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Characterizations of h-hemiregular and h-semisimple Hemirings by Interval Valued Fuzzy h-Ideals

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Abstract: In this paper we define interval valued fuzzy h-subhemirings, interval valued fuzzy interior h-ideals, interval valued fuzzy prime h-ideals and interval valued fuzzy semiprime h-ideals. We characterize h-hemiregular and h-semisimple hemirings by the properties of these interval valued fuzzy h-ideals.

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INTRODUCTION

In 1965, Zadeh [1] introduced the concept of fuzzy set. Since then fuzzy sets have been used in many branches of Mathematics. The fuzzification of algebraic structures was initiated by Rosenfeld [2], he introduced the notion of fuzzy subgroups. In [3] J. Ahsan *et al*. initiated the study of fuzzy semirings. The fuzzy algebraic structures play an important role in Mathematics with wide applications in theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [4, 5].

Semirings, as generalizations of associative rings and distributive lattices, was first introduced by Vandiver [6] in 1934. Since then semirings have been extensively used for studying many branches of applied mathematics, theoretical computer sciences and information sciences [7-9]. An additively commutative semiring with zero element is called hemiring. Hemirings, appears in a natural manner, in some applications to the theory of automata, the theory of formal languages and in computer sciences [10, 11].

It is well known that the ideals of semirings play a vital role in the structure theory of semirings and are very useful for many purposes. However, in general, they do not coincide with the usual ring ideals. Many results in rings apparently have no analogues in semirings using only ideals. In order to overcome this difficulty in [12], Henriksen defined a more restricted class of ideals in semirings, called k-ideals, with the property that if a semiring R is a ring, then a complex in R is a k-ideal if and only if it is a ring ideal. Another more restricted, but very important, class of ideals in hemirings, called now h-ideals, was given and discussed by Iizuka [13].

In 1975 the concept of interval valued fuzzy sets was introduced by Zadeh [14], as a generalization of the notion of fuzzy sets. In hemirings this concept was initiated by Ma and Zhan in [15]. In [16] Sun *et al.* characterized h-hemiregular and h-intra-hemiregular hemirings by the properties of their interval valued fuzzy left and right h-ideals. In this paper we define interval valued fuzzy h-subhemirings, interval valued fuzzy interior h-ideals, interval valued fuzzy prime h-ideals and interval valued fuzzy semiprime h-ideals. We characterize h-hemiregular and h-semisimple hemirings by the properties of these interval valued fuzzy h-ideals.

PRELIMINARIES

For basic definitions we refer to [10, 18].

The h-closure $\overline{A}$ of a non-empty subset A of a hemiring R is defined as:

$$\overline{A} = \{x \in R \mid x + a + z = b + z, \text{for some } a, b \in A, z \in R\}$$

A fuzzy subset λ of a universe X is a function $\lambda: X \rightarrow [0,1]$. A fuzzy subset of X of the form

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$$\lambda(z)=\begin{cases} t & \text{if } z=x \\ 0 & \text{if } z\neq x \end{cases}$$

is called the fuzzy point with support x and value t, where $t\in(0,1]$. It is usually denoted by x_t. A fuzzy point x_t is said to belong to a fuzzy set λ, written as $x_t\in\lambda$. For any two fuzzy subsets λ and μ of X, $\lambda\le\mu$ means that, for all $x\in X$, $\lambda(x)\le\mu(x)$. The symbols $\lambda\wedge\mu$ and $\lambda\vee\mu$ will mean the following fuzzy subsets of X

$$(\lambda\wedge\mu)(x)=\min\{\lambda(x),\mu(x)\}$$

$$(\lambda\vee\mu)(x)=\max\{\lambda(x),\mu(x)\}$$

Definition [17]: A subset A of a hemiring R, is called h-idempotent if $A=\overline{A^2}$

Definition [17]: A hemiring R, is called h-semisimple if every h-ideal of R is h-idempotent.

Lemma [17]: A hemiring R, is h-semisimple if and only if one of the following holds:

(i) For all $x\in R$, there exists $c_i, d_i, e_i, f_i, c_j', d_j', e_j', f_j'\in R$ such that

$$x+\sum_{i=1}^{m}c_i x\, d_i e_i x\, f_i+z=\sum_{j=1}^{n}c_j' x\, d_j' e_j' x\, f_j'+z$$

(ii) For all $x\in R, x\in\overline{RxRxR}$.

(iii) For all $A\subseteq R, A\subseteq\overline{RARAR}$.

Definition [18]: A hemiring R is said to be h-hemiregular if for each $x\in R$, there exist $a,b,z\in R$ such that $x+xax+z=xbx+z$.

Lemma [18]: A hemiring R is h-hemiregular if and only if for any right h-ideal I and any left h-ideal L of R we have $\overline{IL}=I\cap L$.

INTERVAL VALUED FUZZY SETS

Let Ω be the family of all closed subintervals of [0,1]. By an interval number $\hat{a}$ we mean an interval $[a^-,a^+]\in\Omega$, where $0\le a^-\le a^+\le 1$. The interval [a,a] can be identified by the number $a\in[0,1]$. The element 0 = [0,0] is the minimal element and the element 1 = [1,1] is the maximal element of Ω, according to the partial order $[\alpha,\alpha']\le[\beta,\beta']$ if and only if $\alpha\le\beta, \alpha'\le\beta'$ defined on Ω for all $[\alpha,\alpha'],[\beta,\beta']\in\Omega$.

An interval valued fuzzy subset $\hat{\lambda}$ of a hemiring R is a function $\hat{\lambda}: R\to\Omega$. We write $\hat{\lambda}(x)=[\lambda^-(x),\lambda^+(x)]\subseteq[0,1]$, for all $x\in R$, where $\lambda^-,\lambda^+: R\to[0,1]$ are fuzzy subsets of R such that for each $x\in R$, $0\le\lambda^-(x)\le\lambda^+(x)\le 1$. For simplicity we write $\hat{\lambda}=[\lambda^-,\lambda^+]$. From now to onward we will denote the set of all interval valued fuzzy subsets of R by F(Ω,R).

Let A be a subset of a hemiring R. Then the interval valued characteristic function $\hat{\chi}_A$ of A is defined to be a function $\hat{\chi}_A: R\to\Omega$ such that for all $x\in R$.

$$\hat{\chi}_A(x)=\begin{cases}1 & \text{if } x\in A\\ 0 & \text{if } x\notin A\end{cases}$$

Clearly $\hat{\chi}_A\in F(\Omega,R)$. Note that $\hat{\chi}_R(x)=1$ for all $x\in R$.
An interval valued fuzzy subset of R of the form

$$x_{\hat{t}}(y)=\begin{cases}\hat{t}(\neq 0) & \text{if } y=x\\ 0 & \text{if } y\neq x\end{cases}$$

is said to be interval valued fuzzy point with support x and value $\hat{t}$. An interval valued fuzzy point $x_{\hat{t}}$ is said to belong to interval valued fuzzy subset $\hat{\lambda}$ written as $x_{\hat{t}}\in\hat{\lambda}$ if $\hat{\lambda}(x)\ge\hat{t}$.

For any $\hat{\lambda},\hat{\mu}\in F(\Omega,R)$, union and intersection of $\hat{\lambda}$ and $\hat{\mu}$ are defined as, for all $x\in R$,

$$(\hat{\lambda}\vee\hat{\mu})(x)=[\lambda^-(x)\vee\mu^-(x),\lambda^+(x)\vee\mu^+(x)]$$

$$(\hat{\lambda}\wedge\hat{\mu})(x)=[\lambda^-(x)\wedge\mu^-(x),\lambda^+(x)\wedge\mu^+(x)]$$

Further for any $\hat{\lambda},\hat{\mu}\in F(\Omega,R)$, $\hat{\lambda}\le\hat{\mu}$ if and only if $\hat{\lambda}(x)\le\hat{\mu}(x)$, that is $\lambda^-(x)\le\mu^-(x)$ and $\lambda^+(x)\le\mu^+(x)$, for all $x\in R$.

Definition [16]: Let $\hat{\lambda}$, $\hat{\mu}\in F(\Omega,R)$. Then the h-intrinsic product of $\hat{\lambda}$ and $\hat{\mu}$ is denoted and defined by

$$\left(\hat{\lambda}\otimes\hat{\mu}\right)(x)=\bigvee_{x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a_j'b_j'+z}\left[\left\{\left(\bigwedge_{i=1}^{m}\hat{\lambda}(a_i)\right)\wedge\left(\bigwedge_{i=1}^{m}\hat{\mu}(b_i)\right)\wedge\left(\bigwedge_{j=1}^{n}\hat{\lambda}(a_j')\right)\wedge\left(\bigwedge_{j=1}^{n}\hat{\mu}(b_j')\right)\right\}\right]$$

for all $x\in R$, if x can be expressed as

$$x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a_j'b_j'+z$$

and 0 if x cannot be expressed as

$$x+\Sigma_{i=1}^{m}a_ib_i+z=\Sigma_{j=1}^{n}a_j'b_j'+z$$

INTERVAL VALUED FUZZY h-IDEALS

Definition: Let $\hat{\lambda}\in F(\Omega,R)$. Then $\hat{\lambda}$ is said to be an interval valued fuzzy h-subhemiring of R if it satisfies

(1) $x_{\hat{t}}\in\hat{\lambda}, y_{\hat{r}}\in\hat{\lambda}\Rightarrow(x+y)_{\min\{\hat{t},\hat{r}\}}\in\hat{\lambda}$

(2) $x_{\hat{t}}\in\hat{\lambda}, y_{\hat{r}}\in\hat{\lambda}\Rightarrow(xy)_{\min\{\hat{t},\hat{r}\}}\in\hat{\lambda}$

(3) $x+a+z=b+z,\ a_{\hat{t}}\in\hat{\lambda}, b_{\hat{r}}\in\hat{\lambda}\Rightarrow x_{\min\{\hat{t},\hat{r}\}}\in\hat{\lambda}$

for all $x,y,z,a,b\in R$ and $t,r\in(0,1]$.

Definition: Let $\hat{\lambda}\in F(\Omega,R)$. Then $\hat{\lambda}$ is said to be an interval valued fuzzy left (resp. right) h-ideal of R if it satisfies (1), (3) and (4) $x_{\hat{t}}\in\hat{\lambda}\Rightarrow(yx)_{\hat{t}}\in\hat{\lambda}$ (resp. (5) $(xy)_{\hat{t}}\in\hat{\lambda}$) for all $x,y\in R$ and $t\in(0,1]$. $\hat{\lambda}\in F(\Omega,R)$ is called an interval valued fuzzy h-ideal of R if it is both, interval valued fuzzy left and right h-ideal of R.

Definition: Let $\hat{\lambda}\in F(\Omega,R)$. Then $\hat{\lambda}$ is said to be an interval valued fuzzy interior h-ideal of R if and only if it satisfies (1), (2), (3) and (6) $y_{\hat{t}}\in\hat{\lambda}\Rightarrow(xyz)_{\hat{t}}\in\hat{\lambda}$ for all $x,y,z\in R$ and $t\in(0,1]$.

Definition: Let $\hat{\lambda}\in F(\Omega,R)$ where R is a commutative hemiring with unity. Then $\hat{\lambda}$ is said to be an interval valued fuzzy prime h-ideal of R if and only if it satisfies (1), (3), (4), (5) and (7) $(xy)_{\hat{t}}\in\hat{\lambda}\Rightarrow x_{\hat{t}}\in\vee q\hat{\lambda}$ or $y_{\hat{t}}\in\hat{\lambda}$ for all $x,y\in R$ and $t\in(0,1]$.

Definition: Let $\in F(\Omega,R)$, where R is a commutative hemiring with unity. Then $\hat{\lambda}$ is said to be an interval valued fuzzy semiprime h-ideal of R if and only if it satisfies (1), (3), (4), (5) and (8) $(x^2)_{\hat{t}}\in\hat{\lambda}\Rightarrow x_{\hat{t}}\in\hat{\lambda}$ for all $x\in R$ and $t\in(0,1]$.

Remark: If $\hat{\lambda}$ is any interval valued fuzzy h-subhemiring (right h-ideal, left h-ideal, interior h-ideal, prime h-ideal, semiprime h-ideal) then $\hat{\lambda}(0)\geq\hat{\lambda}(x)$ for all $x\in R$.

Theorem: Let $\hat{\lambda}\in F(\Omega,R)$ Then (1) to (8) are equivalent to (1′) to (1′) respectively, where:

(1′) $\hat{\lambda}(x+y)\geq\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$

(2′) $\hat{\lambda}(xy)\geq\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$

(3′) $x+a+z=b+z\Rightarrow\hat{\lambda}(x)\geq\min\{\hat{\lambda}(a),\hat{\lambda}(b)\}$

(4′) $\hat{\lambda}(xy)\geq\hat{\lambda}(y)$

(5′) $\hat{\lambda}(xy)\geq\hat{\lambda}(x)$

(6′) $\hat{\lambda}(xyz)\geq\hat{\lambda}(y)$

(7′) $\max\{\hat{\lambda}(x),\hat{\lambda}(y)\}\geq\hat{\lambda}(xy)$

(8′) $\hat{\lambda}(x)\geq\hat{\lambda}(x^2)$

for $x,y,z,a,b\in R$.

Proof: We prove (1) is equivalent to (1′), others follow in an analogous way.

(1)⟹(1′) Suppose (1′) does not hold. Then there exist $x,y\in R$, such that $\hat{\lambda}(x+y)<\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$. Then for some $t\in(0,1]$, $\hat{\lambda}(x+y)<\hat{t}\leq\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}$. Then $x_{\hat{t}}\in\hat{\lambda}, y_{\hat{t}}\in\hat{\lambda}$ but $(x+y)_{\hat{t}}\overline{\in}\hat{\lambda}$. Which is again a contradiction. Hence (1′) holds.

(1′)⟹(1′) Let $x,y\in R$ and $t,r\in(0,1]$ be such that $x_{\hat{t}}\in\hat{\lambda}, y_{\hat{r}}\in\hat{\lambda}$. Then by (1′)

$$\hat{\lambda}(x+y)\geq\min\{\hat{\lambda}(x),\hat{\lambda}(y)\}\geq\min\{\hat{t},\hat{r}\}$$

This implies $(x+y)_{\min\{\hat{t},\hat{r}\}}\in\hat{\lambda}$. This proves (1)

Now using Theorem 4.7, we have the following:

Definition: Let $\hat{\lambda} \in F(\Omega,R)$. Then $\hat{\lambda}$ is said to be an interval valued fuzzy h-subhemiring of R if and only if it satisfies $(1')$,$(2')$ and $(3')$.

Definition [16]: Let $\hat{\lambda} \in F(\Omega,R)$. Then $\hat{\lambda}$ is said to be an interval valued fuzzy left (resp. right) h-ideal of R if and only if it satisfies $(1')$,$(3')$ and $(4')$ resp. $(5')$).

$\hat{\lambda} \in F(\Omega,R)$ is called an interval valued fuzzy h-ideal of R if it is both, interval valued fuzzy left and right h-ideal of R.

Definition: Let $\hat{\lambda} \in F(\Omega,R)$. Then $\hat{\lambda}$ is said to be an interval valued fuzzy interior h-ideal of R if and only if it satisfies $(1')$,$(2')$,$(3')$ and $(6')$.

Definition: Let $\hat{\lambda} \in F(\Omega,R)$, where R is commutative hemiring with unity. Then $\hat{\lambda}$ is said to be an interval valued fuzzy prime h-ideal of R if and only if it satisfies $(1')$,$(3')$,$(4')$,$(5')$ and $(7')$.

Definition: Let $\hat{\lambda} \in F(\Omega,R)$, where R is commutative hemiring with unity. Then $\hat{\lambda}$ is said to be an interval valued fuzzy semiprime h-ideal of R if and only if it satisfies $(1')$,$(3')$,$(4')$,$(5')$ and $(8')$.

Remark: If $\hat{\lambda}$ is any interval valued fuzzy h-subhemiring (right h-ideal, left h-ideal, interior h-ideal, prime h-ideal, semiprime h-ideal) then $\hat{\lambda}(0) \geq \hat{\lambda}(x)$ for all $x \in R$.

Example: Consider the hemiring R = {0,1,2,3} with the following Caylay tables

+	0	1	2	3	.	0	1	2	3
0	0	1	2	3	0	0	0	0	0
1	1	1	2	3	1	0	1	1	1
2	2	2	2	3	2	0	1	1	1
3	3	3	2	2	3	0	1	1	1

Let $\hat{\lambda} \in F(\Omega,R)$ defined by

$$\hat{\lambda}(0) = \hat{\lambda}(1) = [0.6,0.7],\ \hat{\lambda}(2) = \hat{\lambda}(3) = [0.3,0.45]$$

Then $\hat{\lambda}$ is an interval valued fuzzy h-ideal of R.

Example: Consider the hemiring N_0 of all non-negative integers under the usual binary operations of ordinary addition and multiplication. Let $\hat{\lambda} \in F(\Omega, N_0)$ defined by:

$$\hat{\lambda}(x) = \begin{cases} 1 & \text{if } x = 0 \\ [0.25,0.35] & \text{if } x \neq 0 \text{ and even} \\ [0.55,0.65] & \text{if } x \text{ is odd} \end{cases}$$

Then $\hat{\lambda}$ is an interval valued fuzzy prime fuzzy h-ideal of R.

Theorem: Let I be a non-empty subset of a hemiring R. Then $\hat{\chi}_I$ is an interval valued fuzzy h-subhemiring (resp. h-ideal, interior h-ideal) of R if and only if I is an h-subhemiring (resp. h-ideal, interior h-ideal) of R.

Theorem: Let I be a non-empty subset of a commutative hemiring R with unity . Then $\hat{\chi}_I$ is an interval valued fuzzy prime h-ideal (resp. semiprime h-ideal) of R if and only if I is a prime h-ideal (resp. semiprime h-ideal) of R.

Theorem: Every interval valued fuzzy h-ideal of a hemiring R. is an interval valued fuzzy interior h-ideal of R.

Proof: Proof is straightforward.

Remark: Converse of the Theorem 4.18 is not true in general.

Example: Consider the hemiring R = {0,a,b,c} with the following Caylay tables

+	0	a	b	c	.	0	a	b	c
0	0	a	b	c	0	0	0	0	0
a	a	0	c	b	a	0	b	0	b
b	b	c	0	a	b	0	0	0	0
c	c	b	a	0	c	0	b	0	b

Let $\hat{\lambda} \in F(\Omega,R)$ defined by

$$\hat{\lambda}(0) = \hat{\lambda}(a) = [0.55,0.7],\ \hat{\lambda}(b) = \hat{\lambda}(c) = [0.25,0.35]$$

Then $\hat{\lambda}$ is an interval valued fuzzy interior h-ideal of R, but it is not an interval valued fuzzy h-ideal of R.

Theorem: Every interval valued fuzzy prime h-ideal of a hemiring R. is a fuzzy semiprime h-ideal of R.

Proof: Proof is straightforward.

Remark: Converse of the Theorem 4.21 is not true in general.

Example: Let $N_0 = \{0\} \cup N$ and $p_1, p_2, p_3, \ldots$ be the distinct prime numbers in N_0. If $J^0 = N_0$ and $J^l = p_1 p_2 p_3 \ldots p_l N_0$, $l = 1,2,3\ldots$, then $J^0 \supset J^1 \supset J^2 \supset \ldots J^n \supset J^{n+1} \supset \ldots$ As every non-zero element of N_0 has unique prime factorization, J^l is a semiprime h-ideal for l = 2,3,… but not a prime h-ideal. Then for such values of l, by Theorem 4.17, $\hat{\chi}_{J^l}$ is an interval valued fuzzy semiprime h-ideal of R, but $\hat{\chi}_{J^l}$ is not an interval valued fuzzy prime h-ideal of R.

Theorem: Let $\{\hat{\lambda}_i : i \in \Omega\}$ be a family of interval valued fuzzy h-subhemirings (resp. h-ideals, interior h-ideals) of R. Then $\hat{\lambda} = \underset{i\in\Omega}{\wedge} \hat{\lambda}_i$ is an interval valued fuzzy h-subhemirings (resp. h-ideals, interior h-ideals) of R.

Proof: Proof is straightforward.

Lemma [19]: If $\hat{\lambda}$ and $\hat{\mu}$ are interval valued fuzzy right and left h-ideals of a hemiring R. respectively, then $\hat{\lambda} \otimes \hat{\mu} \le \hat{\lambda} \wedge \hat{\mu}$.

h-HEMIREGULAR AND h-SEMISIMPLE HEMIRINGS

In this section we characterize h-hemiregular and h-semisimple hemirings by the properties of their interval valued fuzzy h-ideals and interval valued fuzzy interior h-ideals.

Definition: Let $\hat{\lambda}, \hat{\mu} \in F(\Omega,R)$. Then we say $\hat{\lambda}[\in]\hat{\mu}$, if $x_{\hat{t}} \in \hat{\lambda} \Rightarrow x_{\hat{t}} \in \hat{\mu}$.

Definition: Let $\hat{\lambda}, \hat{\mu} \in F(\Omega,R)$. Then we say $\hat{\lambda} \sim \hat{\mu}$, if and only if $\hat{\lambda}[\in]\hat{\mu}$ and $\hat{\mu}[\in]\hat{\lambda}$.
Proofs of the following results are straightforward.

Lemma: The relation "~"on F(Ω,R) is an equivalance relation.

Lemma: Let $\hat{\lambda}_1, \hat{\mu}_1, \hat{\lambda}_2, \hat{\mu}_2 \in F(\Omega,R)$ with $\hat{\mu}_1[\in]\hat{\lambda}_1$ and $\hat{\mu}_2[\in]\hat{\lambda}_2$. Then $(\hat{\mu}_1 \otimes \hat{\mu}_2)[\in](\hat{\lambda}_1 \otimes \hat{\lambda}_2)$.

Theorem: Let $\hat{\lambda}, \hat{\mu} \in F(\Omega,R)$. Then $\hat{\mu}[\in]\hat{\lambda}$ if and only if $\hat{\lambda}(x) \ge \hat{\mu}(x)$, for all $x \in R$.

Proof: Proof is obvious.

Theorem: The following conditions for a hemiring R are equivalent:

(i) R is h-hemiregular.
(ii) For any interval valued fuzzy right h-ideal $\hat{\lambda}$ and interval valued fuzzy left h-ideal $\hat{\mu}$ of R, $\hat{\lambda} \wedge \hat{\mu} \sim \hat{\lambda} \otimes \hat{\mu}$.

Proof: (i)⇒(ii) Let $\hat{\lambda}$ be an interval valued fuzzy right h-ideal and $\hat{\mu}$ be an interval valued fuzzy left h-ideal of R. Then by Lemma 4.25, $(\hat{\lambda} \otimes \hat{\mu}) \le (\hat{\lambda} \wedge \hat{\mu})$. Then by the Theorem 5.5, $(\hat{\lambda} \otimes \hat{\mu})[\in](\hat{\lambda} \wedge \hat{\mu})$.

Next as R is h-hemiregular, so for any $x \in R$, there exist $a,b,z \in R$ such that $x + xax + z = xbx + z$.
Thus

$$(\hat{\lambda} \otimes \hat{\mu})(x) = \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a_j' b_j' + z} \left[\left(\bigwedge_{i=1}^{m} \hat{\lambda}(a_i) \right) \wedge \left(\bigwedge_{i=1}^{m} \hat{\mu}(b_i) \right) \wedge \left(\bigwedge_{j=1}^{n} \hat{\lambda}(a_j') \right) \wedge \left(\bigwedge_{j=1}^{n} \hat{\mu}(b_j') \right) \right]$$

$$\ge \min\{\hat{\lambda}(xa), \hat{\lambda}(xb), \hat{\mu}(x)\}$$

$$\ge \min\{\hat{\lambda}(x), \hat{\mu}(x)\} = (\hat{\lambda} \wedge \hat{\mu})(x) \Rightarrow (\hat{\lambda} \wedge \hat{\mu})[\in](\hat{\lambda} \otimes \hat{\mu})$$

Hence $\hat{\lambda} \wedge \hat{\mu} \sim \hat{\lambda} \otimes \hat{\mu}$.

(ii)⇒(i) Let I be a right and J be a left h-ideal of R. Then by Theorem 4.16, $\hat{\chi}_I$ is an interval valued fuzzy right h-ideal and $\hat{\chi}_J$ is an interval valued fuzzy left h-ideal of R. Then by hypothesis and by Lemma 3.2,

$$\hat{\chi}_{\overline{IJ}} = \hat{\chi}_I \otimes \hat{\chi}_J \sim \hat{\chi}_I \wedge \hat{\chi}_J = \hat{\chi}_{I \cap J}. \Rightarrow \overline{IJ} = I \cap J$$

Hence by Lemma 2.5, R is h-hemiregular.

Theorem: Let R be an h-semisimple hemiring and $\hat{\lambda} \in F(\Omega,R)$ Then $\hat{\lambda}$ is an interval valued fuzzy h-ideal of R if and only if it is an interval valued fuzzy interior h-ideal of R.

Proof: By the Theorem 4.18, an interval valued fuzzy h-ideal of R is an interval valued fuzzy interior h-ideal of R.

Conversely assume $\hat{\lambda}$ is an interval valued fuzzy interior h-ideal of R. Let $x,y \in R$. Then by Lemma 2.3, there exists $c_i, d_i, e_i, f_i, c_j', d_j', e_j', f_j' \in R$ such that

$$x+\sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c_j' x d_j' e_j' x f_j' + z$$

Which implies

$$xy+\sum_{i=1}^{m} c_i x d_i e_i x f_i y + zy = \sum_{j=1}^{n} c_j' x d_j' e_j' x f_j' y + zy.$$

Thus

$$\hat{\lambda}(xy) \geq \min\left\{\hat{\lambda}\left(\sum_{i=1}^{m} c_i x d_i e_i x f_i y\right), \hat{\lambda}\left(\sum_{j=1}^{n} c_j' x d_j' e_j' x f_j' y\right), 0.5\right\}$$
$$\geq \hat{\lambda}(x)$$

Thus $\hat{\lambda}$ is an interval valued fuzzy right h-ideal of R. Similarly $\hat{\lambda}$ is an interval valued fuzzy left h-ideal of R.

Theorem: The following conditions for a hemiring R are equivalent:

(i) R is h-semisimple.
(ii) For any interval valued fuzzy interior h-ideals $\hat{\lambda}$ and $\hat{\mu}$ of R, $\hat{\lambda} \wedge \hat{\mu} \sim \hat{\lambda} \otimes \hat{\mu}$.

Proof: (i)$\Rightarrow$(ii) Let $\hat{\lambda}, \hat{\mu}$ be interval valued $(\in, \in \vee q)$-fuzzy interior h-ideals of R. Then by Theorem 5.7, $\hat{\lambda}, \hat{\mu}$ are interval valued fuzzy h-ideals of R. Then by Theorem 5.6, $(\hat{\lambda} \otimes \hat{\mu}) \sqsubseteq (\hat{\lambda} \wedge \hat{\mu})$. Now as R is h-semisimple, so for any $x \in R$, there exists $c_i, d_i, e_i, f_i, c_j', d_j', e_j', f_j' \in R$ such that

$$x+\sum_{i=1}^{m} c_i x d_i e_i x f_i + z = \sum_{j=1}^{n} c_j' x d_j' e_j' x f_j' + z$$

Now

$$(\hat{\lambda} \otimes \hat{\mu})(x) = \bigvee_{x+\Sigma_{i=1}^{m} a_i b_i + z = \Sigma_{j=1}^{n} a_j' b_j' + z} \left[\begin{array}{c}\left(\bigwedge_{i=1}^{m} \hat{\lambda}(a_i)\right) \wedge \left(\bigwedge_{i=1}^{m} \hat{\mu}(b_i)\right) \wedge \\ \left(\bigwedge_{j=1}^{n} \hat{\lambda}(a_j')\right) \wedge \left(\bigwedge_{j=1}^{n} \hat{\mu}(b_j')\right)\end{array}\right]$$
$$\geq \min\{\hat{\lambda}(c_i x d_i), \hat{\lambda}(c_j' x d_j'), \hat{\mu}(e_i x f_i), \hat{\mu}(e_j' x f_j')\}$$
$$\geq \min\{\hat{\lambda}(x), \hat{\mu}(x)\} = (\hat{\lambda} \wedge \hat{\mu})(x)$$

Hence $\hat{\lambda} \wedge \hat{\mu} \sim \hat{\lambda} \otimes \hat{\mu}$.

(ii)$\Rightarrow$(i) Let A be an h-ideal of R. Then A is an interior h-ideal of R. Then by Theorem 4.16, $\hat{\chi}_A$ is an interval valued fuzzy interior h-ideal of R. Then

$$\hat{\chi}_A = (\hat{\chi}_A \wedge \hat{\chi}_A) \sim \hat{\chi}_A \otimes \hat{\chi}_A = \hat{\chi}_{\overline{AA}} = \hat{\chi}_{\overline{A^2}}. \Rightarrow \overline{A^2} = A$$

Hence R is h-semisimple.

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