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On an Inverse Spectral Problem for an Integro-differential Operator

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Abstract: In this paper, we considered an inverse problem with two given spectrum for a boundary value problem with aftereffect and eigenvalue in the boundary condition and we showed that transformation operator was generalized degeneracy and we obtained a new proof of the Hochstadt's theorem concerning the structure of the difference $\tilde{q}(x)-q(x)$.

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INTRODUCTION

We consider boundary value problem with aftereffect on a finite interval and with eigenvalue in the boundary condition:

$$-y''(x)+q(x)y(x)+\int_0^x M(x-t)y(t)dt =\lambda^2 y(x),\ 0\le x\le\pi$$

with the boundary condition

$$y'(0,\lambda)-\lambda y(0,\lambda)=0$$

$$y'(\pi,\lambda)+Hy(\pi,\lambda)=0$$

Here λ is spectral parameter, $q(x)\in L_2(0,\pi)$ and H are real parameter. The presence of an aftereffect in a mathematical model produces qualitative changes in the study of the inverse problem. In recent years, inverse spectral problems for integro differential operators have been studied by many authors including [1-4].

The generalized degeneracy of transformation operator for Sturm-Liouville and singular Sturm-Liouville operator was showed in [5-7].

In this paper, using Levitan's method, it was shown that kernel K(x,t) of the integral equation is generalized degeneracy.

Now let us consider two boundary value problem with aftereffect on a finite interval and with eigenvalue in the boundary condition:

$$-y''(x)+q(x)y(x)+\int_0^x M(x-t)y(t)dt =\lambda^2 y(x),0\le x\le\pi \tag{1.1}$$

$$y'(0)-\lambda y(0)=0 \tag{1.2}$$

$$y'(\pi,\lambda)+Hy(\pi,\lambda)=0 \tag{1.3}$$

and

$$-y''(x)+\tilde{q}(x)y(x)+\int_0^x \widetilde{M}(x-t)y(t)dt =\lambda^2 y(x),\ 0\le x\le\pi \tag{1.4}$$

$$y'(0)-\lambda y(0)=0$$

$$y'(\pi,\lambda)+\widetilde{H}y(\pi,\lambda)=0 \tag{1.5}$$

where $\tilde{q}(x)\in L_2(0,\pi)$ and $\widetilde{H}$ is real parameter.

Denote the spectrum of the first problem by $\{\lambda_n\}_{n\ge1}$ and the spectrum of the second by $\{\tilde{\lambda}_n\}_{n\ge1}$. Next, we denote by $\varphi(x,\lambda)$ the solution of (1.1) and by $\tilde{\varphi}(x,\lambda)$ the solution of (1.4), satisfying the initial condition (1.2). The representation

$$\tilde{\varphi}(x,\lambda)=\varphi(x,\lambda)+\int_0^x K(x,t)\varphi(t,\lambda)dt \tag{1.6}$$

holds, where K(x,t) satisfies the equation

$$K_{tt}(x,t)-K_{xx}(x,t)+\left(q(x)-\tilde{q}(t)\right)K(x,t) +M(x-t)-\widetilde{M}(x-t) +\int_t^x\left(M(x-\xi)-\widetilde{M}(x-\xi)\right)K(\xi,t)d\xi=0 \tag{1.7}$$

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and the conditions

$$2\frac{dK(x,x)}{dx} = \tilde{q}(x) - q(x) \tag{1.8}$$

$$K(x,0) = 0 \tag{1.9}$$

We put

$$c_n = \int_0^\pi \varphi^2(x,\lambda_n)dx,\ \tilde{c}_n = \int_0^\pi \tilde{\varphi}^2(x,\tilde{\lambda}_n)dx \tag{1.10}$$

$$\rho(\lambda) = \sum_{\lambda_n<\lambda} \frac{1}{c_n},\ \tilde{\rho}(\lambda) = \sum_{\tilde{\lambda}_n<\lambda} \frac{1}{\tilde{c}_n} \tag{1.11}$$

The function $\rho(\lambda)$ $(\tilde{\rho}(\lambda))$ is called the spectral function of problem (1.1)-(1.3) [(1,4), (1.5)]. Problem (1.1)-(1.3) is regarded as an unperturbed problem, while (1.4), (1.5) is considered as a perturbation of (1.1)-(1.3).

It is known fact that in [5] the knowledge of two spectra for given Sturm-Liouville equation makes it possible to recover its spectral function, i.e., to find the numbers $\{c_n\}$. More exactly, suppose that, in addition to the spectrum of problem (1.1)-(1.3), we also know the spectrum $\{\mu_n\}$ of the problem

$$-y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)dt = \lambda^2 y(x),\ 0 \le x \le \pi$$

$$y'(0) - \lambda y(0) = 0 \tag{1.12}$$

$$y'(\pi,\lambda) + H_1 y(\pi,\lambda) = 0,\ H_1 \ne H$$

Knowing $\{\lambda_n\}$ and $\{\mu_n\}$, we can calculate the numbers $\{c_n\}$. Similarly, for (1.4), if, in addition to $\{\tilde{\lambda}_n\}$, we also know the spectrum $\{\tilde{\mu}_n\}$ determined by the boundary conditions

$$y'(0) - \lambda y(0) = 0$$

$$y'(\pi,\lambda) + \tilde{H}_1 y(\pi,\lambda) = 0,\ \tilde{H}_1 \ne \tilde{H} \tag{1.13}$$

then it follows that we can determine the numbers $\{\tilde{c}_n\}$. It is also shown that in [4]

$$\lambda_n = n + \frac{1}{\pi} + \frac{H}{n\pi} + \frac{1}{2n\pi}\int_0^\pi q(t)dt + \frac{k_n}{n}, \{k_n\} \in l_2 \text{ and } n \in \mathbb{N} - \{0\} \tag{1.14}$$

$$\varphi(x,\lambda) = \cos\lambda x + \sin\lambda x + \frac{1}{\lambda}\int_0^x \sin\lambda(x-\tau)\left[q(\tau)\varphi(\tau,\lambda) + \int_0^\tau M(\tau-s)\varphi(s,\lambda)ds\right]d\tau \tag{1.15}$$

Theorem 1. Consider the operator

$$Ly(x) \equiv -y''(x) + q(x)y(x) + \int_0^x M(x-t)y(t)dt = \lambda^2 y(x),\ 0 \le x \le \pi \tag{1.16}$$

subject to the boundary conditions

$$y'(0) - \lambda y(0) = 0 \tag{1.17}$$

$$y'(\pi,\lambda) + Hy(\pi,\lambda) = 0 \tag{1.18}$$

Let $\{\lambda_n\}$ be the spectrum of L subject to (1.17) and (1.18).
If (1.18) is replaced by the new boundary condition

$$y'(\pi,\lambda) + H_1 y(\pi,\lambda) = 0 \tag{1.19}$$

then a new operator and a new spectrum, say $\{\mu_n\}$, result.
Now, consider the second operator

$$\tilde{L}y(x) \equiv -y''(x) + \tilde{q}(x)y(x) + \int_0^x \tilde{M}(x-t)y(t)dt = \lambda^2 y(x),\ 0 \le x \le \pi \tag{1.20}$$

Suppose that $\tilde{L}$ has the spectrum $\{\tilde{\lambda}_n\}$ with $\{\tilde{\lambda}_n\} = \{\lambda_n\}$ for all n under the boundary conditions (1.17) and

$$y'(\pi,\lambda) + \tilde{H}y(\pi,\lambda) = 0 \tag{1.21}$$

$\tilde{L}$ with the boundary conditions (1.17) and

$$y'(\pi,\lambda) + \tilde{H}_1 y(\pi,\lambda) = 0 \tag{1.22}$$

is assumed to have the spectrum $\{\tilde{\mu}_n\}$. Assuming that H, $H_1 \neq H$, $\tilde{H}$ and $\tilde{H}_1 \neq \tilde{H}$ are real numbers that are not infinite.

Denote by Λ_0 the finite index set for which $\tilde{\mu}_n \neq \mu_n$ and by Λ the infinite index set for which $\tilde{\mu}_n = \mu_n$. Under the above assumptions, it follows that the kernel K(x,t) is degenerate in the extended sense:

$$K(x,t) = \sum_{\Lambda_0} c_n \tilde{\varphi}_n(x)\varphi_n(t) \tag{1.23}$$

where φ_n and $\tilde{\varphi}_n$ are suitable solutions of (1.1) and (1.4).

Proof: It follows from (1.6) that

$$\tilde{\varphi}'(x,\lambda) = \varphi'(x,\lambda) + K(x,x)\varphi(x,\lambda) + \int_0^x \frac{\partial K}{\partial x}\varphi(t,\lambda)dt \tag{1.24}$$

and

$$\tilde{\varphi}'(x,\lambda) + \tilde{H}\tilde{\varphi}(x,\lambda) = \varphi'(x,\lambda) + \tilde{H}\varphi(x,\lambda) + K(x,x)\varphi(x,\lambda) + \int_0^x \left(\frac{\partial K}{\partial x} + \tilde{H}K\right)\varphi(t,\lambda)dt \tag{1.25}$$

Substituing x=π and $\lambda = \lambda_n$ into the last equation and using the boundary conditions (1.18), we obtain

$$\left(\tilde{H} - H\right)\varphi(\pi,\lambda_n) + K(\pi,\pi)\varphi(\pi,\lambda_n) + \int_0^\pi \left(\frac{\partial K}{\partial x} + \tilde{H}K\right)_{x=\pi} \varphi(t,\lambda_n)dt = 0 \tag{1.26}$$

As $n\to\infty$ and $\varphi(\pi,\lambda_n) \to (-1)^n(\cos 1 + \sin 1)$, the integral on the right-hand side tends to zero. Therefore, from (1.26) we get

$$K(\pi,\pi) = H - \tilde{H} \tag{1.27}$$

$$\int_0^\pi \left(\frac{\partial K}{\partial x} + \tilde{H}K\right)_{x=\pi} \varphi(t,\lambda_n)dt = 0,\ n = 0,1,... \tag{1.28}$$

Since the systems of functions $\varphi(t,\lambda_n)$ is complete, it follows from the last equation that

$$\left(\frac{\partial K}{\partial x} + \tilde{H}K\right)_{x=\pi} = 0,\ 0 \le t \le \pi \tag{1.29}$$

We now use the equation imposed on the second mentioned spectrum. Using (1.6) again, we obtain

$$\tilde{\varphi}'(x,\lambda) + \tilde{H}_1\tilde{\varphi}(x,\lambda) = \varphi'(x,\lambda) + \tilde{H}_1\varphi(x,\lambda) + K(x,x)\varphi(x,\lambda) + \int_0^x \left(\frac{\partial K}{\partial x} + \tilde{H}_1 K\right)\varphi(t,\lambda)dt \tag{1.30}$$

Setting x=π and $\lambda = \mu_n$ $(n \in \Lambda)$ and using (1.19), we get

$$\int_0^\pi \left(\frac{\partial K}{\partial x} + \tilde{H}_1 K\right)_{x=\pi} \varphi(t,\mu_n)dt + \left(\tilde{H}_1 - H_1\right)\varphi(\pi,\mu_n) + K(\pi,\pi)\varphi(\pi,\mu_n) = 0 \tag{1.31}$$

In the last equation as $n\to\infty$, the left-hand side tends to zero and $\varphi(\pi,\mu_n) \to (-1)^n(\cos 1 + \sin 1)$. Therefore;

$$K(\pi,\pi) = H_1 - \tilde{H}_1 \tag{1.32}$$

$$\int_0^{\pi}\left(\frac{\partial K}{\partial x} + \tilde{H}_1 K\right)_{x=\pi}\varphi(t,\mu_n)\,dt = 0,\ n\in\Lambda \tag{1.33}$$

Comparing (1.27) and (1.32), we obtain $H - \tilde{H} = H_1 - \tilde{H}_1$. For $n\in\Lambda_0$, relation (1.30) for $(x = \pi \text{ and } \lambda = \mu_n)$ yields

$$\int_0^{\pi}\left(\frac{\partial K}{\partial x} + \tilde{H}_1 K\right)_{x=\pi}\varphi(t,\mu_n)\,dt = \tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n) \tag{1.34}$$

It follows from (1.33) and (1.34) that

$$\left(\frac{\partial K}{\partial x} + \tilde{H}_1 K\right)_{x=\pi} = \sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2}\varphi(t,\mu_n),\ 0\le t\le\pi \tag{1.35}$$

We derive from (1.29) and (1.35) the following equations:

$$K(\pi,t) = \frac{1}{\tilde{H}_1 - \tilde{H}}\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2}\varphi(t,\mu_n) \tag{1.36}$$

$$\left(\frac{\partial K(x,t)}{\partial x}\right)_{x=\pi} = -\frac{\tilde{H}}{\tilde{H}_1 - \tilde{H}}\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2}\varphi(t,\mu_n),\ 0\le t\le\pi \tag{1.37}$$

The function K(x,t) satisfies (1.7). Therefore, it follows from the initial conditions (1.36) and (1.37) that, in triangle I (Fig. 1), we have

$$K(x,t) = \frac{1}{\tilde{H}_1 - \tilde{H}}\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2}\left[\tilde{c}(x,\mu_n) - \tilde{H}\tilde{t}(x,\mu_n)\right]\varphi(t,\mu_n) \tag{1.38}$$

where $\tilde{c}(x,\lambda)$ and $\tilde{t}(x,\lambda)$ are the solutions of (1.4) satisfying the initial conditions

$$\tilde{c}(\pi,\lambda) = \tilde{t}'(\pi,\lambda) = 1,\ \tilde{c}'(\pi,\lambda) = \tilde{t}(\pi,\lambda) = 0 \tag{1.39}$$

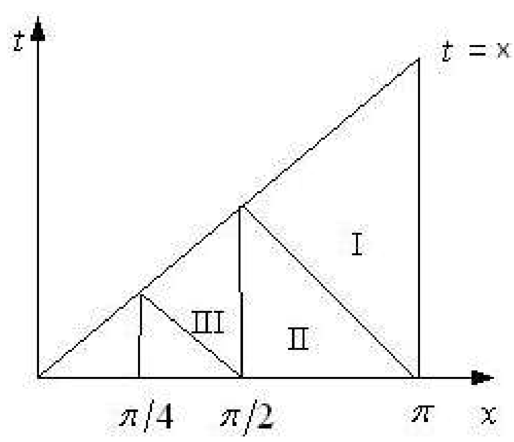

Fig. 1:

The function K(x,t) and sum (1.38) satisfy (1.9); therefore, they coincide in triangle II; consequently, they coincide in triangle III, because solutions of (1.7) satisfy the same initial conditions on the line $x = \frac{\pi}{2}$, etc., i.e., K(x,t) is expressed by (1.38) throughout the triangle $0\le x\le t\le\pi$.

Theorem 2: If the spectra and $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$ coincide and $\{\mu_n\}$ and $\{\tilde{\mu}_n\}$ differ in a finite number of their terms, i.e., $\tilde{\mu}_n = \mu_n$ for $n\in\Lambda$, then

$$\tilde{q}(x) - q(x) = \sum_{\Lambda_0}\tilde{c}_n\frac{d}{dx}\left(\tilde{\varphi}_n.\varphi_n\right) \tag{1.40}$$

where φ_n and $\tilde{\varphi}_n$ are suitable solutions of (1.1) and (1.4)

Proof: We obtain from (1.8) the equation

$$\tilde{q}(x) - q(x) = 2\frac{dK(x,x)}{dx} \tag{1.41}$$

Differentiating (1.38) and setting t = x, we obtain

$$\tilde{q}(x) - q(x) = \frac{2}{\tilde{H}_1 - \tilde{H}}\sum_{\Lambda_0}\frac{\tilde{\varphi}'(\pi,\mu_n) + \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)}{\|\varphi(t,\mu_n)\|^2}\frac{d}{dx}\left\{\left[\tilde{c}(x,\mu_n) - \tilde{H}\tilde{t}(x,\mu_n)\right]\varphi(x,\mu_n)\right\} \tag{1.42}$$

Consequently,

$$\tilde{q}(x) - q(x) = \sum_{\Lambda_0}\tilde{c}_n\frac{d}{dx}\left(\tilde{\varphi}_n.\varphi_n\right) \tag{1.43}$$

where

$$\tilde{c}(x,\mu_n) - \tilde{H}\tilde{t}(x,\mu_n) = \tilde{\varphi}_n,\ \varphi(x,\mu_n) = \varphi_n(x,\mu_n)$$

and

$$\tilde{c}_n = \frac{2\left[\tilde{\varphi}'(\pi,\mu_n) - \tilde{H}_1\tilde{\varphi}(\pi,\mu_n)\right]}{\left(\tilde{H}_1 - \tilde{H}\right)\|\varphi(t,\mu_n)\|^2} \tag{1.44}$$

This completes the proof of Theorem 2. We note that similar problem was investigated [7].

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