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An Improved Numerical Method for Solving Nonlinear Volterra-Fredholm Integral Equations Using Hybrid Functions

M. Roodaki and Z. JafariBehbahani

Department of Mathematics, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran

Abstract: This paper presents an effective numerical method for solving a class of nonlinear Volterra-Fredholm integral equations by using the hybrid functions of Legendre polynomials and block-pulse functions. The proposed method which differs with previous approaches, attemp ts to use the exact forms of known functions in solving the integral equation. So its accuracy appropriately increases and the computational burden decreases. The convergence analysis and associated theorems are also considered. The efficiency and accuracy of proposed method are confirmed by some given test examples.

Key words: Hybrid functions . nonlinear volterra-fredholm integral equations of the second kind . legendre polynomials . block-pulse functions . function expansion . cross product . expansion method

INTRODUCTION

The functional equations and specially integral and differential equations, show up in many engineering and scientific studies, e.g., fluid dynamics, viscoelasticity, heat and mass transfer, torsion of a wire, elasticity, electromagnetic scattering, biomechanics, electrodynamics, electrostatics, nonlinear vibrations of oscillation systems and beams, behavior of electrically exciting nanotubes, etc [1-5].

Generally, the evaluation of the exact solution of functional equations by analytical methods may be difficult. So the appropriate method with a great appeal for mathematicians will be a numerical method.

In recent years, some promising approximate analytical methods have been proposed for solving nonlinear equations in sciences and specially engineering [3]. Barari *et al*. applied variational iteration and parameterized perturbation methods to investigate nonlinear vibration of Euler-Bernoulli beams subjected to the axial loads [3]. Fereidoon *et al*. used frequency-amplitude formulation for studying the periodic solutions of free vibration of mechanical systems with third and fifth-order nonlinearity [4]. The behavior of an electrically exciting nanotube was studies by Kaliji *et al*., using optimal homotopy asymptotic method [5].
Several studies have been considered the nonlinear Volterra-Fredholm integral equation of the form

$$f(s)=g(s)+\int_0^s k_1(s,t)G_1(t,f(t))\,dt+\int_0^1 k_2(s,t)G_2(t,f(t))\,dt,\ 0\le s\le 1 \qquad (1)$$

where $f(s)$ is an unknown function defined on [0,1] and $g(s)$, $k_1(s,t)$, $k_2(s,t)$, $G_1(t,f(t))$ and $G_2(t,f(t))$ are given L^2 functions. Several numerical methods have been proposed for solving Eq. (1). In some of them, an approximate solution has been computed by using Adomian decomposition method [7], He's variational iteration method [8], homotopy perturbation method [9], modified Laplace-Adomian decomposition method [10] and toeplitz-collocation method [11], for general or special types of $G_1(t,f(t))$ and $G_2(t,f(t))$. Also, orthogonal basis functions have been applied to estimate the solution of Eq. (1), [12-14].

Recently, the hybrid functions of Legendre polynomials and block-pulse functions have been applied in solving various types of integral equations, control problems, time-varying descriptor systems and etc., [15-18]. By these functions, the computational advantages of Legendre polynomials are combined with the simplicity of block-pulse functions, making a powerful set of basis functions. A numerical solution for the linear case of Eq. (1), was presented by Hsiao, using the hybrid functions [19]. Moreover, Maleknejad *et al*. used the hybrid functions to

Corresponding Author: M. Roodaki, Department of Mathematics, Marvdasht Branch, Islamic Azad University, Marvdasht, Iran

compute an approximate solution for this equation when $G_1(t,f(t))=[f(t)]^p$ and $G_2(t,f(t))=[f(t)]^q$, with arbitrary positive integers p and q [20].

In this paper, the methods used in [19, 20] are improved and a novelty method which is more accurate and has less burden in computation, is proposed. Using the hybrid functions in this manner, allows us to convert Eq. (1) to a system of mere algebraic equations.

The layout of the paper is as follows: A description of the hybrid functions of Legendre polynomials and block-Pulse functions and some of their properties are mentioned in section 2. In this section, some theorems regarding the convergence of function expansion with respect to the hybrid functions are also proved. In section 3, a method for computing numerical solutions of nonlinear Volterra-Fredholm integral equations by using hybrid functions is proposed. The results of proposed method in solving several numerical examp les are reported in section 4. Finally, section 5 states some conclusions briefly.

A brief review on Legendre polynomials and block-pulse functions

Definition 1: The Legendre polynomials on the interval [-1,1] are given by the following recursive formu la

$$\begin{aligned}
&L_0(s)=1\\
&L_1(s)=s\\
&L_{m+1}(s)=\frac{2m+1}{m+1}sL_m(s)-\frac{m}{m+1}L_{m-1}(s), \qquad m=1,2,3,\cdots
\end{aligned}$$

The set of $\{L_m(s):m=0,1,\cdots\}$ in the Hilbert space $L^2[-1,1]$ is a complete orthogonal set. Orthogonality of Legendre polynomials on the interval [-1,1] implies that

$$<L_i(s),L_j(s)>=\int_{-1}^{1}L_i(s)L_j(s)ds=\begin{cases}\frac{2}{2i+1}, & i=j\\ 0, & i\neq j\end{cases} \tag{2}$$

for i,j = 0,1…, such that <.,.> denotes the inner product.

Definition 2: In an N-set of block-pulse functions over the interval [0,1), each component is defined as

$$\phi_n(s)=\begin{cases}1, & \frac{n-1}{N}\leq s<\frac{n}{N}\\ 0, & \text{otherwise,}\end{cases} \tag{3}$$

where n = 1,2,…,N, for arbitrary positive integer N.

Block-pulse functions have several important properties such as disjointness, orthogonality and completeness [21].

Definition 3: Let $\{L_m(s)\}_{m=0}^{M-1}$ be an M-set of Legendre polynomials and $\{\phi_n(s)\}_{n=1}^{N}$ be an N-set of block-pulse functions over the interval [0,1), too. An MN-set of hybrid functions (HFs) is defined over the interval [0,1) as

$$h_{n,m}(s)=L_m(2Ns-2n+1)\phi_n(s), \quad \begin{matrix}n=1,2,\cdots,N\\ m=0,1,\cdots,M-1\end{matrix} \tag{4}$$

In the above definition, N and M are the order of block-pulse functions and the order of Legendre polynomials, respectively. So, the interval [0,1) is divided to N-subintervals and M Legendre polynomials constructed on each of them. It is clear that $h_{n,m}(s)$ can be written as

$$h_{n,m}(s)=\begin{cases}L_m(2Ns-2n+1), & \frac{n-1}{N}\leq s<\frac{n}{N}\\ 0, & \text{otherwise,}\end{cases} \tag{5}$$

for n = 1,2,…N and m = 0,1,…,M-1.
Throughout the paper, the following abbreviation for the indices in $h_{n,m}(s)$ is used for convenience,

$$h_k(s)=h_{n,m}(s),\, k=1,2,\cdots,NM \tag{6}$$

where k = (n-1)M+m. It is obvious that $\{h_k(s)\}_{k=1}^{NM}$ is a complete orthogonal set in the Hilbert space $L^2[0,1)$. The components of this set can be considered in the following HFs vector

$$H(s)=[h_{1,0}(s),\cdots,h_{1,M-1}(s),\cdots,h_{N,0}(s),\cdots,h_{N,M-1}(s)]^T \tag{7}$$

Function expansion: The truncated HFs expansion of any function $f(s)\in L^2[0,1)$ is defined as

$$f(s)\approx\sum_{n=1}^{N}\sum_{m=0}^{M-1}c_{n,m}h_{n,m}(s)=F^T\cdot H(s) \tag{8}$$

where H(s) defined in (7) and the NM-vector F contains the coefficients $c_{n,m}$ that are defined as

$$F_k=c_{n,m}=\frac{<f(s),h_{n,m}(s)>}{<h_{n,m}(s),h_{n,m}(s)>}=N(2m+1)\cdot\int_{\frac{n-1}{N}}^{\frac{n}{N}}f(s)h_{n,m}(s)ds$$

for k = 1,2,…,NM and called HFs coefficients vector. The uniform convergence and the expected error for (8) are showed in [22] and the results can be summarized in the following theorem.

Theorem 1: Let *f*(s) be a continuous function defined on [0,1) and $\bar{f}(s)$ be its truncated HFs expansion. If $|f''(s)|\le M_2$, then we have the following error estimation

$$\|f(s)-\bar{f}(s)\|_2^2\le\frac{3}{8}M_2^2\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(2m-3)^4}$$

Proof: The above theorem guarantees the uniform convergence of the function expansion with respect to HFs [22].

Integration of hybrid functions: In the previous methods which were based on HFs, the approximation of $\int_0^s H(t)dt$ and $\int_0^1 H(t)H^T(t)dt$ were utilized [19, 20]. But in our novelty method, which will be presented in the next section, the expansions of $\int_0^s k_1(s,t)H(t)dt$ and $\int_0^1 k_2(s,t)H(t)dt$ with respect to HFs for the known functions $k_1(s,t)$ and $k_2(s,t)$ are required. With this approach, the error of the numerical solution of integral equations can be decreased appropriately.
Suppose that $k_1(s,t)$ be an $L^2([0,1)\times[0,1))$ function. It can be concluded from Eq. (5) that

$$\int_0^s k_1(s,t)h_{n,m}(t)dt=\begin{cases}0, & s<\frac{n-1}{N}\\ \int_{\frac{n-1}{N}}^{s}k_1(s,t)h_{n,m}(t)dt, & \frac{n-1}{N}\le s<\frac{n}{N}\\ \int_{\frac{n-1}{N}}^{\frac{n}{N}}k_1(s,t)h_{n,m}(t)dt, & s\ge\frac{n}{N}\end{cases} \tag{9}$$

for n = 1,2,…N and m = 0,1,…,M-1. Expressing $\int_0^s k_1(s,t)h_{n,m}(t)dt$ in terms of HFs results in

$$\int_0^s k_1(s,t)H(t)dt \approx \overline{P}\cdot H(s) \tag{10}$$

where $\overline{P}$ is an NM×NM matrix of the form

$$\overline{P}=\begin{bmatrix} PE_1 & PH_{1,2} & PH_{1,3} & \cdots & PH_{1,N} \\ \mathbf{0} & PE_2 & PH_{2,3} & \ddots & PH_{2,N} \\ \mathbf{0} & \mathbf{0} & PE_2 & \ddots & PH_{3,N} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & PE_N \end{bmatrix} \tag{11}$$

in which PE_{n_p} and PH_{n_p,n_q} are M×M matrices and can be computed as follow

$$(PE_{n_p})_{m_i,m_j} = N(2m_j-1)\int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}}\int_{\frac{n_p-1}{N}}^{s} k_1(s,t)h_{n_p,m_i}(t)h_{n_p,m_j}(s)dtds$$

$$(PH_{n_p,n_q})_{m_i,m_j} = N(2m_j-1)\int_{\frac{n_q-1}{N}}^{\frac{n_q}{N}}\int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} k_1(s,t)h_{n_p,m_i}(t)h_{n_q,m_j}(s)dtds$$

for $n_p = 1,2,\cdots,N$, $n_q = n_p+1,\cdots,N$ and m_i, m_j = 0,1,…,M-1. It is noticeable that if k(s,t) = 1, the matrix $\overline{P}$ is reduced to the operational matrix of integration in HFs domain [19, 20] and $\int_0^s H(t)dt$ can be expanded with respect to the hybrid functions.

Similarly, for $\int_0^1 k_2(s,t)H(t)dt$ it follows that

$$\int_0^1 k_2(s,t)H(t)dt \approx \overline{D}\cdot H(s) \tag{12}$$

where

$$\overline{D}_{NM\times NM}=\begin{bmatrix} DB_{1,1} & DB_{1,2} & \cdots & DB_{1,N} \\ DB_{2,1} & DB_{2,2} & \cdots & DB_{2,N} \\ \vdots & \vdots & & \vdots \\ DB_{N,1} & DB_{N,2} & \cdots & DB_{N,N} \end{bmatrix} \tag{13}$$

and M×M matrices DB_{n_p,n_q} can be computed as follows

$$(DB_{n_p,n_q})_{m_i,m_j} = N(2m_j-1)\int_{\frac{n_q-1}{N}}^{\frac{n_q}{N}}\int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} k_2(s,t)h_{n_p,m_i}(t)h_{n_q,m_j}(s)dtds$$

for $n_p, n_q = 1,2,\cdots,N$ and m_i, m_j = 0,1,…, M-1.

Evaluation of cross products: In continuation of this section, we will encounter to the product of H(s) and $H^T(s)$, which is called the product matrix of the hybrid functions. It is clear that

$$h_{n_p,m_i}(s)h_{n_q,m_j}(s)=\begin{cases} h_{n_p,m_i}(s)h_{n_p,m_j}(s), & n_p = n_q \\ 0, & n_p \neq n_q \end{cases} \tag{14}$$

for n_p, n_q = 1,2,…N and m_i, m_j = 0,1,…,M-1. Therefore

$$\mathrm{H}(s)\cdot \mathrm{H}^T(s)=\begin{bmatrix} H_1 & 0 & \cdots & 0 \\ 0 & H_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & H_N \end{bmatrix} \tag{15}$$

in which the $M\times M$ matrices H_{n_p} are as follows

$$(H_{n_p})_{m_i,m_j} = h_{n_p,m_i}(s)h_{n_p,m_j}(s), \quad \begin{array}{l} n_p = 1,2,\cdots,N \\ m_i, m_j = 0,1,\cdots,M-1 \end{array} \tag{16}$$

Now, let F be an arbitrary NM-vector of the form

$$F=[c_{1,0},\cdots,c_{1,M-1}(s),\cdots,c_{N,0}(s),\cdots,c_{N,M-1}(s)]^T \tag{17}$$

If $\mu=(n_p-1)M+m_i$, the μth component of $\mathrm{H}(s)\cdot \mathrm{H}^T(s)\cdot F$ can be computed as

$$(\mathrm{H}(s)\cdot \mathrm{H}^T(s)\cdot F)_\mu = \sum_{k=0}^{M-1} h_{n_p,m_i}(s)h_{n_p,k}(s)c_{n_p,k} \tag{18}$$

By expanding the components of $\mathrm{H}(s)\cdot \mathrm{H}^T(s)\cdot F$ in terms of HFs, it is obtained as

$$\mathrm{H}(s)\cdot \mathrm{H}^T(s)\cdot F \approx \tilde{F}\cdot \mathrm{H}(s) \tag{19}$$

It is remarkable that the $(NM\times NM)$-matrix $\tilde{F}$ is a block matrix too and we have

$$\tilde{F}=\begin{bmatrix} \tilde{F}_1 & 0 & \cdots & 0 \\ 0 & \tilde{F}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \tilde{F}_N \end{bmatrix} \tag{20}$$

in which the components of the $M\times M$ matrices $\tilde{F}_{n_p}$ can be computed as follows

$$\left(\tilde{F}_{n_p}\right)_{m_i,m_j} = \sum_{k=0}^{M-1} \frac{< h_{n_p,m_i}(s)h_{n_p,k}(s), h_{n_p,m_j}(s)>}{< h_{n_p,m_j}(s), h_{n_p,m_j}(s)>} c_{n_p,k} \tag{21}$$

$$=N(2m_j-1)\sum_{k=0}^{M-1}\left(\int_{\frac{n_p-1}{N}}^{\frac{n_p}{N}} h_{n_p,m_i}(s)h_{n_p,k}(s)h_{n_p,m_j}(s)ds\right)c_{n_p,k} \tag{22}$$

for n_p = 1,2,…N and m_i, m_j = 0,1,…,M-1.

The components of vector F may be considered as the expansion coefficients in Eq. (8). In this situation $\tilde{F}$ is called the coefficients matrix. The calculation procedure of Eq. (19) for N = 2 and M = 8 can be found in [19].

Approximation for power of functions: Now, a truncated expansion of $[f(s)]^{\alpha}$, for positive integer $\alpha \geq 2$ and $f(s) \in L^2[0,1)$ is computed. This idea comes from [18] and is indicated in the following lemma.

Lemma 1: Let NM-vectors F and F_{α} be the HFs coefficients of $f(s)$ and $[f(s)]^{\alpha}$, respectively. Then F_{α} can be computed from the following recursive formula

$$\begin{cases} F_{\alpha} = F^T \cdot \tilde{F}^T_{\alpha-1}, & \alpha = 3,4,\cdots \\ F_2 = F^T \cdot \tilde{F} \end{cases} \tag{23}$$

where $\tilde{F}$ defined in Eq. (20).

Proof: [18].

SOLVING NONLINEAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

The results obtained in the previous section are applied to present an effective method for solving the nonlinear Volterra-Fredholm integral equations, numerically.

Consider the following nonlinear Volterra-Fredholm integral equation

$$f(s)=g(s)+\int_0^s k_1(s,t)[f(t)]^{\alpha}dt+\int_0^1 k_2(s,t)[f(t)]^{\beta}dt, \quad 0\leq s\leq 1 \tag{24}$$

where $\alpha,\beta \geq 1$ and L^2 functions $k_1(s,t)$, $k_2(s,t)$ and g(s) are known but $f(s)$ is not [6]. We can Approximate the functions f, g, $[f(s)]^{\alpha}$ and $[f(s)]^{\beta}$ with respect to the hybrid functions as follows

$$\begin{aligned} f(s) &\approx F^T \cdot H(s) = H^T(s) \cdot F \\ g(s) &\approx G^T \cdot H(s) = H^T(s) \cdot G \\ [f(s)]^{\alpha} &\approx F^T_{\alpha} \cdot H(s) = H^T(s) \cdot F_{\alpha} \\ [f(s)]^{\beta} &\approx F^T_{\beta} \cdot H(s) = H^T(s) \cdot F_{\beta} \end{aligned} \tag{25}$$

where H(s) is defined in Eq. (7) and NM-vectors F, G, F_{α} and F_{β} are HFs coefficients of f, g, $[f(s)]^{\alpha}$ and $[f(s)]^{\beta}$, respectively. Elements of F_{α} and F_{β} are nonlinear combinations of the elements F.

To approximate the first-integral part in Eq. (24), from Eqs. (25) and (10), it is obtained as

$$\int_0^s k_1(s,t)[f(t)]^{\alpha}dt \approx \int_0^s k_1(s,t)\cdot F^T_{\alpha}\cdot H(t)dt = F^T_{\alpha}\cdot\int_0^s k_1(s,t)\cdot H(t)dt \approx F^T_{\alpha}\cdot\overline{P}\cdot H(s) \tag{26}$$

with $\overline{P}$ defined in Eq. (11).

Furthermore, using Eqs. (25) and (12), the second-integral part in Eq. (24), can be approximated as

$$\int_0^1 k_2(s,t)[f(t)]^{\beta}dt \approx \int_0^1 k_2(s,t)\cdot F^T_{\beta}\cdot H(t)dt = F^T_{\beta}\cdot\int_0^1 k_2(s,t)\cdot H(t)dt \approx F^T_{\beta}\cdot\overline{D}\cdot H(s) \tag{27}$$

in which $\overline{D}$ is defined in Eq. (13).

By substituting Eqs. (25), (26) and (27) in Eq. (24) and replacing $\approx$ with =, we have

$$F^T \cdot H(s) = G^T \cdot H(s) + F_\alpha^T \cdot \overline{P} \cdot H(s) + F_\beta^T \cdot \overline{D} \cdot H(s)$$

or

$$F - \overline{P}^T \cdot F_\alpha - \overline{D}^T \cdot F_\beta = G \tag{28}$$

Eq. (28) is a nonlinear system of NM algebraic equations. The NM components of F are unknown and can be computed by solving this system using Newton method or other iterative methods. Hence, an approximate solution $f(s) \approx F^T H(s)$ can be computed for Eq. (24). If the approximate value of $f(\alpha)$, $0 \le a < 1$ is required, we can evaluate it as

$$f(a) \approx \sum_{m=0}^{M-1} c_{n,m} h_{n,m}(a)$$

providing a belongs into the interval $\left[\frac{n-1}{N}, \frac{n}{N}\right)$.

NUMERICAL EXAMPLES

In this section, we implement the proposed method on some examples. A simple integral equation is considered for example 1. This example is presented here to illustrate the matrices $\overline{P}$, $\overline{D}$ and $\tilde{F}$ in details, thought it doesn't show more information about the efficiency of method. Examples 2 and 3 are selected from different references. In this way, we can compare our method with rationalized Haar functions method [12] and the other method based on hybrid functions [19]. The results of these methods and the exact solution of integral equations are compared for 16 terms and the absolute errors reported in Table 1 and 2. Similarly, in example 4, we compare our proposed method with three existing methods [13, 20, 23], in Table 3. Furthermore, the accuracy of the method is studied by computing

$$e_2 = \|\overline{f}(s) - f(s)\|_2 = \left[\sum_{n=1}^{N} \int_{\frac{n-1}{N}}^{\frac{n}{N}} \left(\overline{f}(s) - f(s)\right)^2 ds\right]^{\frac{1}{2}} \tag{29}$$

where $f(s)$ and $\overline{f}(s)$ are the exact and approximate solutions of the integral equation, respectively. The results are tabulated for N = 2 and different values of M in Table 4. The computations associated with the examples were performed using Matlab 7.0 software on a personal computer.

Table 1: The absolute errors for example 2

s	Hybrid method [19] N = 2, M = 11	Haar method [12] m = 64	Proposed method N = 2, M = 11
0.0000	0	0	5.7287508e-14
0.0625	2.4403e-10	1.9905101e-03	5.5511151e-15
0.1250	3.2490e-10	2.6503334e-04	3.3306691e-15
0.1875	2.8085e-10	2.2908322e-03	2.8865799e-15
0.2500	7.4210e-11	6.0534288e-04	4.4408921e-16
0.3125	3.2812e-10	2.6764534e-03	3.5527137e-15
0.3750	1.2778e-10	1.0423090e-03	3.9968029e-15
0.4375	3.8882e-10	3.1716009e-03	5.7731597e-15
0.5000	1.9656e-10	1.6033846e-03	1.5587531e-13
0.5625	4.6678e-10	3.8073828e-03	1.5543122e-14
0.6250	2.8489e-10	2.3238199e-03	8.8817842e-15
0.6875	5.6685e-10	4.6237430e-03	8.4376950e-15
0.7500	3.9829e-10	3.2488771e-03	8.8817842e-16
0.8125	6.9537e-10	5.6719701e-03	9.7699626e-15
0.8750	5.4391e-10	4.4366741e-03	1.0658141e-14
0.9375	8.6037e-10	7.0179205e-03	1.4210855e-14

Table 2: The absolute errors for example 3

s	Hybrid method [19] N = 2, M = 11	Haar method [12] m = 64	Proposed method N = 2, M = 11
0.0000	0	0	0
0.0625	2.9246894e-03	1.1066366e-03	1.7763568e-15
0.1250	3.0416120e-05	8.6150773e-03	1.8873791e-15
0.1875	2.8643120e-03	8.8778609e-03	2.1094237e-15
0.2500	1.1987009e-04	1.4057590e-02	1.9984014e-15
0.3125	2.7470860e-03	1.4812991e-02	1.7763568e-15
0.3750	2.6314821e-04	2.0068848e-02	2.1094237e-15
0.4375	2.5798676e-03	1.6056236e-02	2.1094237e-15
0.5000	4.5184544e-04	1.6042820e-02	4.6629367e-15
0.5625	2.3724526e-03	1.8957844e-02	4.4408921e-16
0.6250	6.7488666e-04	2.5124998e-02	3.3306691e-16
0.6875	2.1370832e-03	2.0579851e-02	2.2204460e-16
0.7500	9.1908048e-04	1.8954521e-02	0
0.8125	1.8877124e-0.3	1.1039383e-02	2.2204460e-16
0.8750	1.1698829e-03	6.4435960e-03	3.8857806e-16
0.9375	1.6392696e-03	7.7188716e-03	4.9960036e-16

Example 1: Consider the following nonlinear Volterra-Fredholm integral equation

$$f(s) = -2s^3 + \frac{3}{4}s^2 + \frac{9}{2}s - 1 + \int_0^s (2s-t)f(t)dt + \int_0^1 (s^2-2s)t[f(t)]^2 dt, 0 \le s < 1 \tag{30}$$

with the exact solution $f(s) = 3s-1$. Choosing N = 2 and M = 4, we have

$$\overline{P} = \begin{bmatrix} \frac{1}{8} & \frac{3}{16} & \frac{1}{16} & 0 & \frac{5}{8} & \frac{1}{4} & 0 & 0 \\ \frac{-1}{24} & \frac{-3}{80} & \frac{1}{48} & \frac{1}{60} & \frac{-1}{24} & 0 & 0 & 0 \\ 0 & \frac{-1}{80} & \frac{-1}{112} & \frac{1}{80} & 0 & 0 & 0 & 0 \\ 0 & \frac{-1}{560} & \frac{-1}{112} & \frac{-1}{240} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{5}{16} & \frac{1}{16} & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{12} & \frac{-3}{80} & \frac{1}{16} & \frac{1}{60} \\ 0 & 0 & 0 & 0 & 0 & \frac{-3}{80} & \frac{-1}{112} & \frac{3}{80} \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{560} & \frac{-3}{112} & \frac{-1}{240} \end{bmatrix}, \quad \overline{D} = \begin{bmatrix} \frac{-5}{96} & \frac{-3}{64} & \frac{1}{192} & 0 & \frac{-11}{96} & \frac{-1}{64} & \frac{1}{192} & 0 \\ \frac{-5}{288} & \frac{-1}{64} & \frac{1}{576} & 0 & \frac{-11}{288} & \frac{-1}{192} & \frac{1}{576} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-5}{32} & \frac{-9}{64} & \frac{1}{64} & 0 & \frac{-11}{32} & \frac{-3}{64} & \frac{1}{64} & 0 \\ \frac{-5}{288} & \frac{-1}{64} & \frac{1}{576} & 0 & \frac{-11}{288} & \frac{-1}{192} & \frac{1}{576} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \tilde{F} = \begin{bmatrix} \tilde{F}_1 & 0 \\ 0 & \tilde{F}_2 \end{bmatrix}$$

$$\tilde{F}_i = \begin{bmatrix} c_{i,0} & c_{i,1} & c_{i,2} & c_{i,3} \\ \frac{1}{3}c_{i,1} & c_{i,0} + \frac{2}{5}c_{i,2} & \frac{2}{3}c_{i,1} + \frac{3}{7}c_{i,3} & \frac{3}{5}c_{i,2} \\ \frac{1}{5}c_{i,2} & \frac{2}{5}c_{i,1} + \frac{9}{35}c_{i,3} & c_{i,0} + \frac{2}{7}c_{i,2} & \frac{3}{5}c_{i,1} + \frac{4}{15}c_{i,3} \\ \frac{1}{7}c_{i,3} & \frac{9}{35}c_{i,2} & \frac{3}{7}c_{i,1} + \frac{4}{21}c_{i,3} & c_{i,0} + \frac{4}{15}c_{i,2} \end{bmatrix}, \quad i = 1,2 \text{ and } G = \begin{bmatrix} \frac{1}{8} & \frac{177}{160} & \frac{-1}{32} & \frac{-1}{80} & \frac{15}{8} & \frac{87}{160} & \frac{-5}{32} & \frac{-1}{80} \end{bmatrix}^T$$

Table 3: The absolute errors for example 4

s	Method in [13] k = 16	Method in [20] M = 8, N = 8	Proposed method M = 4, N = 2
0.0	0.005	8.50e-05	8.8546457e-07
0.1	0.001	1.04e-04	1.3389934e-08
0.2	0.005	1.25e-14	4.2255516e-07
0.3	0.002	1.52e-04	4.2239909e-07
0.4	0.001	1.74e-04	1.3829762e-08
0.5	0.002	1.95e-04	8.8497852e-07
0.6	0.003	2.08e-04	2.1538651e-08
0.7	0.001	2.04e-04	4.0887189e-07
0.8	0.001	1.92e-04	4.0635593e-07
0.9	0.005	1.49e-04	2.8983681e-08

Table 4: The values of e_2 for examples 2-4

M	N	Example 2	Example 3	Example 4
4	2	6.7848e-04	4.9436e-05	4.0521e-07
6	2	1.4204e-06	7.8036e-08	2.6927e-18
8	2	1.5892e-09	6.5687e-11	0
10	2	1.1053e-12	3.4333e-14	-
12	2	7.5409e-16	3.4333e-14	-

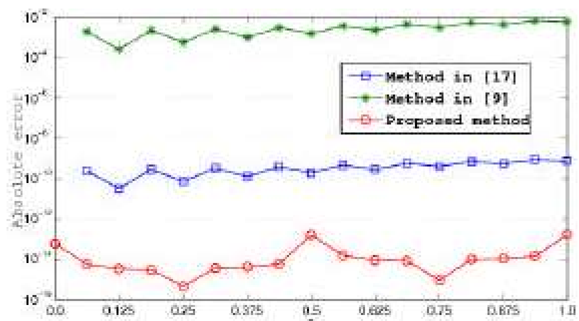

Fig. 1: Absolute error graph for example 2

Solving the nonlinear system (28), the unknown vector F is obtained as

$$F=\begin{bmatrix}\frac{-1}{4} & \frac{3}{4} & 0 & 0 & \frac{5}{4} & \frac{3}{4} & 0 & 0\end{bmatrix}^T$$

which confirms that the proposed method gives the analytical solution of Eq. (30).

Example 2: [12, 19] Consider the following Volterra-Fredholm integral equation

$$f(s)=e^{2s+\frac{1}{3}}-\int_0^1 \frac{1}{3}e^{2s-\frac{1}{3}t}f(t)dt, 0\le s<1 \tag{31}$$

with the exact solution $f(s) = e^{2s}$. Table 1 illustrates the absolute errors in 16 terms for various methods, which confirms that our method gives almost the same solution as the analytic method.

Example 3: [12, 19] For the following Volterra-Fredholm integral equation

$$f(s)=\cos(s)-\int_0^s (s-t)\cos(s-t)f(t)dt,\ 0\le t\le s<1 \tag{32}$$

with the exact solution

$$f(s)=\frac{1}{3}(2\cos\sqrt{3}s+1)$$

the absolute errors for various methods in 16 terms are reported in Table 2.

Example 4: [13, 20, 23] Consider the following nonlinear Volterra-Fredholm integral equation

$$f(s)=g(s)+\int_0^s (s-t)[f(t)]^2 dt+\int_0^1 (s+t)f(t)dt, 0\le s<1 \tag{33}$$

where

$$g(s)=-\frac{1}{30}s^6+\frac{1}{3}s^4-s^2+\frac{5}{3}s-\frac{5}{4}$$

with the exact solution $f(s)=s^2-2$. Table 3 illustrates the absolute errors for our method and two existing methods [13, 20]. Furthermore, for N = 2 and M = 8 the exact solution is obtained by our method, as well as the method in [23].

COMMENTS ON THE RESULTS

In this paper, a numerical method for solving nonlinear Volterra-Fredholm integral equations is presented. The properties of the hybrid functions of block-pulse functions and Lagendre polynomials are discussed and utilized to reduce a nonlinear Volterra-Fredholm integral equation to a nonlinear system of algebraic equations. As it is seen from the numerical examples, the method is easy to implement and provides very accurate solutions. The method is an improvement of other methods based on the hybrid functions and has the following advantages:

I. Since in this approach, only the unknown function $f(s)$ and the function g(s) are expanded by HFs, it provides more accurate solutions than some of other existing methods.
II. Examples 1 and 4 show that the exact solution of the integral equation can be computed by the method with suitable choice of M and N, for the case of piecewise polynomials.
III. Theorem 1 shows that function approximation with respect to HFs is uniformly converges to the function and the error becomes zero faster than

$$\sum_{n=N+1}^{\infty}\sum_{m=M}^{\infty}\frac{1}{n^5(2m-3)^4}$$

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REFERENCES

1. Burton, T.A., 2005. Volterra Integral and Differential Equations. 2nd Edn., B.V. Elsevier.
2. Polyanin, A.D. and A.V. Manzhirov, 2008. Handbook of Integral Equations. 2nd Edn., Chapman & Hall/CRC Press, Boca Raton.
3. Barari, A., H.D. Kaliji, M. Ghadimi and G. Domairry, 2011. Non-linear vibration of Euler-Bernoulli beams. Latin American Journal of Solids and Structures, 8: 139-148.
4. Fereidoon, A., M. Ghadimi, A. Barari, H.D. Kaliji and G. Domairry, 2012. Nonlinear vibration of oscillation systems using frequency-amplitude formulation, Shock and Vibration, 19: 323-332.

5. Kaliji, H.D., M. Ghadimi and M.H. Pashaei, 2012. Study the Behavior of an Electrically Exciting Nanotube Using Optimal Homotopy Asymptotic Method. International Journal of Applied Mechanics, DOI:10.1142/S1758825112001336.
6. Atkinson, K.E., 1997. The Numerical Solutions of Integral Equations of the Second Kind, Cambridge University Press, Cambridge.
7. Wazwaz, A., 2002. A reliable treatment for mixed Volterra-Fredholm integral equations. Appl. Math. Comput., 127: 405-414.
8. Yousefi, SA., A. Lotfi and Mehdi Dehghan, 2009. He's variational iteration method for solving nonlinear mixed Volterra-Fredholm integral equations. Comput. Math. Appl., 58: 2172-2176.
9. Ghasemi, M., M. Tavassoli Kajani and E. Babolian, 2007. Numerical solutions of the nonlinear Volterra-Fredholm integral equations by using homotopy perturbation method. Appl. Math. Comput., 188: 446-449.
10. Fadaei, J. and M.M. Moghadam, 2011. Numerical solution of systems of integral differential equations by using modified Laplace-Adomian decomposition method. World Applied Sciences Journal, 15: 1786-1790.
11. Gherjalar, D.H.-A., A. Arzhang and M. Yousefi, 2012. A numerical method based on toeplitz-collocation method and orthogonal polynomials for solving nonlinear functional integral equations. World Applied Sciences Journal, 16: 253-262.
12. Reihani, M.H. and Z. Abadi, 2007. Rationalized Haar functions method for solving Fredholm and Volterra integral equations. J. Comput. Appl. Math., 200: 12-20.
13. Ordokhani, Y., 2006. Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via rationalized Haar functions. Appl. Math. Comput., 180: 436-443.
14. Maleknejad, K., E. Hashemizadeh and B. Basirat, 2012. Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations. Commun. Nonlinear Sci. Numer. Simulat., 17: 52-61.
15. Maleknejad, K. and M. Tavassoli Kajani, 2003. Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions. Appl. Math. Comput., 145: 623-629.
16. Marzban, H.R. and M. Razzaghi, 2003. Hybrid functions approach for linearly constrained quadratic optimal control problems. Appl. Math. Model., 27: 471-485.
17. Hsiao, C.H., 2010. Numerical solutions of linear time-varying descriptor systems via hybrid functions. Appl. Math. Comput., 216: 1363-1374.
18. Maleknejad, K., B. Basirat and E. Hashemizadeh, 2011. Hybrid Legendre polynomials and Block-Pulse functions approach for nonlinear Volterra-Fredholm integro-differential equations. Appl. Math. Comput., 61: 2821-2828.
19. Hsiao, C.H., 2009. Hybrid function method for solving Fredholm and Volterra integral euations of the second kind. J. Comput. Appl. Math., 230: 59-68.
20. Hashemizadeh, E., K. Maleknejad and B. Basirat, 2011. Hybrid functions approach for the nonlinear Volterra-Fredholm integral equations. Procedia Computer Science, 3: 1189-1194.
21. Ganti, P.R., 1983. Piecewise Constant Orthogonal Functions and Their Application to Systems and Control, Springer-Verlag.
22. Sohrabi, S., 2011. Study on convergence of hybrid functions method for solution of nonlinear integral equations. Appl. Anal., iFirst: 1-13.
23. Marzban, H.R., H.R. Tabrizidooz and M. Razzaghi, 2011. A composite collocation method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations. Commun. Nonlinear. Sci. Numer. Simulat., 16: 1186-1194.