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Downward Sets and their Topological Properties

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Abstract: In this paper, we study downward sets properties and their similarities to the convex sets in a topological vector space. Let G is a downward subset of ordered topological vector space X. we prove that if G be a open downward subset then it's downward hull set is open. We give a characterization for downward sets and get some results about it's closedness, compactness and it's convex hull sets.We show correspondences between concepts of monotonic analysis, like downward sets and their upper extreme points and similar concepts of convex analysis, that is, convex sets and their extreme points.

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INTRODUCTION

Because of their nature and applications, increasing functions have found an important role in optimization research and this research has led to a new branch of mathematics, namely, "Monotonic Analysis." Recent works [2-6, 15] show close relations between monotonic analysis and convex analysis. Tuy [15] has shown some similarities between the concepts of monotonic and convex analysis in R^n: downward vs. convex sets, increasing vs. convex functions, upper extreme points of downward sets vs. extreme points of convex sets and so forth.

Downward sets and increasing functions in a topological vector space and their similarities to the convex sets and convex functions was studied by Mohebi [7]. He characterized closed downward sets by upper boundary and upper exterme points. In section 3, he proved some topological properties of downward sets.

The role of downward sets for characterization of best approximation and best simultaneous approximation and separation properties can be found in [l, 9].

In this paper, we use some of these concepts and also give some more new analysis property of downward sets. To make an order on a topological vector space, we use a closed convex pointed cone K, usually with $intK \neq 0$.

PRELIMINARIES

Let X be a (Hausdorff) topological vector space. We assume that X is equipped with a closed convex pointed cone K in X(the latter means that $Kn\,(-K) = \{O\}$).

We say $x \le y$ or $y \ge x$ if and only if $y-x \in K$.

The topological vector space (X, τ) equipped with the partial order$\le$will be called an ordered topological vector space, sometimes represented by (X, τ, $\le$). Let $u \in intK$. Using u we define the function p_u as follows:

$$P_u(x) := \inf\{\lambda \in R : x \le \lambda u\} \quad (x \in X)$$

Rubinov and Gasimov have investigated the properties of this function in a more general case in [11]. For simplicity we refer to p_u as the p-function.
The following assertions can be easily verified:

(i) p_u is finite,
(ii) $x \le p_u(x)u$ for all $x \in X$,
(iii) p_u is sublinear,
(iv) p_u is continuous,
(v) p_u is topical, namely, p_u is increasing ($x \le y$ = $p_u(x) \le p_u(y)$) and $p_u(x+\lambda u) = p_u(x)+\lambda$ for all $x \in X$ and all $\lambda \in R$.

Now, consider the function

$$\|x\| := \max(p_u(x), p_u(-x)) \qquad (2.1)$$

It is well known [13] that $\|.\|$ is a norm on X. We assume that the norm in (2.1) generates a topology on X that coincides with the original topology of X. In this case, X = (X, $\|.\|$, $\le$) is an ordered normed space. In many applications, these two topologies coincide. Here, we give a simple example [10]. By R^n_+, we mean the

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cone of all n-tuples $(x_1,x_2,...,i_n)$ in R^n such that $x_i \geq 0, 1\leq i\leq n$.

Example 2.1: Let $X = R^n$, $u = 1:= (1,1,...,1)\in R^n$, $K = R^n_+$. If $x = (x_i,x_2,...,x_n)\in R^n$ then $p(x) = \max_{1\leq i\leq n} xi$ and $p(-x) = \max_{i\leq i\leq n}(-x_i)$. In this case, $\|x\|= \max_{i\leq i\leq n}|x_i|$ and the topology induced by the norm is the well-known standard topology on X.
The following definitions can be found in [9, 12, 14].

Definition 2.2: let X be a ordered topological vector space.A set $G\subset X$ is called a downward set if for any two points $x, x'\in X$, $x'\leq x$ and $x\in G$, then $x'\in G$.

The empty set and X are special downward sets in X. It is well known that the intersection and union of a family of downward sets are downward.

For every set $G\subset X$, the whole space X is a downward set containing G and therefore the following concept is well defined.

Definition 2.3: The intersection of all downward sets containing G is called the downward hull of G and denoted by G_*.

An immediate consequence of definition 2.3 is that the downward hull of any set $G\subset X=(X,\tau,\leq)$ is downward. Clearly G_* is the least (by inclusion) downward set, which contain G.
The following proposition has been proved in [7, 14].

Proposition 2.4: The downward hull of a set $G\subset X = (X,\tau,\leq)$ is the set $G_* = G-K = U y\in G\{x: x\leq y\}$,where $G-K:= \{g-k: g\in G, k\in K\}$.

In the following, we give an example of downward hull of a set.

Example 2.5: Let $X:= R^2$, $K:= R^n_+$ and

$$G:= \{(x, y)\in X:-1\leq x\leq 1, 0\leq y\leq 1)$$

Therefore, G is a closed and bounded (compact) subset of X. It is clear that G is not downward. In view of proposition 2.4 we conclude that the downward hull G of the set G_* is the following set:

$$G_* = \{(x,y)\in X:x\leq 2,y\leq 1\}$$

Proposition 2.6: [7] The downward hull G_* of a connected set $G\subseteq X = (x,\tau,\leq)$ is connected.

Mohebi and Kermani [7] showed that every downward set in ordered normed spaces is connected. They showed, also, G_* is closed, whenever G is a compact subset and if G is an upper bounded subset of $(X,\tau,\leq)$, then so is G_*.

Throughout this paper, we assume that the topology induced by $\|.\|$ in (2.1) coincides with the original topology of X. By $(X,\tau,\leq)$we mean the ordered topological vector space X, which is equipped with a closed solid pointed convex cone K, While $(X, \|.\|, \leq)$ represents the ordered normed space X, which enjoys the norm in (2.1).

TOPOLOGICAL PROPERTIES OF DOWNWARD SETS

Now, we are ready to state and prove our main results.

Lemma 3.1: If G be an open subset of $X = (X,\tau,\leq)$, then so is G_*.

Proof: According to proposition 2.4 we have $G_* = G-K = \cup\{G-k:k\in K\}$. For each $k\in K$, $G-k$ is open. Then $\cup\{G-k: k\in K]$ is open and therefore G_* is open.

Lemma 3.2: If G be a weakly compact subset of $X = (X,\tau,\leq)$, then G_* is closed.

Proof: Let $\{x_n\}$ be a sequence in G and $x_n\to x$ (in norm). By proposition 2.4, there are sequences of $\{g_n\}$ in G and $\{k_n\}$ in K such that $x_n = g_n-k_n\to x$ (in norm). Since G is weakly compact, so there is subsequence $\{g_{nk}\}$ such that it is weakly convergence to an element of G as g.Hence for each $f\in X^*$, dual space of X, we have:

$$f(g_n-k_n) = f(g_n)-f(k_n)\to f(x)$$

Also

$$f(g_{nk})\to f(g)$$

Therefore

$$f(k_n)\to f(g)-f(x)=f(g-x)$$

Hence

$$k_n\to g_-x \text{ (weakly)}$$

But, K is convex and closed so K = clA = wk-clK. Therefore $g-x\in K$ and $x = g-(g-x)$. That is $x\in G_*$.

Lemma 3.3: Let G be a subset of $X = (x,\tau,\leq)$. Then G is convex if and only if G_* is convex.

Proof: Let G is convex. Then, by proposition 2.4, $G_* = G-K$ is convex.

Conversely, Let G_* is convex and $g_1, g_2\in G$, $0\leq\lambda\leq 1$. Thus for $k_0\in k$, g_1-k_o and $g_2-k_o\in G_*$. Hence

$$\lambda(g_1-k_o)+(1-\lambda)(g_2-k_o) = \lambda g_1+(1-\lambda)g_2-k_0\in G_*$$

Therefore

$$\lambda g_1+(1-\lambda)g_2 = [\lambda g_1+(1-\lambda) g_2-k_o]+k_o \in G$$

Lemma 3.4: Let G be a subset of X = (X,τ,≤) and G, is compact, then G is compact.

Proof: Let $\{g_n\}\subseteq G$ and $k_0\in K$ Then, by proposition 2.4, $\{g_n-k_o\}\subseteq G_*$. Hence there is a subsequence as $\{g_{nk}-k_0\}$ which is convergence to g in G_*. Therefore $\{g_{nk}\}$ is convergence to $g+k_0\in G$.

Lemma 3.5: let W is a downward set of X = (X,τ,≤). Then co(W), convex hull W, is downward.

Proof: Let $x\in co(W)$, $y\in X$ and $y\leq x$. By the definition of convex hull, there are some $0\leq\alpha_i\leq 1$ and $x_i\in W, i = 1, \ldots, n$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

Therefore

$$y \leq \sum_{i=1}^{n} \alpha_i x_i$$

Let $\alpha_j\neq 0$, then we have

$$\frac{y}{\alpha_j} \quad \frac{\sum_{i\neq j}^{n} \alpha_i x_i}{\alpha_j} \leq x_j.$$

Now, because $x_j\in W$ and W is downward set, it follow that

$$\frac{y}{\alpha_j} \quad \frac{\sum_{i\neq j}^{n} \alpha_i x}{\alpha_j} \in W.$$

and hence by definition of convex hull we have

$$\alpha_j\left(\frac{y}{\alpha_j} - \frac{\sum_{i\neq j}^{n} \alpha_i x_i}{\alpha_j} \leq x_i\right) = y - \sum_{i\neq j}^{n} \alpha_i x_i \in co(W)$$

Therefore there exists $y_1, \ldots y_m\in W$ and $0\leq\beta_d\leq 1$, d=1,…m such that

$$y - \sum_{i\neq j}^{n} \alpha_i x_i - \sum_{d=1}^{m} \beta_d y_d$$

Therefore

$$y - \sum_{i\neq j}^{n} \alpha_i x_i + \sum_{d=1}^{m} \beta_d y_d$$

That is $y\in W$ and the proof is complete.

Recall that a component of X is, by definition, a maximal connected subset of X.

Lemma 3.6: Let W is downward set of X. Then the component of X which contains W is also downward set.

Proof: Let C_w be the component of X which is contains W and C_{w*} be downward hull of C_w. By proposition 2.6 C_w is connected and contains W. Therefore $C_w\subseteq C_{w*}\subseteq C_w$. Hence $C_w = C_{w*}$ and C_w is downward.

Definition 3.7: Let X is a topological vector space and K is a closed pointed convex cone in X such that intK ≠0.
Let

$$K_y^0 = int(y+K) = y+intK \text{ for all } y\in X.$$

We say that a point $x\in X$ is an upper boundary point of a set $G\subseteq X$ if $x\in \overline{G}$ while $K_y^0\subseteq X\backslash G$. The set of upper boundary points of G is called the upper boundary of G and denoted by ∂^+G.

Let $D\subseteq X$. A point $v\in D$ is called an upper extreme point of D if $x\in D$ and $x\geq v$ imply $x = v$.

Obviously, $\partial^+G\subseteq G$ if G is closed. Also, every upper extreme point of a downward set G satisfies $K_y^0\subseteq X\backslash G$ and therefore is an upper boundary point of G. In other words, the set V of all upper extreme points of G is a subset of ∂^+G.

Now, we show correspondences between concepts of monotonic analysis, like downward sets and their upper extreme points and similar concepts of convex analysis, that is, convex sets and their extreme points.

Proposition 3.8: Let G be a non-trivial downward set of X, $a\in G$,K closed pointed convex cone in X and V is the set of upper extreme point of G. Then the following statements are equivalent.

(i) $a\in V$.
(ii) If $x, k\in X$ and $x = a+k$, then either $x\notin G$, or $k\notin intK$, or $k = 0$.
(iii) If $W = \{x_1, \ldots,x_n\}\subseteq G$ and $a\in W_*$, then $a = x_k$ for some k.
(iv) For each $k\in intK$, $G\backslash\{a+K\}$ is downward.

Proof: (1)→(2):

Let $k \in intK$. Then by definition of a, $a+k\in a+intK\subset X\backslash G$.
Therefore $x = a+k\notin G$.

Let $x = a+k$ and $x\in G$. Then $k\notin intK$. Otherwise, $x=a+k\in K_\alpha^0 \subset X\backslash G$. Hence $x\notin G$, which is contradiction.

(In fact $K = 0$. Because $a \leq x = a+k \in G$. But, a is extreme point so $a = a+k$. Therefore

$k = 0$

$(2)\rightarrow(3)$:

By proposition 2.4, there exist $x_k \in W$, $k \in K$ such that $a = x_k - k$.

Then $k = 0$.

$(3)\rightarrow(1)$:

Let $x \in G$, $a \leq x$. Then $a \in \{x\}_*$. Therefore $a = x$.

$(1)\rightarrow(4)$:

Let $a \in V$, then from definition of a it follow that $a+k \notin G$. Therefore $G\setminus\{a+k\} = G$ is downward.

$(4)\rightarrow(1)$:

Let $G\setminus\{a+k\} = G$ is downward set, $x \in G$ and $a \leq x$. Then $a+k \leq x+k$; and $x+k \notin G\setminus\{a+k\}$.Otherwise, since $G\setminus\{a+k\}$ is downward it follow that $a+k \in G\setminus\{a+k\}$ which is contradiction. But, $x+k \in G$. Because if $x+k \notin G$ then $x+k \notin \partial^+ G$. Since $x \in G$, then $x \in \overline{G}$.

Therefore $x+k \notin K_x^0 = x+\text{int}K$ which is contradiction. Therefore $x+k = a+k$. That is $x = a$.

Lemma 3.9: Let G be a downwad set of $X = (X,\tau,\leq)$. Then clG, the closure of G, is downward set.

Proof: Let $x, x' \in X$, $x' \leq x$ and $x \in clG$. If $x \in G$ then $x' \in G$ and so $x' \in clG$. Let $x \in clG\setminus G$. Then there exist a sequence $\{x_n\} \subseteq G$ such that $x_n \rightarrow x$ (in norm), so $x_n - x' \rightarrow x - x'$. There exist $k \in K$, since $\text{int}k \neq \emptyset$ such that $x-x'-k \in \text{int}K$ and $x_n - x' - k \rightarrow x-x'-k$ (in norm). Therefore $x_n - x' - k \in \text{int}K$, for some $n \in N$. This follow that $x' \leq x'+k \leq x_n$. That is $x' \in clG$.

DOWNWARD SETS AND SEPARATION PROPERTIES

Let $X = (X,\tau,\leq)$ be lattice Banach space which is equipped with a closed convex pointed K in X. For any subset A of X We shall use the notation $A^+ = \{a^+ : a \in A\}$, where $a^+ = \sup(a, 0)$. We also use notation $a^- = -\inf(a, 0)$.

A subset $Z \subseteq X$ is called upward if for each $w \in Z$ and $x \in X$, the inequality $x \geq w$ implies $x \in Z$.
The following lemma is clear.

Lemma 4.1: let G be a subset of $X = (X,\tau,\leq)$.

(1) G is a downward set if and only if $-G = \{x \in X: -x \in G)$ is a upward set.
(2) G is a downward set if and only if G', complement of G, is a upward set.

Proof: It is obvious.

A linear functional f on X is called positive if $f(x) \geq 0$ holds for each $x \in K$. If f be a positive linear functional on X then, it is easy to see that, for each $\alpha \in R$, $A = \{x: f(x) \geq \alpha\}$ is an upward set of X. We recall that, a subset A of X is called a closed half-space if there is a continuous linear functional $f: X \rightarrow R$ such that $A = \{x: f(x) \geq \alpha\}$ for some $\alpha \in R$. Therefore if f be a positive linear functional then the closed half-space, depend it, is upward. In other word, any positive closed half-space is upward.
Now, we have a conversely case:

Proposition 4.2: Let W be a upward set of X. Then there exists a positive linear functional f such that $W \subseteq \{x \in X: f(x) \geq \alpha\}$ for some α. (In other word, any upward set is a subset of a positive closed half-space.

Proof: Suppose that W is a upward set of X and $v \notin \overline{CO}(W)$. We can suppose $v \leq 0$ (Indeed, if $v \notin W$, then-$v^- \notin W$. Otherwise,if-$v^- \in W$ then from $v = v^+ - v^-$we have-$v^- = v - v^+ \in W$ and from

$$v = v^+ - v^- = v^+ + (v - v^+) \geq v - v^+$$

result that $v \in W$, which is a contradiction.).Now, by Hahn-Banach theorem there exists a continuous linear functional $g: X \rightarrow R$ and a real scalar α such that:

$$g(v) < \alpha \leq g(W) \text{ for all } w \in \overline{CO}(W)$$

If $x \in K$, then $w+x \geq w$ for all $w \in W$. Since W is upward so $x+w \in W$. Since $w+x \in W \subseteq \overline{CO}(W)$ therefore from $g(v) < \alpha \leq g(w+x)$, result that $g(w-v+x) \geq 0$. For fixed $w \in W$ we define,

$$f(x) = g(w^+ + x) - g(v) = g(w^+ - v + x)$$

Hence if $x \geq 0$ then $w^+ - v + x \in W$ and therefore $f(x) = g(w^+ - v + x) \geq 0$ with $f(x) \geq \alpha$. That is, f is positive linear functional and $W \subseteq \{x \in X: f(x) \geq \alpha\}$.

Thus the consequences of Hahn-Banach Theorem are applicable to upward sets. For example, we have the following proposition:

Proposition 4.3: Let W be a upward subset of X then $\overline{CO}\,W$ is the intersection of the closed positive half-spaces containing W.

Proof: Let H = {A: W⊆A = {x∈X: f(x)≥α}} be the collection of all closed positive half-spaces containing W.By proposition 4.2 H≠ø. Since each set in H is closed and convex, $\overline{CO}$ W⊆∩{A: A∈H}. On the other hand, if $X_0 \notin \overline{CO}$ W, then by Hhan-Banach theorem there is a continuous linear functional f: X R and α in R such that f(x)≥α and $f(x_0)$ <α for all x in $\overline{CO}$ W. Thus A = {x: f(x)≥α] belong to H and X_0∈A. That is $X_0 \notin \cap_{A \in H} A$.

Corollary 4.4: Let W be a upward subset of X then $\overline{CO}$ W is upward.

Proof: Since each closed positive half-spaces is upward and the intersections of upward sets are upward, the state following from the preceding proposition.

CONCLUSIONS

In this paper, we proved some topological properties and separation properties of downward sets. Also, we characterized upward sets by continuous positive linear functional.

However, convexity is sometimes a very restrictive assumption, so there is a clear need to study monotonic analysis concepts (such as: Best approximation, increasing and positively homogeneous functions (IPH, DPH), NTU games in economics, analysis of topical functions, ICAR (increasing convex-along rays) functions, optimization) by not necessarily convex sets. Downward sets and upward sets are a tool in the study of the monotonic analysis.

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