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Generalized Intuitionistic Fuzzy Bi-Ideals in Semigroups

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Abstract: In this paper, using t-norm Δ and s-norm ∇ we introduce the notion of generalized intuitionistic fuzzy bi-ideal, generalized intuitionistic fuzzy generalized bi-ideal and generalized intuitionistic fuzzy (1,2) ideal of a semigroup. We characterize different classes of semigroups by the properties of these fuzzy ideals.

Key words: Generalized intuitionistic fuzzy bi-ideal . generalized intuitionistic fuzzy generalized bi-ideal and generalized intuitionistic fuzzy (1,2) ideal of a semigroup

INTRODUCTION

The notion of fuzzy set was introduced by Zadeh [1], deals with the application of fuzzy technology. The information processing is already important and it will certainly increase in importance in the future. Mordeson *et al*. gave a systematic exposition of fuzzy semigroups in [2], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. The monograph by Mordeson and Malik [3], deals with the applications of fuzzy approach to the concepts of automata and formal languages. In [4], Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideal of semigroups. The concept of (1,2) ideals in semigroups was introduced by Lajos [5]. Lajos and Jun in [6] consider the fuzzification of (1,2) ideals in semigroups.

After the introduction of fuzzy sets by Zadeh, there have been a number of generalizations of this fundamental concept. Atanassov [7] introduced the notion of intuitionistic fuzzy set which is a generalization of fuzzy set [8, 9]. Intuitionistic fuzzy set theory has been applied in different fields, for example logic programming, decision making problems, etc. De *et al*. in [10] applied intutionistic fuzzy set theory in medical diagnosis. In [11], Kim and Jun introduced the concept of intuitionistic fuzzy ideals of semigroups and in [12] Kim and Lee studied intuitionistic fuzzy bi-ideals of semigroups. In [13], Hur *et al*. introduced the concept of intuitionistic fuzzy generalized bi-ideals of semigroups. Shabir *et al*. introduced the notion of intuitionistic fuzzy prime bi-ideals of semigroups in [14].

Kim in [15] considered the fuzzification of R-subgroups of Near-Rings with respect to an s-Norm. In [12], Kim and Lee gave the concept of intuitionistic (T, S) normed fuzzy ideals of Γ-Rings. In [16], Zhan studied the fuzzy left h-ideals in hemirings with t-norms. Interval valued intuitionistic (S, T)-fuzzy H_v-submodules were studied by Zhan and Dudek in [17]. Akram and Dar in [18] introduced the idea of fuzzy left h-ideal in hemirings with respect to an s-norm. In this paper we consider the generalization of intuitionistic fuzzy bi-ideals, (1,2) ideals in a semigroups S and investigate some properties of such ideals.

PRELIMINARIES

Throughout this paper S will denote a semigroup. By a subsemigroup of S we mean a non-empty subset A of S such that $AA \subseteq A$. By a left (right) ideal of S we mean a non-empty subset A of S such that $SA \subseteq A\,(AS \subseteq A)$. By a two sided ideal or simply an ideal, we mean a non-empty subset of S which is both a left and a right ideal of S. A subsemigroup A of a semigroup S is called a bi-ideal of S if $ASA \subseteq A$. A subsemigroup A of S is called a (1,2)-ideal of S if $ASA^2 \subseteq A$. A semigroup S is said to be regular if, for each $x \in S$ there exists $y \in S$ such that $x = xyx$

An intuitionistic fuzzy set A in S is an object having the form

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in S\}$$

where the functions $\mu_A : S \to [0,1]$ and $\gamma_A : S \to [0,1]$ denote the degree of membership and the degree of non-membership of each element $x \in S$ to A and $0 \le \mu_A(x) + \gamma_A(x) \le 1$ for all $x \in S$.

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For the sake of simplicity, we shall use the symbol $A = (\mu_A, \gamma_A)$ for the intuitionistic fuzzy set

$$A = \{(x, \mu_A(x), \gamma_A(x)) : x \in S\}$$

Im (μ_A) denotes the image set of μ_A. Similarly Im (γ_A) denotes the image set of γ_A.

Definition [7]: Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be non-empty intuitionistic fuzzy sets in a set S. Then

(1) $A \subseteq B$ if and only if $\mu_A \leq \mu_B$ and $\gamma_A \geq \gamma_B$.

(2) $A^c = (\gamma_A, \mu_A)$.

(3) $A \cap B = (\mu_A \wedge \mu_B, \gamma_A \vee \gamma_B)$.

(4) $A \cup B = (\mu_A \vee \mu_B, \gamma_A \wedge \gamma_B)$.

(5) $A = (\mu_A, \overline{\mu}_A)$ where $\overline{\mu}_A = 1 - \mu_A$.

(6) $A = (\overline{\gamma}_A, \gamma_A)$ where $\overline{\gamma}_A = 1 - \gamma_A$.

Definition [17]: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy subsemigroup of S if

$$\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(xy) \leq \gamma_A(x) \vee \gamma_A(y)$$

for all $x, y \in S$.

Definition [17]: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy left (right) ideal of S if

$$\mu_A(xy) \geq \mu_A(y) \quad (\mu_A(xy) \geq \mu_A(x))$$

and

$$\gamma_A(xy) \leq \gamma_A(y) \quad (\gamma_A(xy) \leq \gamma_A(x))$$

for all $x, y \in S$.

An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy ideal of S if it is both an intuitionis tic fuzzy left and right ideal of S.

Definition [17]: An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy bi-ideal of S if

$$\mu_A(xwy) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(xwy) \leq \gamma_A(x) \vee \gamma_A(y)$$

for all $x, y, w \in S$.

Definition [11]: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called an intuitionistic fuzzy generalized intuitionistic fuzzy bi-ideal of S if

$$\mu_A(xwy) \geq \mu_A(x) \wedge \mu_A(y)$$

and

$$\gamma_A(xwy) \leq \gamma_A(x) \vee \gamma_A(y)$$

for all $x, y, w \in S$.

Definition [17]: An intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of a semigroup S is called an intuitionistic fuzzy (1,2) -ideal of S if

$$\mu_A(xw(yz)) \geq \min\{\mu_A(x), \mu_A(y), \mu_A(z)\}$$

and

$$\gamma_A(xw(yz)) \leq \max\{\gamma_A(x), \gamma_A(y), \gamma_A(z)\}$$

for all $x, y, z, w \in S$.

Definition [10]: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in S and let $t \in [0,1]$. Then the sets

$$U(\mu_A : t) = \{x \in S : \mu_A(x) \geq t\}$$

and

$$L(\gamma_A : t) = \{x \in S : \gamma_A(x) \leq t\}$$

are called μ-level t-cut and γ-level t-cut of A, respectively.

Definition [18]: By a t-norm Δ, we mean a function $\Delta : [0,1] \times [0,1] \to [0,1]$ satis fying the following conditions

(t1) $x \, \Delta \, 1 = x$

(t2) $x \, \Delta \, y = y \, \Delta \, x$

(t3) $x \, \Delta \, (y \, \Delta \, z) = (x \, \Delta \, y) \, \Delta \, z$

(t4) if $w \leq x$ and $y \leq z$ then $w \, \Delta \, y \leq x \, \Delta \, z$

for all $x, y, z, w \in [0,1]$.

Remark [18]: Every t-norm Δ has a useful property

$$(x \, \Delta \, y) \leq \min(x, y)$$

for all $x, y \in [0,1]$.

Definition [18]: By an s-norm ∇, we mean a function $\nabla : [0,1] \times [0,1] \to [0,1]$ satisfying the following conditions (s1) $x \nabla 0 = x$

(s2) $x \nabla y = y \nabla x$

(s3) $x \nabla (y \nabla z) = (x \nabla y) \nabla z$

(s4) if $w \le x$ and $y \le z$ then $w \nabla y \le x \nabla z$

for all $x,y,z,w \in [0,1]$.

Remark [18]: Every s-norm has a useful property

$$\max(x,y) \le x \nabla y$$

for all $x,y \in [0,1]$.

Definition [18]: A mapping $\eta : [0,1] \to [0,1]$ is called a negation if it satisfies

(η1) $\eta(0) = 1, \eta(1) = 0$

(η2) η is non-increasing.

(η3) $\eta(\eta(x)) = x$.

The most frequently used negation is $x \to 1-x$.

Remark [18]: The t-norm and s-norm are said to be dual with respect to the negation $\eta(x) = 1 - x$, if

$$x \nabla y = \eta(\eta(x) \Delta \eta(y))$$

This holds if and only if $x \Delta y = \eta(\eta(x) \nabla \eta(y))$.

Generalized intuitionistic fuzzy bi-ideals in a semigroup s

In this paper we denote by Δ and ∇, the t-norm and s-norm which are dual with respect to the negation $\eta(x) = 1 - x$.

Definition: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy subsemigroup of S if

$$\mu_A(xy) \ge \mu_A(x) \Delta \mu_A(y)$$

and

$$\gamma_A(xy) \le \gamma_A(x) \nabla \gamma_A(y)$$

for all $x,y \in S$.

Example: Let S = {a,b,c,d} be a semigroup with the following Cayley table

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ as

$$\mu_A(a) = 0.6,\ \mu_A(b) = 0.5,\ \mu_A(c) = 0.7,\ \mu_A(d) = 0.7$$

and

$$\gamma_A(a) = 0.3,\ \gamma_A(b) = 0.4,\ \gamma_A(c) = 0.4,\ \gamma_A(d) = 0.3$$

Let $\Delta : [0,1] \times [0,1] \to [0,1]$ be defined by

$$x \Delta y = \max(x + y - 1, 0)$$

and $\nabla : [0,1] \times [0,1] \to [0,1]$ be defined by

$$x \nabla y = \min(x + y, 1)$$

for all $x,y \in [0,1]$. Then Δ is a t-norm and ∇ is an s-norm. By routine calculations we check that the intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy subsemigroup of S.

Definition: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy left (right) ideal of S if

$$\mu_A(xy) \ge \mu_A(y) \quad (\mu_A(xy) \ge \mu_A(x))$$

and

$$\gamma_A(xy) \le \gamma_A(y) \quad (\gamma_A(xy) \le \gamma_A(x))$$

for all $x,y \in S$

An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy two sided ideal (or generalized intuitionistic fuzzy ideal) of S if it is both generalized intuitionistic fuzzy left ideal and generalized intuitionistic fuzzy right ideal of S.

Definition: A generalized intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ in S is called a generalized intuitionistic fuzzy bi-ideal of S if

$$\mu_A(xwy) \ge \mu_A(x) \Delta \mu_A(y)$$

and

$$\gamma_A(xwy) \le \gamma_A(x) \nabla \gamma_A(y)$$

for all $x,y,w \in S$.

Example: Let S = {0,a,b,c} be a semigroup with the following multiplication table

·	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	a
c	0	0	a	b

Define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ as

$$\mu_A(0) = 0.4,\ \mu_A(a) = 0.4,\ \mu_A(b) = 0.6,\ \mu_A(c) = 0.2$$

and

$$\gamma_A(0) = 0.1,\ \gamma_A(a) = 0.5,\ \gamma_A(b) = 0.4,\ \gamma_A(c) = 0.6.$$

Let $(\Delta,\nabla): [0,1]\times[0,1]\to[0,1]$ be defined as

$$x\,\Delta\, y = xy \text{ and } x\,\nabla\, y = x + y - xy$$

for all $x,y\in[0,1]$. Then Δ is a t-norm and ∇ is an s-norm. The intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy bi-ideal of S.

But if we define t-norm Δ and s-norm ∇ as

$$x\,\Delta\, y = \frac{xy}{x+y-xy} \text{ and } x\,\nabla\, y = \frac{x+y-2xy}{1-xy}$$

then $A = (\mu_A, \gamma_A)$ is not a generalized intuitionistic fuzzy bi-ideal of S under the norm (Δ,∇) as

$$0.4 = \mu_A(0) = \mu_A(bb) \ngeq \mu_A(b)\,\Delta_2\mu_A(b) = 0.4285$$
$$0.1 = \gamma_A(0) = \gamma_A(bb) \leq \gamma_A(b)\,\nabla_2\gamma_A(b) = 0.5714$$

Definition: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is called a generalized intuitionistic fuzzy generalized bi-ideal of S if

$$\mu_A(xwy) \geq \mu_A(x)\Delta\,\mu_A(y)$$

and

$$\gamma_A(xwy) \leq \gamma_A(x)\nabla\,\gamma_A(y)$$

for all $x,y,w\in S$.

Obviously every generalized intuitionistic fuzzy bi-ideal of S is a generalized intuitionistic fuzzy generalized bi-ideal of S but the converse is not true in general.

Example: In Example 3.5, if we define an intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ as:

$$\mu_A(0) = 0.3,\ \mu_A(a) = 0.4,\ \mu_A(b) = 0.2,\ \mu_A(c) = 0.4$$

and

$$\gamma_A(0) = 0.1,\ \gamma_A(a) = 0.5,\ \gamma_A(b) = 0.8,\ \gamma_A(c) = 0.6$$

Let $(\Delta,\nabla): [0,1]\times[0,1]\to[0,1]$ be defined as

$$\alpha\,\Delta\,\beta = \frac{\alpha\beta}{\alpha+\beta-\alpha\beta} \text{ and } \alpha\,\nabla\,\beta = \frac{\alpha+\beta-2\alpha\beta}{1-\alpha\beta}$$

for all $\alpha,\beta\in[0,1]$. Then Δ is a t-norm and ∇ is an s-norm. The intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy generalized bi-ideal of S. But

$$0.2 = \mu_A(b) = \mu_A(cc) \ngeq \mu_A(c)\,\Delta\,\mu_A(c) = 0.25$$
$$0.8 = \gamma_A(b) = \gamma_A(cc)\,?\,\gamma_A(c)\,\nabla\,\gamma_A(c) = 0.75$$

Hence $A = (\mu_A, \gamma_A)$ is not a generalized intuitionistic fuzzy bi-ideal of S.

Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be two intuitionistic fuzzy subsets of a semigroup S. The product $A\odot B = (\mu_A\odot\mu_B, \gamma_A\odot\gamma_B)$ is defined as

$$\mu_A\odot\mu_B = \begin{cases} \bigvee_{x=yz} \mu_A(y)\Delta\,\mu_B(z) & \text{if } \exists\, y,z\in S, \text{ such that } x = yz \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma_A\odot\gamma_B = \begin{cases} \bigwedge_{x=yz} \gamma_A(y)\nabla\,\gamma_B(z) & \text{if } \exists\, y,z\in S, \text{ such that } x = yz \\ 1 & \text{otherwise} \end{cases}$$

The operation $\odot$ is associative.

Theorem: Let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be the intuitionistic fuzzy sets in a semigroup S. If $A\subseteq B$ (i.e. $\mu_A \leq \mu_B$, $\gamma_A \geq \gamma_B$) then $A\odot C\subseteq B\odot C$ and $C\odot A\subseteq C\odot B$.

Proof: Let $a\in S$. If a is not expressible as $a = pq$ for some $p, q\in S$. Then

$$(\mu_A\odot\mu_C)(a) = 0 = (\mu_B\odot\mu_C)(a)$$

and

$$(\gamma_A\odot\gamma_C)(a) = 1 = (\gamma_B\odot\gamma_C)(a)$$

Otherwise

$$(\mu_A\odot\mu_C)(a) = \bigvee_{a=pq}(\mu_A(p)\Delta\mu_C(q)) \leq \bigvee_{a=pq}(\mu_B(p)\Delta\mu_C(q)) = (\mu_B\odot\mu_C)(a)$$

and

$$(\gamma_A\odot\gamma_C)(a) = \bigwedge_{a=pq}(\gamma_A(p)\nabla\,\gamma_C(q)) \geq \bigwedge_{a=pq}(\gamma_B(p)\nabla\,\gamma_C(q)) = (\gamma_B\odot\gamma_C)(a)$$

Thus $A\odot C\subseteq B\odot C$. Similarly we can show that $C\odot A\subseteq C\odot B$.

We define an intuitionistic fuzzy set $S = (S,S')$ in S as $S(x) = 1$ and $S'(x) = 0$ for all $x\in S$.

Let A be a non-empty subset of a semigroup S. Then the intuitionistic characteristic function of A is denoted by $\ddot{A} = (\Phi_A, \Psi_A)$ and is defined as

$$\Phi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}, \quad \Psi_A(x) = \begin{cases} 0 & \text{if } x \in A \\ 1 & \text{if } x \notin A \end{cases}$$

for all x∈S.

Theorem: Let A be a non-empty subset of a semigroup S. Then A is a subsemigroup of S if and only if the intuitionistic characteristic function $\ddot{A} = (\Phi_A, \Psi_A)$ of A is a generalized intuitionistic fuzzy subsemigroup of S.

Proof: Assume that A is a subsemigroup of S. We show that $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy subsemigroup of S. Suppose that there exist x,y∈S, such that

$$\Phi_A(xy) < \Phi_A(x) \Delta \Phi_A(y)$$

Take

$$t = \tfrac{1}{2}\left(\Phi_A(xy) + \Phi_A(x) \Delta \Phi_A(y)\right)$$

then t∈[0,1], such that

$$\Phi_A(xy) < t < \Phi_A(x) \Delta \Phi_A(y) \le \min\left(\Phi_A(x), \Phi_A(y)\right)$$

Thus $\Phi_A(x) > t$ and $\Phi_A(y) > t$. This implies x,y∈A. Since A is a subsemigroup of S, xy∈A. This implies $\Phi_A(xy) = 1 > t$. This is a contradiction. Hence

$$\Phi_A(xy) \ge \Phi_A(x) \Delta \Phi_A(y)$$

Similarly, if there exist x,y∈S, such that

$$\Psi_A(xy) > \Psi_A(x) \nabla \Psi_A(y)$$

Take

$$t' = \tfrac{1}{2}\left(\Psi_A(xy) + \Psi_A(x) \nabla \Psi_A(y)\right)$$

Then $t' \in (0,1]$ such that

$$\Psi_A(xy) > t' > \Psi_A(x) \nabla \Psi_A(y) \ge \max\left(\Psi_A(x), \Psi_A(y)\right)$$

This implies $\Psi_A(x) < t'$ and $\Psi_A(y) < t'$ that is x,y∈A. So xy∈A. This implies $\Psi_A(xy) = 0 < t'$ which is a contradiction. Hence

$$\Psi_A(xy) \le \Psi_A(x) \nabla \Psi_A(y)$$

for all x,y∈S.

This shows that $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy subsemigroup of S.

Conversely, let $\ddot{A} = (\Phi_A, \Psi_A)$ be a generalized intuitionistic fuzzy subsemigroup of S. We show that A is a subsemigroup of S. Let x,y∈A, $\Phi_A(x) = 1,\ \Phi_A(y) = 1$ and $\Psi_A(x) = 0,\ \Psi_A(y) = 0$. Since

$$\Phi_A(xy) \ge \Phi_A(x) \Delta \Phi_A(y) = 1 \Delta 1 = 1$$

and

$$\Psi_A(xy) \le \Psi_A(x) \nabla \Psi_A(y) = 0 \nabla 0 = 0$$

so $\Phi_A(xy) = 1$ and $\Psi_A(xy) = 0$. This implies xy∈A. Hence A is a subsemigroup of S.

Similarly, we can show that,

Theorem: A non-empty subset A of a semigroup S is a left (right) ideal of S if and only if the intuitionistic characteristic function $\ddot{A} = (\Phi_A, \Psi_A)$ of A is a generalized intuitionistic fuzzy left (right) ideal of S.

Theorem: A non-empty subset A of a semigroup S is a bi-ideal of S if and only if the intuitionistic characteristic function $\ddot{A} = (\Phi_A, \Psi_A)$ of A is a generalized intuitionistic fuzzy bi-ideal of S.

Proof: Assume that A is a bi-ideal of S. Then by Theorem 3.9, $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy subsemigroup of S.

Let x,y,z∈S be such that

$$\Phi_A(xyz) < \Phi_A(x) \Delta \Phi_A(z)$$

and

$$\Psi_A(xyz) > \Psi_A(x) \nabla \Psi_A(z)$$

Take

$$t = \tfrac{1}{2}\left(\Phi_A(xyz) + \Phi_A(x) \Delta \Phi_A(z)\right)$$

and

$$t' = \tfrac{1}{2}\left(\Psi_A(xyz) + \Psi_A(x) \nabla \Psi_A(z)\right)$$

Then $t, t' \in (0,1]$ such that

$$\Phi_A(xyz) < t < \Phi_A(x) \Delta \Phi_A(z) \le \min\left(\Phi_A(x), \Phi_A(z)\right)$$

and

$$\Psi_A(xyz) > t' > \Psi_A(x) \nabla \Psi_A(z) \ge \max\left(\Psi_A(x), \Psi_A(z)\right)$$

This implies $\Phi_A(x) > t$, $\Phi_A(z) > t$ and $\Psi_A(x) < t'$, $\Psi_A(z) < t'$. This shows that x,z∈A. Since

A is a bi-ideal of S, we have xyz∈A. Thus $\Phi_A(xyz) = 1 > t$ and $\Psi_A(xyz) = 0 < t'$. This is a contradiction. Hence

$$\Phi_A(xyz) \geq \Phi_A(x) \Delta \Phi_A(z)$$

and

$$\Psi_A(xyz) \leq \Psi_A(x) \nabla \Psi_A(z)$$

for all x,y,z∈S. This proves that $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy bi-ideal of S.

Conversely, assume that $\ddot{A} = (\Phi_A, \Psi_A)$ be a generalized intuitionistic fuzzy bi-ideal of S. By Theorem 3.9 A is a subsemigroup of S. Let x,y∈A and z∈S, Then $\Phi_A(x) = 1, \Phi_A(y) = 1$ and $\Psi_A(x) = 0, \Psi_A(y) = 0$. Since

$$\Phi_A(xzy) \geq \Phi_A(x) \Delta \Phi_A(y) = 1 \Delta 1 = 1$$

and

$$\Psi_A(xzy) \leq \Psi_A(x) \nabla \Psi_A(y) = 0 \nabla 0 = 0$$

Thus xzy∈A. Hence A is a bi-ideal of S.

Theorem: A non-empty subset A of a semigroup S is a generalized bi-ideal of S if and only if $\ddot{A} = (\Phi_A, \Psi_A)$ is a generalized intuitionistic fuzzy generalized bi-ideal of S.

Proof: The proof is similar to the proof of the Theorem 3.11.

Theorem: An intuitionistic fuzzy set A = (μ_A, γ_A) in a semigroup S is a generalized intuitionistic fuzzy subsemigroup of S if and only if $A \odot A \subseteq A$ that is $\mu_A \odot \mu_A \leq \mu_A$ and $\gamma_A \odot \gamma_A \geq \gamma_A$.

Proof: Let A = (μ_A, γ_A) be a generalized intuitionistic fuzzy subsemigroup of S and a∈S. If

$$(\mu_A \odot \mu_A)(a) = 0 \text{ and } (\gamma_A \odot \gamma_A)(a) = 1$$

then

$$\mu_A \odot \mu_A \leq \mu_A \text{ and } \gamma_A \odot \gamma_A \geq \gamma_A$$

Otherwise, there exist elements x,y∈S such that $a = xy$. Then,

$$(\mu_A \odot \mu_A)(a) = \vee_{a=xy} \mu_A(x) \Delta \mu_A(y) \leq \vee_{a=xy} \mu_A(xy) = \mu_A(a)$$

and

$$(\gamma_A \odot \gamma_A)(a) = \wedge_{a=xy} \gamma_A(x) \nabla \gamma_A(y) \geq \vee_{a=xy} \gamma_A(xy) = \gamma_A(a)$$

Hence $\mu_A \odot \mu_A \leq \mu_A$ and $\gamma_A \odot \gamma_A \geq \gamma_A$.

Conversely, assume that $\mu_A \odot \mu_A \leq \mu_A$ and $\gamma_A \odot \gamma_A \geq \gamma_A$. Let x and y be any elements of S. Then

$$\mu_A(xy) \geq (\mu_A \odot \mu_A)(xy) = \vee_{xy=bc} \mu_A(b) \Delta \mu_A(c) \geq \mu_A(x) \Delta \mu_A(y)$$

and

$$\gamma_A(xy) \leq (\gamma_A \odot \gamma_A)(xy) = \wedge_{xy=bc} \gamma_A(b) \nabla \gamma_A(c) \leq \gamma_A(x) \nabla \gamma_A(y)$$

Hence A = (μ_A, γ_A) is a generalized intuitionistic fuzzy subsemigroup of S.

Theorem: Let A = (μ_A, γ_A) be an intuitionistic fuzzy set in a semigroup S. Then A = (μ_A, γ_A) is a generalized intuitionistic fuzzy generalized bi-ideal of S if and only if $A \odot S \odot A \subseteq A$ that is

$$\mu_A \odot S \odot \mu_A \leq \mu_A \text{ and } \gamma_A \odot S' \odot \gamma_A \geq \gamma_A$$

Proof: Suppose A = (μ_A, γ_A) be a generalized intuitionistic fuzzy generalized bi-ideal of S. Let a be any element of S.

Case 1: If a is not expressible as xy for all x,y∈S, then

$$(\mu_A \odot S \odot \mu_A)(a) = 0 \leq \mu_A(a)$$

and

$$(\gamma_A \odot S' \odot \gamma_A)(a) = 1 \geq \gamma_A(a)$$

Case 2: If a is expressible as $a = xy$ for some x,y∈S and $x = pq$ for some p,q∈S, then

$$\mu_A(pqy) \geq \mu_A(p) \Delta \mu_A(y)$$

and

$$\gamma_A(pqy) \leq \gamma_A(p) \nabla \gamma_A(y)$$

Therefore

$$\begin{aligned}(\mu_A \odot S \odot \mu_A)(a) &= \vee_{a=xy} ((\mu_A \odot S)(x) \Delta \mu_A(y)) \\ &= \vee_{a=xy} \left(\vee_{x=pq} (\mu_A(p) \Delta S(q)) \Delta \mu_A(y) \right) \\ &= \vee_{a=xy} \left(\vee_{x=pq} (\mu_A(p) \Delta 1) \Delta \mu_A(y) \right) \\ &= \vee_{a=xy} \vee_{x=pq} (\mu_A(p) \Delta \mu_A(y)) \\ &\leq \vee_{a=bcd} (\mu_A(b) \Delta \mu_A(d)) \\ &\leq \vee_{a=bcd} \mu_A(bcd) = \mu_A(a)\end{aligned}$$

and

$$(\gamma_A \odot S \odot \gamma_A)(a) = \underset{a=xy}{\wedge}((\gamma_A \odot S')(x) \nabla \gamma_A(y))$$
$$= \underset{a=xy}{\wedge}\left(\underset{x=pq}{\wedge}\left((\gamma_A(p) \nabla S'(q)) \nabla \gamma_A(y)\right)\right)$$
$$= \underset{a=xy}{\wedge}\left(\underset{x=pq}{\wedge}\left((\gamma_A(p) \nabla 0) \nabla \gamma_A(y)\right)\right)$$
$$= \underset{a=xy}{\wedge}\underset{x=pq}{\wedge}\left((\gamma_A(p) \nabla \gamma_A(y)\right)$$
$$\geq \underset{a=bcd}{\wedge}\left((\gamma_A(b) \nabla \gamma_A(d)\right)$$
$$\geq \underset{a=bcd}{\wedge}\gamma_A(bcd) = \gamma_A(a)$$

and we have $\mu_A \odot S \odot \mu_A \leq \mu_A$ and $\gamma_A \odot S' \odot \gamma_A \geq \gamma_A$.

Conversely, assume that $\mu_A \odot S \odot \mu_A \leq \mu_A$ and $\gamma_A \odot S \odot \gamma_A \geq \gamma_A$. Let x,y,z∈S. Then

$$\mu_A(xyz) \geq (\mu_A \odot S \odot \mu_A)(xyz)$$
$$= \underset{xyz=bc}{\vee}\{(\mu_A \odot S)(b) \Delta \mu_A(y)\}$$
$$\geq (\mu_A \odot S)(xy) \Delta \mu_A(z)$$
$$= \underset{xy=pq}{\vee}\{\mu_A(p) \Delta S(q)\} \Delta \mu_A(z)$$
$$\geq \{\mu_A(x) \Delta S(y)\} \Delta \mu_A(z)$$
$$= (\mu_A(x) \Delta 1) \Delta \mu_A(z)$$
$$= \mu_A(x) \Delta \mu_A(z)$$

and

$$\gamma_A(xyz) \leq (\gamma_A \odot S \odot \gamma_A)(xyz)$$
$$= \underset{xyz=bc}{\wedge}\left\{(\gamma_A \odot S')(b) \nabla \gamma_A(c)\right\}$$
$$\leq (\gamma_A \odot S')(xy) \nabla \gamma_A(z)$$
$$= \underset{xy=pq}{\wedge}\left\{(\gamma_A(p) \nabla S'(q)\right\} \nabla \gamma_A(z)$$
$$\leq \left\{\gamma_A(x) \nabla S'(y)\right\} \nabla \gamma_A(z)$$
$$= \left\{\gamma_A(x) \nabla 0\right\} \nabla \gamma_A(z) = \gamma_A(x) \nabla \gamma_A(z)$$

Thus A = (μ_A, γ_A) is a generalized intuitionistic fuzzy generalized bi-ideal of S.

Corollary: Let A = (μ_A, γ_A) be an intuitionistic fuzzy set in a semigroup S. Then A = (μ_A, γ_A) is a generalized intuitionistic fuzzy bi-ideal of S if and only if $A \odot A \subseteq A$ and $A \odot S \odot A \subseteq A$.

Proof: The proof follows from Theorem 3.13 and Theorem 3.14.

Lemma: Let A = (μ_A, γ_A) be an intuitionistic fuzzy set in a semigroup S and B = (μ_B, γ_B) be any generalized intuitionistic fuzzy bi-ideal of S. Then

$$A \odot B = (\mu_A \odot \mu_B, \gamma_A \odot \gamma_B)$$

and

$$B \odot A = (\mu_B \odot \mu_A, \gamma_B \odot \gamma_A)$$

are both generalized intuitionistic fuzzy bi-ideals of S.

Proof: Since B = (μ_B, γ_B) is a generalized intuitionistic fuzzy bi-ideal of S, we have from Theorem 3.14

$$\mu_B \odot S \odot \mu_B \leq \mu_B \text{ and } \gamma_B \odot S' \odot \gamma_B \geq \gamma_B$$

Now

$$(\mu_A \odot \mu_B) \odot (\mu_A \odot \mu_B) \leq (\mu_A \odot \mu_B) \odot (S \odot \mu_B)$$
$$\leq \mu_A \odot (\mu_B \odot S \odot \mu_B) \leq \mu_A \odot \mu_B$$

and

$$(\gamma_A \odot \gamma_B) \odot (\gamma_A \odot \gamma_B) \geq (\gamma_A \odot \gamma_B) \odot (S' \odot \gamma_B)$$
$$\geq \gamma_A \odot (\gamma_B \odot S' \odot \gamma_B) \geq \gamma_A \odot \gamma_B$$

Hence it follows that

$$A \odot B = (\mu_A \odot \mu_B, \gamma_A \odot \gamma_B)$$

is a generalized intuitionistic fuzzy subsemigroup of S. Also we have

$$(\mu_A \odot \mu_B) \odot S \odot (\mu_A \odot \mu_B) = (\mu_A \odot \mu_B) \odot (S \odot \mu_A) \odot \mu_B$$
$$\leq (\mu_A \odot \mu_B) \odot (S \odot \mu_B)$$
$$= \mu_A \odot (\mu_B \odot S \odot \mu_B)$$
$$\leq \mu_A \odot \mu_B$$

and

$$(\gamma_A \odot \gamma_B) \odot S' \odot (\gamma_A \odot \gamma_B) = (\gamma_A \odot \gamma_B) \odot (S' \odot \gamma_A) \odot \gamma_B$$
$$\geq (\gamma_A \odot \gamma_B) \odot (S' \odot \gamma_B)$$
$$= \gamma_A \odot (\gamma_B \odot S' \odot \gamma_B)$$
$$\geq \gamma_A \odot \gamma_B$$

Thus it follows from Corollary 3.15 that $A \odot B = (\mu_A \odot \mu_B, \gamma_A \odot \gamma_B)$ is a generalized intuitionistic fuzzy bi-ideal of S. Similarly, it can be seen that $B \odot A = (\mu_B \odot \mu_A, \gamma_B \odot \gamma_A)$ is a generalized intuitionistic fuzzy bi-ideal of S.

Theorem: Every intuitionistic fuzzy left (right) ideal of a semigroup S is a generalized intuitionistic fuzzy bi-ideal of S.

Proof: Let A = (μ_A, γ_A) be an intuitionistic fuzzy left ideal of S and x,y,w∈S. Then

$$\mu_A(xwy) = \mu_A((xw)y) \geq \mu_A(y) \geq \mu_A(x) \Delta \mu_A(y)$$

because $1 \geq \mu_A(x)$ and $\mu_A(y) = \mu_A(y)$ so

$$\mu_A(y) = 1 \Delta \mu_A(y) \geq \mu_A(x) \Delta \mu_A(y)$$

Similarly

$$\gamma_A(xwy) = \gamma_A((xw)y) \leq \gamma_A(y) \leq \gamma_A(x) \nabla \gamma_A(y)$$

because $\gamma_A(y) = \gamma_A(y)$ and $0 \leq \gamma_A(x)$ so

$$\gamma_A(y) = \gamma_A(y) \nabla 0 \leq \gamma_A(y) \nabla \gamma_A(x) = \mu_A(x) \Delta \mu_A(y)$$

Similarly we can show that

$$\mu_A(xy) \geq \mu_A(x) \Delta \mu_A(y) \text{ and } \gamma_A(xy) \leq \gamma_A(x) \nabla \gamma_A(y)$$

Thus A = (μ_A, γ_A) is a generalized intuitionistic fuzzy bi-ideal of S.

Theorem: Let A = (μ_A, γ_A) be a generalized intuitionistic fuzzy bi-ideal of S. Then

(i) $A = (\mu_A, \overline{\mu}_A)$ is a generalized intuitionistic fuzzy bi-ideal of S.

(ii) $A = (\overline{\gamma}_A, \gamma_A)$ is a generalized intuitionistic fuzzy bi-ideal of S.

Proof: (i) It is sufficient to prove that $\overline{\mu}_A$ satisfies

$$\overline{\mu}_A(xy) \leq \overline{\mu}_A(x) \nabla \overline{\mu}_A(y) \text{ and } \overline{\mu}_A(xay) \leq \overline{\mu}_A(x) \nabla \overline{\mu}_A(y)$$

for all a,x,y∈S. For any a,x,y∈S, we have

$$\begin{aligned}\overline{\mu}_A(xy) &= 1 - \mu_A(xy) \leq 1 - \{\mu_A(x) \Delta \mu_A(y)\} \\ &= \eta\{\mu_A(x) \Delta \mu_A(y)\} \\ &= \eta\left[\eta\eta\{\mu_A(x)\} \Delta \eta\eta\{\mu_A(y)\}\right] \\ &= \eta\left[\eta\{\eta(\mu_A(x)\} \Delta \eta\{\eta(\mu_A(y)\}\right] \\ &= \eta\left[\eta\{1-\mu_A(x)\} \Delta \eta\{1-\mu_A(y)\}\right] \\ &= \{1-\mu_A(x)\} \nabla \{1-\mu_A(y)\} \\ &= \overline{\mu}_A(x) \nabla \overline{\mu}_A(y)\end{aligned}$$

and

$$\begin{aligned}\overline{\mu}_A(xay) &= 1 - \mu_A(xay) \leq 1 - \{\mu_A(x) \Delta \mu_A(y)\} \\ &= \eta\{\mu_A(x) \Delta \mu_A(y)\} \\ &= \eta\left[\eta\eta\{\mu_A(x)\} \Delta \eta\,\eta\{\mu_A(y)\}\right] \\ &= \eta\left[\eta\{\eta(\mu_A(x)\} \Delta \eta\{\eta(\mu_A(y)\}\right] \\ &= \eta\left[\eta\{1-\mu_A(x)\} \Delta \eta\{1-\mu_A(y)\}\right] \\ &= \{1-\mu_A(x)\} \nabla \{1-\mu_A(y)\} \\ &= \overline{\mu}_A(x) \nabla \overline{\mu}_A(y).\end{aligned}$$

Therefore A is a generalized intuitionistic fuzzy bi-ideal of S.
Similarly we can prove (ii).

Definition: A fuzzy set μ in a semigroup S is called a generalized fuzzy bi-ideal of S if

$$\mu(xy) \geq \mu(x) \Delta \mu(y) \text{ and } \mu(xwy) \geq \mu(x) \Delta \mu(y)$$

for all x,y,w∈S.

Theorem: An intuitionistic fuzzy set A = (μ_A, γ_A) in a semigroup is a generalized intuitionistic fuzzy bi-ideal of S if and only if the fuzzy sets μ_A and $\overline{\gamma}_A$ are generalized fuzzy bi-ideals of S. Where $\overline{\gamma}_A(x) = 1 - \gamma_A(x)$.

Proof: Let A = (μ_A, γ_A) be a generalized intuitionistic fuzzy bi-ideal of S. Then clearly μ_A is a generalized fuzzy bi-ideal of S. Let x,w,y∈S. Then

$$\begin{aligned}\overline{\gamma}_A(xy) &= 1 - \gamma_A(xy) \\ &\geq 1 - (\gamma_A(x) \nabla \gamma_A(y)) = \eta(\gamma_A(x) \nabla \gamma_A(y)) \\ &= \eta\left[\eta(1-\gamma_A(x)) \nabla \eta(1-\gamma_A(y))\right] \\ &= (1-\gamma_A(x)) \Delta (1-\gamma_A(y)) = \overline{\gamma}_A(x) \Delta \overline{\gamma}_A(y)\end{aligned}$$

and

$$\begin{aligned}\overline{\gamma}_A(xwy) &= 1 - \gamma_A(xwy) \\ &\geq 1 - \gamma_A(x) \nabla \gamma_A(y) = \eta(\gamma_A(x) \nabla \gamma_A(y)) \\ &= \eta\left[\eta(1-\gamma_A(x)) \nabla \eta(1-\gamma_A(y))\right] \\ &= (1-\gamma_A(x)) \Delta (1-\gamma_A(y)) = \overline{\gamma}_A(x) \Delta \overline{\gamma}_A(y)\end{aligned}$$

Hence $\mu_A, \overline{\gamma}_A$ are generalized fuzzy bi-ideals of S.

Conversely, suppose that μ_A and $\overline{\gamma}_A$ are generalized fuzzy bi-ideals of S. Let w,x,y∈S. Then

$$\begin{aligned}1 - \gamma_A(xy) &= \overline{\gamma}_A(xy) \geq \overline{\gamma}_A(x) \Delta \overline{\gamma}_A(y) \\ &= (1-\gamma_A(x)) \Delta (1-\gamma_A(y)) \\ &= \eta\left[\eta(1-\gamma_A(x)) \nabla \eta(1-\gamma_A(y))\right] \\ &= \eta(\gamma_A(x) \nabla \gamma_A(y))\end{aligned}$$

Thus

$$1-\gamma_A(xy) \geq 1-\left(\gamma_A(x) \nabla \gamma_A(y)\right)$$

which implies that

$$\gamma_A(xy) \leq \gamma_A(x) \nabla \gamma_A(y)$$

and

$$\begin{aligned} 1-\gamma_A(xwy) &= \overline{\gamma}_A(xwy) \geq \overline{\gamma}_A(x) \Delta \overline{\gamma}_A(y) \\ &= (1-\gamma_A(x)) \Delta (1-\gamma_A(y)) \\ &= \eta\left[\eta(1-\gamma_A(x)) \nabla \eta\left(1-\gamma_A(y)\right)\right] \\ &= \eta\left(\gamma_A(x) \nabla \gamma_A(y)\right) \end{aligned}$$

Thus

$$1-\gamma_A(xwy) \geq 1-\left(\gamma_A(x) \nabla \gamma_A(y)\right)$$

which implies that

$$\gamma_A(xwy) \leq \gamma_A(x) \nabla \gamma_A(y)$$

Therefore $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy bi-ideal of S.

Corollary: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy bi-ideal of S if and only if $A = (\mu_A, \overline{\mu}_A)$ and $A = (\overline{\gamma}_A, \gamma_A)$ are generalized intuitionistic fuzzy bi-ideals of S.

Proof: Proof follows from Theorem 3.20.

Definition: A generalized intuitionistic fuzzy subsemigroup $A = (\mu_A, \gamma_A)$ of a semigroup S is called a generalized intuitionistic fuzzy (1,2) ideal of S if

$$\mu_A(xw(yz)) \geq \mu_A(x) \Delta \left\{\mu_A(y) \Delta \mu_A(z)\right\}$$

and

$$\gamma_A(xw(yz)) \leq \gamma_A(x) \nabla \left\{\gamma_A(y) \nabla \gamma_A(z)\right\}$$

for all $x,y,z,w \in S$.

Theorem: Every generalized intuitionistic fuzzy bi-ideal of a semigroup S is a generalized intuitionistic fuzzy (1,2) ideal.

Proof: Let $A = (\mu_A, \gamma_A)$ be a generalized intuitionistic fuzzy bi-ideal of S and let $w,x,y,z \in S$. Then

$$\begin{aligned} \mu_A(xw(yz)) &= \mu_A((xwy)z) \geq \mu_A(xwy) \Delta \mu_A(z) \\ &\geq \left\{\mu_A(x) \Delta \mu_A(y)\right\} \Delta \mu_A(z) \\ &= \mu_A(x) \Delta \left\{\mu_A(y) \Delta \mu_A(z)\right\} \end{aligned}$$

and

$$\begin{aligned} \gamma_A(xw(yz)) &= \gamma_A((xwy)z) \leq \gamma_A(xwy) \nabla \gamma_A(z) \\ &\leq \left\{\gamma_A(x) \nabla \gamma_A(y)\right\} \nabla \gamma_A(z) \\ &= \gamma_A(x) \nabla \left\{\gamma_A(y) \nabla \gamma_A(z)\right\} \end{aligned}$$

Hence $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy (1,2) ideal of S.

Theorem: A non-empty subset A of a semigroup S is a (1,2) ideal of S if and only if the intuitionistic characteristic function $\ddot{A} = (\Phi_A, \Psi_A)$ of A is a generalized intuitionistic fuzzy (1,2) ideal of S.

Proof: Proof follows from Theorem 3.11.

Theorem: Let $A = (\mu_A, \gamma_A)$ be an intuitionistic fuzzy set in a semigroup S. Then $A = (\mu_A, \gamma_A)$ is a generalized intuitionistic fuzzy (1,2) ideal of S if and only if $A \odot A \subseteq A$ and $A \odot S \odot (A \odot A) \subseteq A$, that is

$$\mu_A \odot S \odot \mu_A \leq \mu_A \text{ and } \gamma_A \odot S' \odot \gamma_A \geq \gamma_A$$

Proof: Proof follows from Theorem 3.13 and Theorem 3.14.

Theorem: Let $A = (\mu_A, \gamma_A)$ be a generalized intuitionistic fuzzy (1,2) ideal of S. Then

(i) $A = (\mu_A, \overline{\mu}_A)$ is a generalized intuitionistic fuzzy (1,2) ideal of S.
(ii) $A = (\overline{\gamma}_A, \gamma_A)$ is a generalized intuitionistic fuzzy (1,2) ideal of S.

Proof: Proof follows from Theorem 3.18.

Definition: A fuzzy subsemigroup μ of a semigroup S is called a generalized fuzzy (1,2) ideal of S if

$$\mu_A(xw(yz)) \geq \mu_A(x) \Delta \left\{\mu_A(y) \Delta \mu_A(z)\right\}$$

for all $x,y,z,w \in S$.

Theorem: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is a generalized intuitionistic fuzzy (1,2) ideal of S if and only if the fuzzy sets μ_A and $\overline{\gamma}_A$ are generalized fuzzy (1,2) ideals of S. Where $\overline{\gamma}_A(x) = 1-\gamma_A(x)$.

Proof: Proof follows from Theorem 3.20.

Corollary: An intuitionistic fuzzy set $A = (\mu_A, \gamma_A)$ in a semigroup S is a generalized intuitionistic fuzzy (1,2) ideal of S if and only if $A = (\mu_A, \overline{\mu}_A)$ and $A = (\overline{\gamma}_A, \gamma_A)$ are generalized intuitionistic fuzzy (1,2) ideals of S.

Proof: The Proof follows from the Theorem 3.20.

Theorem: Let A = (μ_A, γ_A) be a generalized intuitionistic fuzzy bi-ideal (respectively (1,2) ideal) of S. Then for each $t \in [0,1]$;

(a) if t = 1, then the upper level set $U(\mu_A : t)$ is either empty or a bi-ideal (respectively (1,2) ideal) of S.
(b) if t = 0, then the lower level set $L(\gamma_A : t)$ is either empty or a bi-ideal (respectively (1,2) ideal) of S.

Proof:

(a) Let $x, y \in U(\mu_A : 1)$. Then $\mu_A(x) = 1 = \mu_A(y)$. Since

$$\mu_A(xy) \geq \mu_A(x) \Delta \mu_A(y) = 1 \Delta 1 = 1$$

we have $xy \in U(\mu_A : 1)$. Now let $w \in S$ and $x, y \in U(\mu_A : 1)$. Then

$$\mu_A(xwy) \geq \mu_A(x) \Delta \mu_A(y) = 1 \Delta 1 = 1$$

Hence $xwy \in U(\mu_A : 1)$. Hence $U(\mu_A : t)$ is a bi-ideal of S.

(b) Let $x, y \in L(\gamma_A : 0)$. Then $\gamma_A(x) = 0 = \gamma_A(y)$. Since

$$\gamma_A(xy) \leq \gamma_A(x) \nabla \gamma_A(y) = 0 \nabla 0 = 0$$

we have $xy \in L(\gamma_A : 0)$. Now let $w \in S$ and $x, y \in L(\gamma_A : t)$. Then,

$$\gamma_A(xwy) \leq \gamma_A(x) \nabla \gamma_A(y) = 0 \nabla 0 = 0$$

Hence $xwy \in L(\gamma_A : 0)$. Hence $L(\gamma_A : t)$ is a bi-ideal of S.

Similarly we can prove these result for (1,2) ideal of S.

Theorem: Let A = (μ_A, γ_A) be an intuitionistic fuzzy set in a semigroup S such that the non-empty sets $U(\mu_A : t)$ and $L(\gamma_A : t)$ are bi-ideals of S for all $t \in [0,1]$. Then A = (μ_A, γ_A) is a generalized intuitionistic fuzzy bi-ideal of S.

Proof: Assume that the non-empty sets $U(\mu_A : t)$ and $L(\gamma_A : t)$ are bi-ideals of S for all $t \in [0,1]$. Suppose $x, y \in S$ such that

$$\mu_A(xy) < \mu_A(x) \Delta \mu_A(y) \text{ or } \gamma_A(xy) > \gamma_A(x) \nabla \gamma_A(y)$$

Taking

$$t' = \tfrac{1}{2}(\mu_A(xy) + \mu_A(x) \Delta \mu_A(y))$$

and

$$t'' = \tfrac{1}{2}(\gamma_A(xy) + \gamma_A(x) \nabla \gamma_A(y))$$

then $t_1, t_1' \in (0,1]$, such that

$$\mu_A(xy) < t' < \mu_A(x) \Delta \mu_A(y) \leq \mu_A(x) \wedge \mu_A(y)$$

or

$$\gamma_A(x) \vee \gamma_A(y) \leq \gamma_A(x) \nabla \gamma_A(y) < t'' < \gamma_A(xy)$$

This implies that $\mu_A(x) > t'$, $\mu_A(y) > t'$ or $\gamma_A(x) < t''$, $\gamma_A(y) < t''$. Thus $x, y \in U(\mu_A : t')$ or $x, y \in L(\gamma_A : t'')$ but $xy \notin U(\mu_A : t')$ or $xy \notin L(\gamma_A : t'')$. This shows that $U(\mu_A : t')$ or $L(\gamma_A : t'')$ is not a subsemigroup of S. Which is a contradiction. Hence A = (μ_A, γ_A) satisfies the inequality

$$\mu_A(xy) \geq \mu_A(x) \Delta \mu_A(y) \text{ and } \gamma_A(xy) \leq \gamma_A(x) \nabla \gamma_A(y)$$

for all $x, y \in S$. Similarly, for $w, x, y \in S$ it satisfies

$$\mu_A(xwy) \geq \mu_A(x) \Delta \mu_A(y) \text{ and } \gamma_A(xwy) \leq \gamma_A(x) \nabla \gamma_A(y)$$

Hence A = (μ_A, γ_A) is a generalized intuitionistic fuzzy bi-ideal of S.

REGULAR SEMIGROUPS

In this section we characterize regular semigroups by the properties of their generalized intuitionistic fuzzy ideals and generalized intuitionistic fuzzy bi-ideals.

Definition: Let S be a semigroup and $a \in S$. Then a is called regular if there exists an element x in S such that $a = axa$. S is called regular if every element of S is regular.
It is well known that

Theorem: A semigroup S is regular if and only if $R \cap L = RL$ for every right ideal R and left ideal L of S.

Lemma: Let A and B be non empty subsets of a semigroup S. Then

(i) $\Phi_A \Delta \Phi_B = \Phi_{A \cap B}$ and $\Psi_A \nabla \Psi_B = \Psi_{A \cap B}$

and

(ii) $\Phi_A \odot \Phi_B = \Phi_{AB}$ and $\Psi_A \odot \Psi_B = \Psi_{AB}$

where $\ddot{A} = (\Phi_A, \Psi_A)$ and $\ddot{B} = (\Phi_B, \Psi_B)$ are the intuitionistic characteristic functions of A and B, respectively.

Proof: (i) Let A and B be any non empty subsets of S and $a \in S$. Then

$$(\Phi_A \Delta \Phi_B)(a) = \Phi_A(a) \Delta \Phi_B(a)$$
$$= \begin{cases} 1 & \text{if } a \in A \text{ and } a \in B \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 & \text{if } a \in A \cap B \\ 0 & \text{if } a \notin A \cap B \end{cases} = \Phi_{A \cap B}(a)$$

and

$$(\Psi_A \nabla \Psi_B)(a) = \Psi_A(a) \nabla \Psi_B(a)$$
$$= \begin{cases} 0 & \text{if } a \in A \text{ and } a \in B \\ 1 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0 & \text{if } a \in A \cap B \\ 1 & \text{if } a \notin A \cap B \end{cases} = \Psi_{A \cap B}(a)$$

(ii) Let $a \in AB$. Then $a = xy$ for some $x \in A$ and $y \in B$. Thus

$$(\Phi_A \odot \Phi_B)(a) = \bigvee_{a=pq} \left(\Phi_A(p) \Delta \Phi_B(q)\right)$$
$$\geq \left(\Phi_A(x) \Delta \Phi_B(y)\right) = 1 \Delta 1 = 1$$

This implies

$$(\Phi_A \odot \Phi_B)(a) = 1 = \Phi_{AB}(a)$$

Similarly

$$(\Psi_A \odot \Psi_B)(a) = \bigwedge_{a=pq} \left(\Psi_A(p) \nabla \Psi_B(q)\right)$$
$$\leq \left(\Psi_A(x) \nabla \Psi_B(y)\right) = 0 \nabla 0 = 0$$

This implies

$$(\Psi_A \odot \Psi_B)(a) = 0 = \Psi_{AB}(a)$$

If $a \notin AB$, we have $a\ z \neq xy$ for all $x \in A$ and $y \in B$. If $a = uv$ for some $u, v \in S$, then we have

$$(\Phi_A \odot \Phi_B)(a) = \bigvee_{a=uv} \left(\Phi_A(u) \Delta \Phi_B(v)\right) = 0 = \Phi_{AB}(a)$$

and

$$(\Psi_A \odot \Psi_B)(a) = \bigwedge_{a=uv} \left(\Psi_A(u) \nabla \Psi_B(v)\right) = 1 = \Psi_{AB}(a)$$

If $a \neq uv$ for all $u, v \in S$, then

$$(\Phi_A \odot \Phi_B)(a) = 0 = \Phi_{AB}(a)$$

and

$$(\Psi_A \odot \Psi_B)(a) = 1 = \Psi_{AB}(a)$$

In any case, we have

$$(\Phi_A \odot \Phi_B)(a) = \Phi_{AB}(a)$$

and

$$(\Psi_A \odot \Psi_B)(a) = \Psi_{AB}(a)$$

Theorem: For a semigroup S, the following conditions are equivalent

(1) S is regular.
(2) $\mu_A \Delta \mu_B \leq \mu_A \odot \mu_B$ and $\gamma_A \nabla \gamma_B \geq \gamma_A \odot \gamma_B$

for every generalized intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy left ideal $B = (\mu_B, \gamma_B)$ of S.

Proof: First assume that (1) holds. Let a be any element of S. Then, there exists an element $x \in S$ such that $a = axa$. Hence we have

$$(\mu_A \odot \mu_B)(a) = \bigvee_{a=pq} \{\mu_A(p) \Delta \mu_B(q)\} \geq \mu_A(ax) \Delta \mu_B(a)$$
$$\geq \mu_A(a) \Delta \mu_B(a) \geq (\mu_A \Delta \mu_B)(a)$$

and

$$(\gamma_A \odot \gamma_B)(a) = \bigwedge_{a=pq} \{\gamma_A(p) \nabla \gamma_B(q)\} \leq \gamma_A(ax) \nabla \gamma_B(a)$$
$$\leq \gamma_A(a) \nabla \gamma_B(a) \leq (\gamma_A \nabla \gamma_B)(a)$$

Therefore

$$\mu_A \Delta \mu_B \leq \mu_A \odot \mu_B$$

and

$$\gamma_A \nabla \gamma_B \geq \gamma_A \odot \gamma_B .$$

Conversely, assume that (2) holds. Let R and L be any right and left ideal of S, respectively. Then intuitionistic characteristic function $\ddot{R} = (\Phi_R, \Psi_R)$ and $\ddot{L} = (\Phi_L, \Psi_L)$ of R and L are generalized intuitionistic fuzzy right ideal and generalized intuitionistic fuzzy left ideal of S, respectively. Thus by hypotheses

$$\Phi_R \Delta \Phi_L \leq \Phi_R \odot \Phi_L \text{ and } \Psi_R \nabla \Psi_L \geq \Psi_R \odot \Psi_L$$

By Lemma 4.3, this implies that

$$\Phi_{R\cap L} \leq \Phi_{RL} \text{ and } \Psi_{R\cap L} \geq \Psi_{RL}$$

Hence $R\cap L\subseteq RL$. But $RL\subseteq R\cap L$ always holds. This implies $R\cap L = RL$, that is S is regular.

Theorem: For a semigroup S, the following conditions are equivalent

(1) S is regular.
(2) $\mu_A \Delta \mu_B \leq \mu_A \odot \mu_B$ and $\gamma_A \nabla \gamma_B \geq \gamma_A \odot \gamma_B$ for every generalized intuitionistic fuzzy bi-ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy left ideal $B = (\mu_B, \gamma_B)$ of S.
(3) $\mu_A \Delta \mu_B \leq \mu_A \odot \mu_B$ and $\gamma_A \nabla \gamma_B \geq \gamma_A \odot \gamma_B$ for every generalized intuitionistic fuzzy generalized bi-ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy left ideal $B = (\mu_B, \gamma_B)$ of S.

Proof: Assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any generalized intuitionistic fuzzy generalized bi-ideal and generalized intuitionistic fuzzy left ideal of S respectively. Let a be any element of S. Since S is regular, there exists an element $x\in S$ such that $a = axa$. Thus we have

$$(\mu_A \odot \mu_B)(a) = \bigvee_{a=pq}\{\mu_A(p)\Delta\mu_B(q)\} \geq \mu_A(a)\Delta\mu_B(xa)$$
$$\geq \mu_A(a)\Delta\mu_B(a) \geq (\mu_A\Delta\mu_B)(a)$$

and

$$(\gamma_A \odot \gamma_B)(a) = \bigwedge_{a=pq}\{\gamma_A(p)\nabla\gamma_B(q)\} \leq \gamma_A(a)\nabla\gamma_B(xa)$$
$$\leq \gamma_A(a)\nabla\gamma_B(a) \leq (\gamma_A\nabla\gamma_B)(a)$$

and so $\mu_A\Delta\mu_B \leq \mu_A \odot \mu_B$ and $\gamma_A \nabla \gamma_B \geq \gamma_A \odot \gamma_B$. Hence we obtain that (1) implies (3).
It is clear that (3)$\Rightarrow$(2).

Since every generalized intuitionistic fuzzy right ideal of S is a generalized intuitionistic fuzzy bi-ideal of S, So

$$\mu_A\Delta\mu_B \leq \mu_A \odot \mu_B \text{ and } \gamma_A \nabla \gamma_B \geq \gamma_A \odot \gamma_B$$

for every generalized intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy left ideal $B = (\mu_B, \gamma_B)$ of S. Thus it follows from the Theorem 4.4, S is regular.

Theorem: For a semigroup S, the following conditions are equivalent

(1) S is regular.
(2) $\mu_A \Delta \mu_B \leq \mu_B \odot \mu_A$ and $\gamma_A \nabla \gamma_B \geq \gamma_B \odot \gamma_A$ for every generalized intuitionistic fuzzy bi-ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy right ideal $B = (\mu_B, \gamma_B)$ of S.
(3) $\mu_A \Delta \mu_B \leq \mu_B \odot \mu_A$ and $\gamma_A \nabla \gamma_B \geq \gamma_B \odot \gamma_A$ for every generalized intuitionistic fuzzy generalized bi-ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy right ideal $B = (\mu_B, \gamma_B)$ of S.

Proof: The proof follows from Theorem 4.5.

Theorem: For a semigroup S, the following conditions are equivalent

(1) S is regular.
(2) $\mu_A \Delta \mu_B \Delta \mu_C \leq \mu_A \odot \mu_B \odot \mu_C$ and $\gamma_A \nabla \gamma_B \nabla \gamma_C \geq \gamma_A \odot \gamma_B \odot \gamma_C$ for every generalized intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$, every generalized intuitionistic fuzzy bi-ideal $B = (\mu_B, \gamma_B)$ and every generalized intuitionistic fuzzy left ideal $C = (\mu_C, \gamma_C)$ of S.
(3) $\mu_A \Delta \mu_B \Delta \mu_C \leq \mu_A \odot \mu_B \odot \mu_C$ and $\gamma_A \nabla \gamma_B \nabla \gamma_C \geq \gamma_A \odot \gamma_B \odot \gamma_C$ for every generalized intuitionistic fuzzy right ideal $A = (\mu_A, \gamma_A)$, every generalized intuitionistic fuzzy generalized bi-ideal $B = (\mu_B, \gamma_B)$ and every generalized intuitionistic fuzzy left ideal $C = (\mu_C, \gamma_C)$ of S.

Proof: First assume that (1) holds. Let $A = (\mu_A, \gamma_A)$, $B = (\mu_B, \gamma_B)$ and $C = (\mu_C, \gamma_C)$ be any generalized intuitionistic fuzzy right ideal, generalized intuitionistic fuzzy generalized bi-ideal and generalized intuitionistic fuzzy left ideal of S, respectively. Let a be any element of S. Since S is regular, there exists an element $x\in S$ such that $a = axa$. Thus we have

$$(\mu_A \odot \mu_B \odot \mu_C)(a) = \bigvee_{a=yz}\{\mu_A(y)\Delta(\mu_B \odot \mu_C)(a)\}$$
$$\geq \mu_A(ax)\Delta(\mu_B \odot \mu_C)(a)$$
$$\geq \mu_A(a)\Delta\left\{\bigvee_{a=pq}\{\mu_B(p)\Delta\mu_C(q)\}\right\}$$
$$\geq \mu_A(a)\Delta\{\mu_B(a)\Delta\mu_C(xa)\}$$
$$\geq \mu_A(a)\Delta\{\mu_B(a)\Delta\mu_C(a)\}$$
$$= (\mu_A\Delta\mu_B\Delta\mu_C)(a)$$

and

$$(\gamma_A \odot \gamma_B \odot \gamma_C)(a) = \bigwedge_{a=yz}\{\gamma_A(p)\nabla(\gamma_B \odot \gamma_C)(z)\}$$
$$\leq \gamma_A(ax)\nabla(\gamma_B \odot \gamma_C)(a)$$
$$\leq \gamma_A(a)\nabla\left\{\bigwedge_{a=pq}\{\gamma_B(p)\nabla\gamma_C(q)\}\right\}$$
$$\leq \gamma_A(a)\nabla\{\gamma_B(a)\nabla\gamma_C(xa)\}$$
$$\leq \gamma_A(a)\nabla\{\gamma_B(a)\nabla\gamma_C(a)\}$$
$$\leq (\gamma_A\nabla\gamma_B\nabla\gamma_C)(a)$$

and so

$$\mu_A \Delta \mu_B \Delta \mu_C \le \mu_A \odot \mu_B \odot \mu_C$$

and

$$\gamma_A \nabla \gamma_B \nabla \gamma_C \ge \gamma_A \odot \gamma_B \odot \gamma_C$$

Hence we obtain that (1) implies (3).

(3)⇒(2) straight forward.

(2)⇒(1)

Let $R = (\mu_R, \gamma_R)$ and $L = (\mu_L, \gamma_L)$ be any generalized intuitionistic fuzzy right ideal and any generalized intuitionistic fuzzy left ideal of S, respectively. Since $S = (S, S)$ is a generalized intuitionistic fuzzy bi-ideal of S, by assumption, we have

$$\mu_R \Delta \mu_L = \mu_R \Delta 1 \Delta \mu_L = \mu_R \Delta S \Delta \mu_L \le \mu_R \odot S \odot \mu_L \le \mu_R \odot \mu_L$$

and

$$\gamma_R \Delta \gamma_L = \gamma_R \nabla 0 \nabla \gamma_L$$
$$= \gamma_R \nabla S' \nabla \gamma_L \ge \gamma_R \odot S' \odot \gamma_L \ge \gamma_R \odot \gamma_L$$

Thus it follows that

$$\mu_R \Delta \mu_L \le \mu_R \odot \mu_L \text{ and } \gamma_R \nabla \gamma_L \ge \gamma_R \odot \gamma_L$$

for every generalized intuitionistic fuzzy right ideal $R = (\mu_R, \gamma_R)$ and every generalized intuitionistic fuzzy left ideal $L = (\mu_L, \gamma_L)$ of S. Thus it follows from Theorem 4.4, S is regular.

INTRA-REGULAR SEMIGROUPS

In this section we characterize intra-regular semigroups by the properties of their generalized intuitionistic fuzzy ideals.

Definition: A semigroup S is called intra-regular semigroup, if for each $a \in S$ there exist $x, y \in S$ such that $a = xa^2y$.

Theorem: The following conditions are equivalent for a semigroup S

(1) is intra-regular.

(2) $L \cap R \subseteq LR$ for every left ideal L and every right ideal R of S.

(3) $\mu_A \Delta \mu_B \le \mu_A \odot \mu_B$, $\gamma_A \Delta \gamma_B \ge \gamma_A \odot \gamma_B$ for every generalized intuitionistic fuzzy left ideal $A = (\mu_A, \gamma_A)$ and every generalized intuitionistic fuzzy right ideal $B = (\mu_B, \gamma_B)$ of S.

Proof: (1)⇔(2) Well known.

(1)⇒(3)

First assume that (1) holds. Let $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ be any generalized intuitionistic fuzzy left ideal and generalized intuitionistic fuzzy right ideal of S, respectively. Let a be any element of S. Since S is intra-regular, there exist elements $x, y \in S$ such that $a = xa^2y$. Hence

$$(\mu_A \odot \mu_B)(a) = \bigvee_{a=pq} \{\mu_A(p) \Delta \mu_B(q)\}$$
$$\ge \mu_A(xa) \Delta \mu_B(ay)$$
$$\ge \mu_A(a) \Delta \mu_B(a) \ge (\mu_A \Delta \mu_B)(a)$$

and

$$(\gamma_A \odot \gamma_B)(a) = \bigwedge_{a=pq} \{\gamma_A(p) \nabla \gamma_B(q)\}$$
$$\le \gamma_A(xa) \nabla \gamma_B(ay)$$
$$\le \gamma_A(a) \nabla \gamma_B(a) \le (\gamma_A \nabla \gamma_B)(a)$$

So $\mu_A \Delta \mu_B \le \mu_A \odot \mu_B$ and $\gamma_A \Delta \gamma_B \ge \gamma_A \odot \gamma_B$. Thus (1) implies (3).

(3)⇒(2)

Next assume that (3) holds. Let L and R be any left ideal and any right ideal of S, respectively. Then intuitionistic characteristic functions (Φ_L, Ψ_L) and (Φ_R, Ψ_R) are generalized intuitionistic fuzzy left ideal and generalized intuitionistic fuzzy right ideal of S, respectively. Thus by hypotheses

$$\Phi_L \Delta \Phi_R \le \Phi_L \odot \Phi_R \text{ and } \Psi_L \nabla \Psi_R \ge \Psi_L \odot \Psi_R$$

By Lemma 4.3, this implies that

$$\Phi_{L \cap R} \le \Phi_{LR} \text{ and } \Psi_{L \cap R} \ge \Psi_{LR}$$

Thus $L \cap R \subseteq LR$ for every left ideal L and every right ideal R of S. So (3) implies (2).

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