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Multipliers in d-algebras

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Abstract: In this paper, we introduce the concept of a multiplier on a d-algebra and obtain some properties of multipliers of d-algebras.

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INTRODUCTION

BCK and BCI-algebras, two classes of algebras of logic, were introduced by Imai and Iseki [1], Iseki [2] and Iseki and Tanaka [3] and have been extensively studied by various other researchers [4, 5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [6, 7] a wider class of abstract algebras was introduced by Q.P.Hu and X.Li.They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. BCH-algebras have also been studied by Chaudhry and some other researchers [8-10].

The notion of d-algebras, another generalization of the notion of BCK-algebras, was introduced by Neggers and Kim [11]. They studied some properties of this class of algebras. Since then many researchers have extensively studied these algebras [12-16].

The study of multipliers have been made by various researchers in the context of C^*-algebras, rings and semigroups [17]. They have studied the properties of multipliers on them as well as properties of these algebraic structures using the notion of a multiplier on them. But the properties of multipliers on d-algebras, an important class of algebras containing the class of BCK-algebras, have not been investigated so far. So with this motivation, in this paper we introduce the concept of a multiplier on a d-algebra and obtain some properties of multipliers of d-algebras.

PRELIMINARIES

In this section we describe some definitions and notions that will be used in the sequel.

Definition 2.1: [8] Let X be a set with binary operation * and a constant 0. Then (X, *, 0) is called a BCK-algebra if it satisfies the following axioms:

(1) $((x*y)*(x*z))*(z*y)=0$
(2) $(x*(x*y))*y=0$
(3) $x*x=0$
(4) $0*x=0$
(5) $x*y=0$ *and* $y*x=0$ imply x = y for all x,y,z∈X

Definition 2.2: [10] A d-algebra is a non-empty set X with a constant 0 and a binary operation * satisfying the following axioms:

(1) $x*x=0$
(2) $0*x=0$
(3) $x*y=0$ *and* $y*x=0$ imply x = y for all x,y∈X

Remark 2.3: It is obvious from above definitions that every BCK-algebra is a d-algebra. The following shows that converse is not true, in general.

Example 2.4: [10] Let X = {0,1,2,3,4,5} with the binary operation * defined by:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	2	2	0	0	0	0
3	3	3	1	0	0	0
4	4	2	1	1	0	0
5	5	5	3	3	1	0

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Then X is a d-algebra, but it is not BCK-algebra. This is because condition(2) of Definition(2.1) is not satisfied as shown:

$$(5*(5*2))*2 = (5*3)*2 = 3*2 = 1 \neq 0$$

This example shows that the class of BCK-algebras is a proper subclass of the class of d-algebras.

Definition 2.5: Let S be a non-empty subset of a d-algebra X, then S is called a subalgebra of X if x*y∈S for all x,y∈S.

Definition 2.6: Let X be a d-algebra and I a subset of X, then I is called an ideal of X if it satisfies the following conditions:

(1) $0 \in I$
(2) $x*y \in I$ and $y \in I$ imply x∈I

Definition 2.7: Let X be a d-algebra and I a nonempty subset of X, then I is called a d-ideal of X if it satisfies the following conditions:

(1) $x*y \in I$ and $y \in I$ imply x∈I and
(2) $x \in I$ and $y \in X$ imply x*y∈I.

From condition (2) it is obvious that for $x \in I \subseteq X, 0 = x*x \in I$.

MULTIPLIERS ON d-ALGEBRAS

In the sequel X will denote a d-algebra with constant 0 and binary operation *, unless otherwise specified. We now prove our results.

Definition 3.1: A self map *f*:X→X satisfying $f(x*y) = f(x)*y$, for all x,y∈X, is called a multiplier on X.

Example 3.2: Let X = {0,a,b} with the binary operation * defined by

*	0	a	b
0	0	0	0
a	a	0	0
b	b	a	0

Then X is a d-algebra. Let *f*:X→X be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a, \\ a & \text{if } x = b. \end{cases}$$

Then *f* is a multiplier on X.

Remark 3.3: If X is a d-algebra with binary operation *, then we define a binary relation ≤ on X by:

$$x \leq y \text{ if and only if } x*y = 0, x, y \in X.$$

Proposition 3.4: Let X be a d-algebra and *f* a multiplier on X, then

(1) $f(0) = 0$,
(2) $f(x) \leq x$ for all x∈X and
(3) if x≤y then $f(x) \leq y$ for all x,y∈X.

Proof:

(1) $f(0) = f(0*f(0)) = f(0)*f(0) = 0$.
(2) Let x∈X. Then $0 = f(0) = f(x*x) = f(x)*x$. So $f(x) \leq x$.
(3) Let $x, y \in X$ and $x \leq y$. Then x*y = 0. So, $0 = f(0) = f(x*y) = f(x)*y$. Thus $f(x) \leq y$.

Proposition 3.5: Let *f* and g be multipliers on X, then their composition *f*°g is a multiplier on X.

Proof: Let x,y∈X. Then

$$\begin{aligned}(f \circ g)(x*y) &= f(g(x*y)) = f(g(x)*y) \\ &= f(g(x))*y = (f \circ g)(x)*y\end{aligned}$$

So *f*°g is a multiplier on X.

Definition 3.6: A d-algebra X is said to be positive implicative if

$$(x*y)*z = (x*z)*(y*z)$$

for all x,y,z∈X.

Let M(X) denotes the collection of all multipliers on X. Obviously O:X→X defined by O(x) = 0 for all x∈X and I: X→X defined by I(x) = x for all x∈X are in M(X). So M(X) is non-empty.

Definition 3.7: Let X be a positive implicative d-algebra and M(X) be the collection of all multipliers on X. We define a binary operation * on M(X) by:

$$(f*g)(x) = f(x)*g(x) \text{ for } x \in X \text{ and } f, g \in M(X).$$

Theorem 3.8: Let X be a positive implicative d-algebra. Then (M(X),*,0) is a positive implicative d-algebra.

Proof: Let X be implicative d-algebra. Let g,f∈M(X). Then

$$\begin{aligned}(g*f)(x*y) &= (g(x*y))*(f(x*y))\\ &= (g(x)*y)*(f(x)*y)\\ &= (g(x)*f(x))*y = ((g*f)(x))*y.\end{aligned}$$

So g*f∈M(X). Let f∈M(X). Then

$$(O*f)(x) = O(x)*f(x) = 0*f(x) = 0 = O(x)$$

for all x∈X. So O*f = O for all f∈M(X).
Now for f∈M(X), we have

$$(f*f)(x) = f(x)*f(x) = 0 = O(x)$$

for all x∈X. So f*f = O.
Let g,f∈M(X) be such that

$$f*g = O \text{ and } g*f = O$$

This implies

$$(f*g)(x) = 0 \text{ and } (g*f)(x) = 0$$

for all x∈X. That is,

$$f(x)*g(x) = 0 \text{ and } g(x)*f(x) = 0$$

which implies f(x) = g(x) for all x∈X. Thus f = g. Hence M(X) is a d-algebra.

Now we show that it is a positive implicative. Let f, g and h∈M(X). Then

$$\begin{aligned}((f*g)*h)(x) &= ((f*g)(x))*h(x) = (f(x)*g(x))*h(x)\\ &= (f(x)*h(x))*(g(x)*h(x))\\ &= ((f*h)(x))*((g*h)(x))\\ &= ((f*h)*(g*h))(x)\end{aligned}$$

for all x∈X
Hence

$$(f*g)*h = (f*g)*(f*h)$$

Thus M(X) is an implicative d-algebra.

Proposition 3.9: Let X be a d-algebra and f a multiplier on X. If f is one-to-one, then f is the identity map on X.

Proof: Let f be one-to-one. Let x∈X. Then

$$f(x*f(x)) = f(x)*f(x) = 0 = f(0)$$

Thus $x*f(x) = 0$, which implies $x \le f(x)$. Since $f(x) \le x$, by proposition 3.4 (2), for all x, therefore f(x) = x. Hence f is the identity map.
Let f be a multiplier on X. We define Ker(f) by:

$$Ker(f) = \{x : x \in X \text{ and } f(x) = 0\}.$$

Proposition 3.10: Let X be a d-algebra and f a multiplier on X. Then

(1) Ker(f) is a subalgebra of X and
(2) If f is one-to-one, then Ker(f) = {0}.

Proof:
(1) Let x,y∈Ker(f). Then f(x) = 0 and f(y) = 0. So $f(x*y) = f(x)*y = 0*y = 0$. Thus x*y∈Ker($f$), which implies Ker($f$) is a subalgebra of X.
(2) Let f be one-to-one. Let x∈Ker(f). So $f(x) = 0 = f(0)$. Thus x = 0. So Ker(f) = {0}.

Definition 3.11: A d-algebra X is called commutative if $x*(x*y) = y*(y*x)$ for all x,y∈X.

Proposition 3.12: Let X be a commutative d-algebra satisfying $x = 0 = x, x \in X$. Let f be a multiplier on X. If x∈Ker(f) and y≤x, then y∈Ker(f).

Proof: Let x∈Ker(f) and y≤x. Then $f(x) = 0$ and $y*x = 0$. Now

$$\begin{aligned}f(y) &= f(y*0) = f(y*(y*x)) = f(x*(x*y))\\ &= f(x)*(x*y) = 0*(x*y) = 0\end{aligned}$$

So x∈Ker(f).

Theorem 3.13: Let X be a d-algebra satisfying x = 0 = x for all x∈X. Let f be a multiplier on X, which is also an endomorphism on X. Then Ker(f) is a d-ideal of X.

Proof: Obviously, 0∈Ker(f). So Ker(f) is nonempty. Let x*y∈Ker(f) and y∈Ker(f). Then f(y) = 0. Also $f(x*y) = 0$, which implies

$$0 = f(x)*f(y) = f(x)*0 = f(x)$$

Thus x∈Ker(f).

Let x∈Ker(f) and y∈X. Then

$$f(x*y) = f(x)*y = 0*y = 0$$

So x*y∈Ker(f). Hence Ker(f) is a d-ideal of X.

Definition 3.14: Let X be a d-algebra and f a multiplier on X. Then the set

$$Fix(f) = \{x : x \in X \; and \; f(x) = x\}$$

is called the set of fixed points of f.

Proposition 3.15: Let X be a d-algebra and f a multiplier on X. Then Fix(f) is a subalgebra of X.

Proof: Since $f(0) = 0$, so Fix(f) is non-empty. Let $x, y \in Fix(f)$. Then $f(x) = x, f(y) = y$. Thus

$$f(x * y) = f(x) * y = x * y$$

So $x * y \in Fix(f)$. Hence Fix(f) is a subalgebra of X.

Definition 3.16: Let X be a d-algebra and f a multiplier on X. f is called idempotent if $f \circ f = f$. $f \circ f$ will be denoted by f^2.

Theorem 3.17: Let X be a positive implicative d-algebra which satisfies x*0 = x for all x∈X. Let f_1, f_2 be two idempotent multipliers on X. If $f_1 \circ f_2 = f_2 \circ f_1$, then $f_1 * f_2$ is an idempotent multiplier on X.

Proof: By Theorem 3.8, we get that $f_1 * f_2$ is a multiplier on X. Now

$$\begin{aligned}
&((f_1 * f_2) \circ (f_1 * f_2))(x) = (f_1 * f_2)((f_1 * f_2)(x)) = (f_1 * f_2)(f_1(x) * f_2(x)) = \\
&(f_1(f_1(x) * f_2(x))) * (f_2(f_1(x) * f_2(x))) = ((f_1 \circ f_1)(x) * f_2(x)) * \\
&((f_2 \circ f_1)(x) * f_2(x)) = (f_1(x) * f_2(x)) * ((f_1 \circ f_2)(x) * f_2(x)) = (f_1(x) * f_2(x)) * \\
&(f_1(f_2(x) * f_2(x))) = ((f_1 * f_2)(x)) * (f_1(0)) = ((f_1 * f_2)(x)) * 0 = (f_1 * f_2)(x).
\end{aligned}$$

Thus $(f_1 * f_2) \circ (f_1 * f_2) = f_1 * f_2$. Hence $f_1 * f_2$ is idempotent.

CONCLUSION

We have initiated a study of multipliers on d-algebras. We have shown that the collection M(X) of multipliers on a d-algebra X is a d-algebra. We have also investigated the conditions under which Ker(f) of a multiplier $f \in$M(X) is an ideal and the product $f_1 * f_2$, $f_1, f_2 \in$M(X), is an idempotent multiplier on X.

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