ with $|\tau|\le 1$, $\tau^2+\rho^2 \le 1$ then we obtain orthonormal vectors and satisfy

$$\langle V_1, JV_2\rangle = \tau, \langle V_2, JV_3\rangle = \rho, \quad \langle V_1, JV_3\rangle = 0$$

Corollary 1: The complex torsions $\tau_{i,j}$ of a H-helix γ, $\tau_{i,j} = 0$ when i+j is even [7].

Theorem 2: The complex torsions of a holomorphic helix of odd and even proper order d on a Kähler manifold satisfy the following relations [1].

$$\begin{array}{lll} \tau_{i,j+2k} = 0 & i=1,2,\ldots,d-2k, & k=1,2,\ldots,(d-1)/2(d\text{odd}) \\ & & k=1,2,\ldots,(d-2)/2(d\text{even}) \\ \kappa_1\tau_{2,d} = \kappa_{d-1}\tau_{1,d-1} & & \\ \kappa_1\tau_{2,j} + \kappa_j\tau_{1,j+1} = \kappa_{j-1}\tau_{1,j-1} & j=3,5,\ldots,d-2(d\text{odd}), & j=j=3,5,\ldots,d-1(d\text{even}) \\ \kappa_{i-1}\tau_{i-1,d} + \kappa_{d-1}\tau_{i,d-1} = \kappa_i\tau_{i+1,d} & i=3,5,\ldots,d-2(d\text{odd}), & i=2,4,\ldots,d-2(d\text{even}) \\ \kappa_{i-1}\tau_{i-1,j} + \kappa_{j-1}\tau_{i,j-1} = \kappa_j\tau_{i,j+1} + \kappa_i\tau_{i+1,j} & i=2,3,\ldots,d-3 & j=i+2,i+4,\ldots,d-1 \end{array}$$

Holomorphic helices of order 3: Theorem 3 For $\{V_1,V_2,V_3\}$ orthonormal frame and κ_1, κ_2 positive constant on a K-manifold M. There is a H-helix γ with curvatures κ_1, κ_2 if and only if [1].

$$\begin{cases} \kappa_1\tau_{3,2} + \kappa_2\tau_{1,2} = 0 \\ \tau_{1,3} = 0 \end{cases}, \quad \tau_{1,2} \le \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

for n≥3 and

Theorem 4: K-manifold M of order 2 and all complex torsions of H-helix of order 3 with curvatures κ_1 and κ_2 satisfy [1].

$$\tau_{1,2} = \frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \quad \tau_{1,3} = 0, \quad \tau_{2,3} = \frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

or

$$\tau_{1,2} = -\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}}, \quad \tau_{1,3} = 0, \quad \tau_{2,3} = -\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}}$$

A classical result stated by M. A. Lancert in 1802 and first proved by B. De Saint Venant in 1845 is a necessary and sufficient condition that a curve be a general helix is the ratio of curvature of torsion to be constant [2, 3]. Adhering to this definition we will give the following definition.

Definition 2: For Frenet curve γ on a K-manifold of order 3, if the ratio of $\frac{\tau_{1,2}}{\tau_{2,3}}$ is constant, then γ is called a holomorphic helix.

Theorem 5: If γ is a general helices of order 3 on K-manifold, then $\frac{\kappa_1}{\kappa_2}$ is constant.

Proof: $\tau_{i,j} = -\tau_{j,i}$, $\tau_{i,j} = 0$ (i+j even), $-\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3} = 0$. then $\frac{\tau_{1,2}}{\tau_{2,3}} = \frac{\kappa_1}{\kappa_2}$ from hypothesis $\frac{\tau_{1,2}}{\tau_{2,3}}$ = constant then $\frac{\kappa_1}{\kappa_2}$ = constant [1].

Theorem 6: γ be a general helix on K-manifold of order 3. Then γ is a general helix if and only if

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \lambda D_{\dot\gamma}^{(2)}\tau_{1,2} + \mu D_{\dot\gamma}\tau_{1,2} = 0$$

here $\lambda = -\frac{3\kappa_2'}{\kappa_2}$ and

$$\mu = (\kappa_1^2+\kappa_2^2) - \frac{\kappa_2''}{\kappa_2} - \frac{3(\kappa_2')^2}{\kappa_2^2}$$

Proof: if γ is a general helix

$$\begin{aligned}
D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\
D_{\dot\gamma}^{(2)}\tau_{1,2} &= \kappa_2'\tau_{1,3} - \kappa_2^2\tau_{1,2} + \kappa_1\kappa_2\tau_{2,3}\\
D_{\dot\gamma}^{(3)}\tau_{1,2} &= \kappa_2''\tau_{1,3} + \kappa_2'(-\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}) - 2\kappa_2\kappa_2'\tau_{1,2}\\
&= 3\kappa_2'\left(\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}\tau_{1,2} + \frac{1}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}\right) + \left\{\frac{\kappa_2''}{\kappa_2} - (\kappa_1^2+\kappa_2^2)\right\}D_{\dot\gamma}\tau_{1,2}
\end{aligned}$$

and we obtained

$$D_{\dot\gamma}^{(3)}\tau_{1,2} - \frac{3\kappa_2'}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2} + \left\{(\kappa_1^2+\kappa_2^2) - \frac{\kappa_2''}{\kappa_2} - \frac{3(\kappa_2')^2}{\kappa_2^2}\right\}D_{\dot\gamma}\tau_{1,2} = 0$$

conversely

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3} \Rightarrow \tau_{1,3} = \frac{1}{\kappa_2}D_{\dot\gamma}\tau_{1,2}, \quad D_{\dot\gamma}\tau_{1,3} = -\frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}\tau_{1,2} + \frac{1}{\kappa_2}D_{\dot\gamma}^{(2)}\tau_{1,2}$$

and

$$D_{\dot\gamma}^{(2)}\tau_{1,2} = \left(-\frac{\kappa_2'}{\kappa_2^2}\right)' D_{\dot\gamma}\tau_{1,2} - \frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2} - \frac{\kappa_2'}{\kappa_2^2}D_{\dot\gamma}^{(2)}\tau_{1,2} + \frac{1}{\kappa_2}D_{\dot\gamma}^{(3)}\tau_{1,2} \tag{3.1}$$

we know that

$$\begin{aligned}
D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\
D_{\dot\gamma}^{(2)}\tau_{1,2} &= \kappa_2'\tau_{1,3} - \kappa_2^2\tau_{1,2} + \kappa_1\kappa_2\tau_{2,3}\\
D_{\dot\gamma}^{(3)}\tau_{1,2} &= 3\kappa_2'D_{\dot\gamma}\tau_{1,3} + \Delta D_{\dot\gamma}\tau_{1,2}
\end{aligned}$$

where

$$\Delta = \frac{\kappa_2''}{\kappa_2} - (\kappa_2^2 + \kappa_1^2)$$

from equation (3.1)

$$D_{\dot\gamma}^{(2)}\tau_{1,3} = \left\{\left(-\frac{\kappa_2'}{\kappa_2^2}\right)' + \frac{\Delta}{\kappa_2}\right\}D_{\dot\gamma}\tau_{1,2} - \kappa_2'\tau_{1,2} - \frac{2(\kappa_2')^2}{\kappa_2}\tau_{1,3} + \frac{\kappa_2'\kappa_1}{\kappa_2}\tau_{2,3} \tag{3.2}$$

$$D_{\dot\gamma}\tau_{1,3} = -\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}$$

if we find the derivative of the given equation

$$D^{(2)}_{\dot\gamma}\tau_{1,3} = -\kappa_2'\tau_{1,2} - \kappa_2 D_{\dot\gamma}\tau_{1,2} + \kappa_1'\tau_{2,3} + \kappa_1 D_{\dot\gamma}\tau_{2,3}$$

and using $D_{\dot\gamma}\tau_{2,3} = -\kappa_1\tau_{1,3}$ we have

$$D^{(2)}_{\dot\gamma}\tau_{1,3} = -\kappa_2'\tau_{1,2} - \kappa_2 D_{\dot\gamma}\tau_{1,2} + \kappa_1'\tau_{2,3} - \kappa_1^2\tau_{1,3} \tag{3.3}$$

By using the equations (3.2) and (3.3) we have

$$-\kappa_2'\tau_{1,2} - \kappa_2 D_{\dot\gamma}\tau_{1,2} + \kappa_1'\tau_{2,3} - \kappa_1^2\tau_{1,3} = \left\{\left(-\frac{\kappa_2'}{\kappa_2^2}\right)' + \frac{\Delta}{\kappa_2}\right\} D_{\dot\gamma}\tau_{1,2} - \kappa_2'\tau_{1,2} - \frac{2(\kappa_2')^2}{\kappa_2}\tau_{1,3} + \frac{\kappa_2'\kappa_1}{\kappa_2}\tau_{2,3}$$

If we product the both sides of the equation with $\tau_{2,3}$ we have the $\kappa_1' = \frac{\kappa_2'\kappa_1}{\kappa_2}$ and then $\kappa_1'\kappa_2 - \kappa_2'\kappa_1 = 0$ and since $\frac{\kappa_1}{\kappa_2}$ is constant then we have $\frac{\kappa_1}{\kappa_2} = \frac{\tau_{1,2}}{\tau_{2,3}} =$ constant.

Theorem 7: If γ is a helix of order 3 on K-manifold, then

$$D^{(3)}_{\dot\gamma}\tau_{1,2} + (\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2} = 0$$

Proof: Since κ_1, κ_2 are constants and for $d = 3$.

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3}, \quad D_{\dot\gamma}\tau_{1,3} = -\kappa_2\tau_{1,2} + \kappa_1\tau_{2,3}, \quad D_{\dot\gamma}\tau_{2,3} = -\kappa_1\tau_{1,3}$$

then we obtain

$$\begin{aligned}
D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\
D^{(2)}_{\dot\gamma}\tau_{1,2} &= \kappa_2 D_{\dot\gamma}\tau_{1,3}\\
&= -\kappa_2^2\tau_{1,2} + \kappa_1\kappa_2\tau_{2,3}\\
D^{(3)}_{\dot\gamma}\tau_{1,2} &= -\kappa_2^2(\kappa_2\tau_{1,3}) + \kappa_1\kappa_2(-\kappa_1\tau_{1,3})\\
&= -(\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2}
\end{aligned}$$

where

$$D^{(3)}_{\dot\gamma}\tau_{1,2} + (\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2} = 0$$

Corollary 2: If γ is a holomorphic helix κ_1, κ_2 separately constants then $\kappa_1' = 0, \kappa_2' = 0$ From there we find

$$D^{(3)}_{\dot\gamma}\tau_{1,2} + (\kappa_2^2 + \kappa_1^2)D_{\dot\gamma}\tau_{1,2} = 0$$

Holomorphic helices of order 4: From

$$D_{\dot\gamma}V_j(s) = -\kappa_{j-1}V_{j-1}(s) + \kappa_j V_{j+1}(s)$$

$j = 1,2,...,d$ and $\tau_{i,j} = \langle V_i, JV_j\rangle$ also for curve of order 4 ($i = 1,2,3$ $j = 1,2,3,4$) then we have

$$\begin{aligned}
D_{\dot\gamma}\tau_{1,2} &= \kappa_2\tau_{1,3}\\
D_{\dot\gamma}\tau_{1,3} &= -\kappa_2\tau_{1,2} + \kappa_3\tau_{1,4} + \kappa_1\tau_{2,3}\\
D_{\dot\gamma}\tau_{1,4} &= -\kappa_3\tau_{1,3} + \kappa_1\tau_{2,4}\\
D_{\dot\gamma}\tau_{2,3} &= -\kappa_1\tau_{1,3} + \kappa_3\tau_{2,4}\\
D_{\dot\gamma}\tau_{2,4} &= -\kappa_1\tau_{1,4} - \kappa_3\tau_{2,3} + \kappa_2\tau_{3,4}\\
D_{\dot\gamma}\tau_{3,4} &= -\kappa_2\tau_{2,4}
\end{aligned}$$

So, the matrix form is

$$\begin{bmatrix} D_{\dot\gamma}\tau_{1,2}\\ D_{\dot\gamma}\tau_{1,3}\\ D_{\dot\gamma}\tau_{1,4}\\ D_{\dot\gamma}\tau_{2,3}\\ D_{\dot\gamma}\tau_{2,4}\\ D_{\dot\gamma}\tau_{3,4}\end{bmatrix} = \begin{bmatrix} 0 & \kappa_2 & 0 & 0 & 0 & 0\\ -\kappa_2 & 0 & \kappa_3 & \kappa_1 & 0 & 0\\ 0 & -\kappa_3 & 0 & 0 & \kappa_1 & 0\\ 0 & -\kappa_1 & 0 & 0 & \kappa_3 & 0\\ 0 & 0 & -\kappa_1 & -\kappa_3 & 0 & \kappa_2\\ 0 & 0 & 0 & 0 & -\kappa_2 & 0\end{bmatrix}\begin{bmatrix}\tau_{1,2}\\ \tau_{1,3}\\ \tau_{1,4}\\ \tau_{2,3}\\ \tau_{2,4}\\ \tau_{3,4}\end{bmatrix}$$

and

$$\begin{aligned}
\tau_{3,1} &= \tau_{4,2} = 0\\
\kappa_2\tau_{2,1} &= \kappa_3\tau_{4,1} + \kappa_1\tau_{3,2}\\
\kappa_2\tau_{4,3} &= \kappa_1\tau_{4,1} + \kappa_3\tau_{3,2}
\end{aligned}$$

Theorem 8 Let M is 2-dimensional K-manifold. For all H-helix of order 4 of complex torsions is given curvatures κ_1, κ_2 and κ_3, satisfy following equations

$$\tau_{1,2} = \tau_{3,4} = \tau, \quad \tau_{2,3} = \tau_{1,4} = \frac{\kappa_2\tau}{\kappa_1 + \kappa_3}, \quad \tau_{1,3} = \tau_{2,4} = 0$$

where

$$\tau = \pm\frac{\kappa_1 + \kappa_3}{\sqrt{\kappa_2^2 + (\kappa_1 + \kappa_3)^2}}$$

$$\tau_{1,2} = -\tau_{3,4} = \tau, \quad \tau_{2,3} = -\tau_{1,4} = \frac{\kappa_2\tau}{\kappa_1 - \kappa_3}, \quad \tau_{1,3} = \tau_{2,4} = 0$$

when $\kappa_1 \neq \kappa_3$ [1].

$$\tau = \pm \frac{\kappa_1 - \kappa_3}{\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}}$$

or

$$\tau_{1,2} = \tau_{3,4} = \tau_{1,3} = \tau_{2,4} = 0,\ \tau_{2,3} = -\tau_{1,4} = \pm 1$$

here $\kappa_1 = \kappa_3$

Theorem 9: Let γ be a general helix on K-manifold M of order 4, so,

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \lambda D_{\dot\gamma}^{(2)}\tau_{1,2} + \mu D_{\dot\gamma}\tau_{1,2} = 0$$

where

$$\lambda = -\frac{3\kappa_2'}{\kappa_2}$$

and

$$\mu = \frac{3(\kappa_2')^2}{\kappa_2^3} - \frac{\kappa_2''}{\kappa_2} + \kappa_1^2 + \kappa_2^2 - \kappa_3^2$$

Proof

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3}$$
$$D_{\dot\gamma}^{(2)}\tau_{1,2} = \kappa_2'\tau_{1,3} - \kappa_2^2\tau_{1,2} + \kappa_2\kappa_3\tau_{1,4} + \kappa_2\kappa_1\tau_{2,3}$$

and

$$D_{\dot\gamma}^{(3)}\tau_{1,2} = 3\kappa_2' D_{\dot\gamma}\tau_{1,3} + (\kappa_2'' - \kappa_2^3 - \kappa_2\kappa_3^2 - \kappa_1^2\kappa_2)\tau_{1,3} + 2\kappa_1\kappa_2\kappa_3\tau_{2,4}$$

$\kappa_1\tau_{2,d} = \kappa_{d-1}\tau_{1,d-1}$ using this relation, $\kappa_1\tau_{2,4} = \kappa_3\tau_{1,3}$ is obtained and in the above expression

$$2\kappa_1\kappa_2\kappa_3\tau_{2,4} = 2\kappa_2\kappa_3^2\tau_{1,3}$$

is written,

$$D_{\dot\gamma}^{(3)}\tau_{1,2} = 3\kappa_2' D_{\dot\gamma}\tau_{1,3} + (\kappa_2'' - \kappa_2^3 - \kappa_2\kappa_3^2 - \kappa_1^2\kappa_2)\tau_{1,3} + 2\kappa_2\kappa_3^2\tau_{1,3}$$
$$= 3\kappa_2' D_{\dot\gamma}\tau_{1,3}(\kappa_2'' - \kappa_2^3 + \kappa_2\kappa_3^2 - \kappa_1^2\kappa_2)\tau_{1,3}$$

is obtained and for

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3} \Rightarrow \tau_{1,3} = \frac{1}{\kappa_2} D_{\dot\gamma}\tau_{1,2}$$
$$\Rightarrow D_{\dot\gamma}\tau_{1,3} = \left(\frac{1}{\kappa_2}\right)' D_{\dot\gamma}\tau_{1,2} + \frac{1}{\kappa_2} D_{\dot\gamma}^{(2)}\tau_{1,2}$$

we find

$$D_{\dot\gamma}^{(3)}\tau_{1,2} = \frac{3\kappa_2'}{\kappa_2} D_{\dot\gamma}^{(2)}\tau_{1,2} + \left\{ -\frac{3(\kappa_2')^2}{\kappa_2^3} + \frac{\kappa_2''}{\kappa_2} - \kappa_2^2 + \kappa_3^2 - \kappa_1^2 \right\} D_{\dot\gamma}\tau_{1,2}$$

or

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \lambda D_{\dot\gamma}^{(2)}\tau_{1,2} + \mu D_{\dot\gamma}\tau_{1,2} = 0$$

Where

$$\lambda = -\frac{3\kappa_2'}{\kappa_2}$$

and

$$\mu = \frac{3(\kappa_2')^2}{\kappa_2^3} - \frac{\kappa_2''}{\kappa_2} + \kappa_1^2 + \kappa_2^2 - \kappa_3^2$$

Theorem 10: If γ a helix on K-manifold of order 4

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \left\{\kappa_1^2 + \kappa_2^2 - \kappa_3^2\right\} D_{\dot\gamma}\tau_{1,2} = 0$$

Proof:

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3}$$
$$D_{\dot\gamma}^{(2)}\tau_{1,2} = -\kappa_2^2\tau_{1,2} + \kappa_2\kappa_3\tau_{1,4} + \kappa_2\kappa_1\tau_{2,3}$$
$$D_{\dot\gamma}^{(3)}\tau_{1,2} = -\kappa_2^2 D_{\dot\gamma}\tau_{1,2} - \kappa_2\kappa_3^2\tau_{1,3} - \kappa_1^2\kappa_2\tau_{1,3} + 2\kappa_1\kappa_2\kappa_3\tau_{2,4}$$

$\kappa_1\tau_{2,d} = \kappa_{d-1}\tau_{1,d-1}$ using the equation, $\kappa_1\tau_{2,4} = \kappa_3\tau_{1,3}$ is obtained and from the above

$$2\kappa_1\kappa_2\kappa_3\tau_{2,4} = 2\kappa_2\kappa_3^2\tau_{1,3}$$

and

$$D_{\dot\gamma}\tau_{1,2} = \kappa_2\tau_{1,3} \Rightarrow \tau_{1,3} = \frac{1}{\kappa_2} D_{\dot\gamma}\tau_{1,2}$$

using equations,

$$D_{\dot\gamma}^{(3)}\tau_{1,2} = -\kappa_2^2 D_{\dot\gamma}\tau_{1,2} - \frac{\kappa_2\kappa_3^2}{\kappa_2} D_{\dot\gamma}\tau_{1,2} - \frac{\kappa_2\kappa_1^2}{\kappa_2} D_{\dot\gamma}\tau_{1,2} + 2\kappa_2\kappa_3^2 \frac{1}{\kappa_2} D_{\dot\gamma}\tau_{1,2}$$
$$= (-\kappa_2^2 + \kappa_3^2 - \kappa_1^2) D_{\dot\gamma}\tau_{1,2}$$

is obtained

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \left\{\kappa_1^2 + \kappa_2^2 - \kappa_3^2\right\} D_{\dot\gamma}\tau_{1,2} = 0$$

Corollary 3: If γ is a helix,because of κ_1, κ_2 will be constants separately, $\kappa_1' = 0$, $\kappa_2' = 0$. Then we obtain

$$D_{\dot\gamma}^{(3)}\tau_{1,2} + \left\{\kappa_1^2 + \kappa_2^2 - \kappa_3^2\right\} D_{\dot\gamma}\tau_{1,2} = 0$$

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