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R-valued Ideals of Ordered Hemirings

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Abstract: The notion of R-valued ideals of an ordered hemiring is considered and some of its properties are given, where R is a commutative multiplicative idempotent bounded CLO-hemiring in which multiplication distributes over arbitrary joins. It is shown that the set of all R-valued ideals of a positive ordered hemiring forms an additively idempotent bounded CLO-hemiring and that some ordered hemirings can be embedded in it.

Key words: Ordered hemiring (semiring); Poe-hemiring (Poe-semiring); CLO-hemiring(semiring).

INTRODUCTION

Given a non-empty set X, a fuzzy subset of X, by definition, is an arbitrary mapping $f : X \to R$ where R is the unit segment $[0, 1]$ of the real line. If the set X bears some structure, one may distinguish some fuzzy subsets of X in terms of that additional structure. This important concept of a fuzzy set was first introduced by Zadeh [1] in his ground-breaking work, which enlarged the concept of a crisp subset. The theory of fuzzy sets has provided a useful mathematical tool for describing the behavior of systems that are too complex or ill-defined to admit precise mathematical analysis by classical methods and tools. Since then, many papers on fuzzy sets appeared showing the importance of the concept and its applications to artificial intelligence, computer science, control engineering, expert systems, management science, operations research, logic, set theory, group theory, groupoids, real analysis, measure theory, topology, and others. Many notions of mathematics are extended to such sets, and various properties of these notions in the context of fuzzy sets are established. In recent years, there has been considerable interest in the connections between fuzzy sets and algebraic structures theory. The readers are referred to [2-16].

In the original conception given in Zadeh [1], the unit interval $[0, 1]$ was equipped with the operations $\vee$ (maximum) and $\wedge$ (minimum). An important extension of this situation, however, was realized when the latter was replaced by an arbitrary *triangular norm* in the sense of Menger [17], namely an associative operation $*$ on $[0, 1]$ satisfying the condition that $([0, 1], \vee, *)$ is a semiring. Similarly, we have the notion of a *triangular conorm* on $[0, 1]$ satisfying the condition that $([0, 1], \wedge, *)$ is a semiring. In [18], Menger interpreted triangular norms in the context of continuum physics as rules for generating new probabilistically-determined objects from existing ones in the psychophysical continuum space. In [19], Butnariu and Klement used triangular norms for interaction rules for economic agents in fuzzy games.

Goguen [20] extended Zadeh's construction by replacing unit interval with an arbitrary bounded distributive lattice, a lead which has been followed by many others (see [21,22]). One may note that any bounded distributive lattice $(R, \vee, \wedge)$ is a commutative multiplicatively idempotent bounded lattice-ordered semiring in which multiplication distributes over arbitrary joins if we take $x + y = x \vee y$ and $x \cdot y = x \wedge y$ for all $x, y \in R$.

This paper investigates ideal theory of an ordered hemiring in a broader framework of the R-valued function, where R is a commutative multiplicatively idempotent bounded CLO-semiring in which multiplication distributes over arbitrary joins. It shows that the set of all R-valued ideals of a positive ordered hemiring forms an additively idempotent bounded CLO-hemiring and that some ordered hemirings can be embedded in it.

PRELIMINARIES

In this section, we summarize some basic concepts (see [23-26]) which will be used throughout the paper.

A *semiring* is an algebraic system $(R, +, \cdot)$ consisting of a non-empty set R together with two binary operations on R called addition and multiplication (denoted in the usual manner) satisfying the following conditions:

(1) $(R, +)$ is a commutative monoid with identity element 0_R;
(2) $(R, \cdot)$ is a monoid with identity element 1_R;
(3) Multiplication distributes over addition from either side;
(4) $0_R \cdot x = 0_R = x \cdot 0_R$ for all $x \in R$;
(5) $0_R \neq 1_R$.

If we do not have a multiplicative identity 1_R, then the structure is called a *hemiring*. R is *commutative* if $(R, \cdot)$ is commutative. A semiring (resp., hemiring) $(R, +, \cdot)$ is called *additively idempotent* if $x + x = x$ for all $x \in R$, and *multiplicatively idempotent* if $x \cdot x = x$ for all $x \in R$. For the sake of simplicity, we shall omit the symbol " $\cdot$ ", writing xy for $x \cdot y$ $(x, y \in R)$.

A semiring (resp., hemiring) $(R, +, \cdot)$ is called an *ordered semiring* (resp., *hemiring*) if and only if there exists a partial relation " $\leq$ " on R satisfying the following conditions for elements x, y and z of R:

(1) if $x \leq y$ then $x + z \leq y + z$;
(2) if $x \leq y$ and $z \geq 0_R$ then $xz \leq yz$ and $zx \leq zy$.

In particular, an ordered semiring (resp., hemiring) is called a *totally ordered semiring* (resp., *hemiring*) if $(R, \leq)$ is a totally ordered set. And an ordered semiring (resp., hemiring) is called *positive* if $x \geq 0_R$ for all $x \in R$.

A semiring (resp., hemiring) $(R, +, \cdot)$ is a *complete-lattice-order semiring* (resp., *hemiring*), denoted as CLO-semiring (resp. CLO-hemiring), if and only if R has the structure of a complete lattice satisfying the conditions that $x+y = x \vee y$ and $xy \leq x \wedge y$ for all $x, y \in R$.

Note that if the lattice structure of a CLO-semiring R is bounded, then the maximal element and minimal element must be 1_R and 0_R, respectively.

Let $(T, \oplus, \odot, \preceq)$ and $(R, +, \cdot, \leq)$ be ordered hemirings, $f : T \to R$ a mapping from T into R. f is called *isotone* if $x, y \in T, x \preceq y$ implies $f(x) \leq f(y)$. f is said to be *inverse isotone* if $x, y \in T, f(x) \leq f(y)$ implies $x \preceq y$ (each inverse isotone mapping is (1-1)). f is called a *homomorphism* if it is isotone and satisfies $f(x \oplus y) = f(x) + f(y)$ and $f(x \odot y) = f(x)f(y)$ for all $x, y \in T$. A homomorphism is called an *isomorphism* if it is surjective and inverse isotone T and R are called isomorphic if there exists an isomorphism between them. T is embedded in R if T is isomorphic to a subset of R, i.e., if there exists a mapping $f : T \to R$ which is homomorphism and inverse isotone.

In the sequel, unless otherwise stated, $(T, \oplus, \odot, \preceq)$ and $(R, +, \cdot, \leq)$ denote a positive ordered hemiring and a commutative multiplicatively idempotent bounded CLO-semiring in which multiplication distributes over arbitrary joins, respectively.

A non-empty subset A of T is called an ideal of T if it satisfies the following conditions:

(1) $A \oplus A \subseteq A$;
(2) $T \odot A \cup A \odot T \subseteq A$;
(3) if $x \in A$ and $T \ni y \preceq x$ then $y \in A$.

For $A \subseteq T$, we denote

$$(A] = \{x \in T | x \preceq y \text{ for some } y \in A\}.$$

Note that condition (3) is equivalent to the condition $(A] = A$. Let $x \in T$. It is obvious that $I(x) = (Nx \oplus T \odot x \oplus x \odot T \oplus T \odot x \odot T]$, where $N = \{0, 1, 2, \cdots\}$, is the principal ideal of T generated by x. In particular, $I(x) = (Nx \oplus T \odot x]$ if T is commutative.

Denote by R^T the set of all functions from T into R. For any $A \subseteq R$, the *characteristic function* of A, denoted by χ_A, is defined by

$$\chi_A(x) = \begin{cases} 1_R & \text{if } x \in A, \\ 0_R & \text{otherwise} \end{cases}$$

for all $x \in R$. A function $f \in R^T$ of the form

$$f(y) = \begin{cases} r & \text{if } y = x, \\ 0_R & \text{otherwise} \end{cases}$$

is said to be a *R-valued point with support x and value r* and is denoted by x_r, where $r \in R \backslash \{0_R\}$.

For $f, g \in R^T$, the *intersection* of f and g, denoted by $f \cap g$, is defined by $(f \cap g)(x) = f(x) \wedge g(x)$ for all $x \in T$.

R-VALUED IDEALS OF AN ORDERED HEMIRING

Let $f, g \in R^T$. We introduce the *sum*, *product* and *intrinsic product* of f and g as follows.

Definition 1. Let $f, g \in R^T$. The *sum, product,* and *intrinsic product* of f and g, denoted by $f \boxplus g$, $f \boxdot g$ and $f\langle\boxdot\rangle g$, are defined by: $\forall x \in T$

$$(f \boxplus g)(x) = \bigvee_{y,z\in T,\, x\preceq y\oplus z} f(y)g(z)$$

$$(f\boxdot g)(x) = \begin{cases} \bigvee_{y,z\in T,\, x\preceq y\odot z} f(y)g(z) & \text{if } \exists\, y,z \in R \\ \text{such that } x \preceq y \odot z, & \\ 0_R \ \text{otherwise.} & \end{cases}$$

and

$$(f\langle\boxdot\rangle g)(x) = \begin{cases} \bigvee_{x\preceq\sum_{i=1}^m y_i\odot z_i} \prod_i^m f(y_i)g(z_i) & \text{if } \exists\, y_i, z_i \in T \\ & \text{such that} \\ x \preceq y_1 \odot z_1 \oplus \cdots \oplus y_m \odot z_m, & \\ 0_R & \text{otherwise.} \end{cases}$$

Define an order relation " $\subseteq$ " on R^T by: $\forall f, g \in R^T$

$$f \subseteq g \Leftrightarrow f(x) \le g(x) \text{ for all } x \in T.$$

Lemma 2. *Let $f, g, h \in R^T$. Then*

(1) $(f \boxdot g) \boxdot h = f \boxdot (g \boxdot h)$ and $(f\langle\boxdot\rangle g)\langle\boxdot\rangle h = f\langle\boxdot\rangle(g\langle\boxdot\rangle h)$.
(2) $(f \boxplus g) \boxplus h = f \boxplus (g \boxplus h)$.
(3) $f \boxdot g \subseteq f\langle\boxdot\rangle g$.
(4) $0_{T_{1_R}} \boxplus f = f = f \boxplus 0_{T_{1_R}}$ and $0_{T_{1_R}}\langle\boxdot\rangle f = 0_{T_{1_R}} = f\langle\boxdot\rangle 0_{T_{1_R}} = f$.
(5) $f \boxplus g = g \boxplus f$. If T is commutative, then $f \boxdot g = g \boxdot f$.

Proof. We only show $(f \boxdot g) \boxdot h = f \boxdot (g \boxdot h)$. The other properties can be similarly proved. If there does not exist $y, z \in R$ such that $x \preceq y \odot z$, then

$$((f \boxdot g) \boxdot h)(x) = 0_R = (f \boxdot (g \boxdot h))(x).$$

Otherwise, we have

$$\begin{aligned} &((f \boxdot g) \boxdot h)(x) \\ &= \bigvee_{x\preceq y\odot z} (f \boxdot g)(y)h(z) \\ &= \begin{cases} \bigvee_{x\preceq y\odot z} \left(\bigvee_{y\preceq u\odot v} f(u)g(v)h(z) \right) & \text{if there exist } u, \\ & v \in T \text{ such that} \\ & y \preceq u \odot v, \\ 0_R & \text{otherwise.} \end{cases} \\ &= \begin{cases} \bigvee_{x\preceq u\odot v\odot z} f(u)g(v)h(z) & \text{if there exist } u, v, z \in T \\ & \text{such that } x \preceq u \odot v \odot z, \\ 0_R & \text{otherwise.} \end{cases} \\ &= (f \boxdot (g \boxdot h))(x). \end{aligned}$$

It follows $(f \boxdot g) \boxdot h = f \boxdot (g \boxdot h)$. $\square$

Lemma 3. *Let $f, g, h \in R^T$. Then*

(1) $f \boxdot (g \boxplus h) \subseteq f \boxdot g \boxplus f \boxdot h$ and $(g \boxplus h) \boxdot f \subseteq g \boxdot f \boxplus h \boxdot f$.
(2) $f\langle\boxdot\rangle(g \boxplus h) \subseteq f\langle\boxdot\rangle g \boxplus f\langle\boxdot\rangle h$ and $(g \boxplus h)\langle\boxdot\rangle f \subseteq g\langle\boxdot\rangle f \boxplus h\langle\boxdot\rangle f$.

Proof. We only show (1). (2) can be similarly proved. Let $x \in T$. If there does not exist $y, z \in R$ such that $x \preceq y \odot z$, then

$$(f \boxdot (g \boxplus h))(x) = 0_R \le (f \boxdot g \boxplus f \boxdot h)(x).$$

Otherwise, we have

$$\begin{aligned} &(f \boxdot (g \boxplus h))(x) \\ &= \bigvee_{x\preceq y\odot z} f(y)(g \boxplus h)(z) \\ &= \bigvee_{x\preceq y\odot z} \left(\bigvee_{z\preceq u\oplus v} f(y)g(u)h(v) \right) \\ &\le \bigvee_{x\preceq y\odot u\oplus y\odot v} (f(y)g(u))(f(y)h(v)) \\ &\le \bigvee_{x\preceq y\odot u\oplus y\odot v} (f \boxdot g)(y \odot u)(f \boxdot g)(y \odot v) \\ &\le \bigvee_{x\preceq a\oplus b} (f \boxdot h)(a)(g \boxdot h)(b) \\ &= (f \boxdot g \boxplus f \boxdot h)(x). \end{aligned}$$

It follows $f \boxdot (g \boxplus h) \subseteq f \boxdot g \boxplus f \boxdot h$. In a similar way we may prove that $(g \boxplus h) \boxdot f \subseteq g \boxdot f \boxplus h \boxdot f$. $\square$

Lemma 4. *Let $f, g, h \in R^T$ be such that $f \subseteq g$. Then*

(1) $f \boxplus h \subseteq g \boxplus h$.
(2) $f \boxdot h \subseteq g \boxdot h$ and $h \boxdot f \subseteq h \boxdot g$.
(3) $f\langle\boxdot\rangle h \subseteq g\langle\boxdot\rangle h$ and $h\langle\boxdot\rangle f \subseteq h\langle\boxdot\rangle g$.

Proof. We only show (1). (2) and (3) can be similarly proved. Let $x \in T$. Then

$$\begin{aligned} (f \boxplus h)(x) &= \bigvee_{x\preceq y\oplus z} f(y)h(z) \\ &\le \bigvee_{x\preceq y\oplus z} g(y)h(z) = (g \boxplus h)(x). \end{aligned}$$

It follows $f \boxplus h \subseteq g \boxplus h$. $\square$

Next, we introduce the concept of *R-valued ideals* of T as follows.

Definition 5. Let $f \in R^T$. f is called an *R-valued ideal* of T if it satisfies the following conditions: $\forall x, y \in T$

(F1a) $f(0_T) = 1_R$;
(F2a) $f(x \oplus y) \ge f(x)f(y)$;
(F3a) $f(x \odot y) \ge f(x) + f(y)$;
(F4a) if $x \preceq y$ then $f(x) \ge f(y)$.

Denote by RI^T the set of all R-value ideals of T.

Example 6. Define on the set $T = \{0, x, y, z\}$ a totally order relation by $0 \prec x \prec y \prec z$ and an addition operation "$\oplus$", a multiplication operation "$\odot$" by the tables:

$\oplus$	0	x	y	z
0	0	x	y	z
x	x	x	y	z
y	y	y	z	x
z	z	z	x	y

and

$\odot$	0	x	y	z
0	0	0	0	0
x	0	x	x	x
y	0	x	x	x
z	0	x	x	x

Then $(T, \oplus, \odot, \preceq)$ is a positive totally order hemiring. Let $(R, \vee, \wedge, 0, 1)$ be a complete distribute bounded lattice which is clearly a CLO-semiring if we define $r+s = r \vee s$ and $rs = r \wedge s$ for all $r, s \in R$. Define f be the function from T into R as follows:

$$f(0) = 1,\ f(x) = r,\ f(y) = s \text{ and } f(z) = s,$$

where $0 \leq s < r \leq 1$. Then f is an R-valued ideal of T.

Let $f \in R^T$ and $r \in R$. Define a set $f_r = \{x \in T | f(x) > r\}$, called the *$r$-strong cut set* of f. Then it is not difficult to see that the following results are valid, which present the relationships between crisp ideals and R-valued ideals of T.

Theorem 7. *Let $A \subseteq T$. Then A is an ideal of T if and only if χ_A is an R-valued ideal of T.*

Theorem 8. *Let $f \in R^T$. Then f is an R-valued ideal of T if and only if f_t ($f_t \neq \emptyset$) is an ideal of T for all $t \in R \setminus \{1_R\}$.*

Theorem 9. *Let $f \in RI^T$. Then $\hat{f}$ is an ideal of T, where $\hat{f} = \{x \in T | f(x) = 1_R\}$.*

Lemma 10. *Let $f \in R^T$ be such that $x \preceq y$ implies $f(x) \geq f(y)$. Then (F2a) holds if and only if the following condition holds:*
(F2b) $f \boxplus f = f$.

Proof. (F2a)$\Rightarrow$(F2b) Let $x \in T$. If $x \preceq y \oplus z$ for some $y, z \in T$, then $f(x) \geq f(y \oplus z)$. Hence

$$\begin{aligned}(f \boxplus f)(x) = \bigvee_{x \preceq y \oplus z} f(y)f(z) &\leq \bigvee_{x \preceq y \oplus z} f(y \oplus z)\\ &\leq \bigvee_{x \preceq y \oplus z} f(x) = f(x)\end{aligned}$$

It follows $f \boxplus f \subseteq f$.

On the other hand,

$$\begin{aligned}(f \boxplus f)(x) = \bigvee_{x \preceq y \oplus z} f(y)f(z) \geq f(x)f(0_T) &= f(x)1_R\\ &= f(x).\end{aligned}$$

It follows $f \subseteq f \boxplus f$. Hence $f \boxplus f = f$.

(F2b)$\Rightarrow$(F2a) Let $x, y \in T$. Then

$$\begin{aligned}f(x \oplus y) &= (f \boxplus f)(x \oplus y)\\ &= \bigvee_{x \oplus y \preceq u \oplus v} f(u)f(v)\\ &\geq f(x)f(y).\end{aligned}$$

Hence (F2a) holds. $\square$

Lemma 11. *Let $f \in R^T$ be such that $x \preceq y$ implies $f(x) \geq f(y)$. Then (F3a) holds if and only if the following condition holds:*
(F3b) $\chi_T \boxdot f \subseteq f$ and $f \boxdot \chi_T \subseteq f$.

Proof. The proof is analogous to that of Lemma 7. $\square$

Lemma 12. *Let $f, g \in RI^T$. Then $f \subseteq f \boxplus g$ and $g \subseteq f \boxplus g$.*

Proof. Let $x \in T$. Then

$$(f \boxplus g)(x) = \bigvee_{x \preceq y \oplus z} f(y)g(z) \geq f(0_T)g(x) = 1_R g(x) = g(x).$$

It follows $f \subseteq f \boxplus g$ and then $g \subseteq f \boxplus g$. $\square$

Theorem 13. *Let $f, g \in RI^T$. Then so are $f \boxplus g$, $f\langle\boxdot\rangle g$ and $f \cap g$.*

Proof. We only show that $f\langle\boxdot\rangle g$ is an R-valued ideal of T. The cases for $f \boxplus g$ and $f \cap g$ can be similarly proved.

(1) $(f\langle\boxdot\rangle g)(0_T) = \bigvee_{0_R \preceq \sum_{i=1}^m y_i \odot z_i} \prod_i^m f(y_i)g(z_i) \geq f(0_T)g(0_T) = 1_R$.

(2) Let $x, y \in T$. If there does not exist $u_i, v_i \in T$ such that $x \preceq u_1 \odot v_1 \oplus \cdots \oplus u_m \odot v_m$ or $y \preceq u_1 \odot v_1 \oplus \cdots \oplus u_m \odot v_m$, then

$$(f\langle\boxdot\rangle g)(x)(f\langle\boxdot\rangle g)(y) = 0_R \leq (f\langle\boxdot\rangle g)(x \oplus y).$$

Otherwise, we have

$$\begin{aligned}&(f\langle\boxdot\rangle g)(x \oplus y)\\ &= \bigvee_{x \oplus y \preceq \sum_{i=1}^m u_i \odot v_i} \prod_i^m f(u_i)g(v_i)\\ &\geq \bigvee_{x \preceq \sum_{i=1}^{m_1} a_i \odot b_i,\, y \preceq \sum_{j=1}^{m_2} c_j \odot d_j} \prod_i^{m_1} f(a_i)g(b_i) \prod_j^{m_2} f(c_j)g(d_j)\\ &= \left(\bigvee_{x \preceq \sum_{i=1}^{m_1} a_i \odot b_i} \prod_i^{m_1} f(a_i)g(b_i)\right)\left(\bigvee_{y \preceq \sum_{j=1}^{m_2} c_j \odot d_j} \prod_j^{m_2} f(c_j)g(d_j)\right)\\ &= (f\langle\boxdot\rangle g)(x)(f\langle\boxdot\rangle g)(y).\end{aligned}$$

Hence, in any case, we have $(f\langle\boxdot\rangle g)(x \oplus y) \geq (f\langle\boxdot\rangle g)(x)(f\langle\boxdot\rangle g)(y)$.

(3) Let $x, y \in T$. If there does not exist $u_i, v_i \in T$ such that $x \preceq u_1 \odot v_1 \oplus \cdots \oplus u_m \odot v_m$, then

$$(f\langle\boxdot\rangle g)(x) = 0_R \leq (f\langle\boxdot\rangle g)(x \odot y).$$

Otherwise, we have

$$\begin{aligned}&(f\langle\boxdot\rangle g)(x \odot y)\\ &\geq \bigvee_{x \preceq \sum_{i=1}^{m} u_i \odot v_i} \prod_{i}^{m} f(u_i) g(v_i \odot y)\\ &\geq \bigvee_{x \preceq \sum_{i=1}^{m} u_i \odot v_i} \prod_{i}^{m} f(u_i) g(u_i) = (f\langle\boxdot\rangle g)(x).\end{aligned}$$

In a similar way, we may prove that $(f\langle\boxdot\rangle g)(x \odot y) \geq (f\langle\boxdot\rangle g)(y)$.

(4) Let $x \preceq y$. Then $y \preceq u_1 \odot v_1 \oplus \cdots \oplus u_m \odot v_m$ for some $u_i, v_i \in T$ gives $x \preceq u_1 \odot v_1 \oplus \cdots \oplus u_m \odot v_m$, and so $(f\langle\boxdot\rangle g)(x) \geq (f\langle\boxdot\rangle g)(y)$.

Summing up the above arguments, $f\langle\boxdot\rangle g \in RI^T$. $\square$

Lemma 14. *Let $f, g, h \in RI^T$. Then $f\langle\boxdot\rangle(g \boxplus h) = f\langle\boxdot\rangle g \boxplus f\langle\boxdot\rangle h$ and $(g \boxplus h)\langle\boxdot\rangle f = g\langle\boxdot\rangle f \boxplus h\langle\boxdot\rangle f$.*

Proof. We only present a proof for the first equality. By Lemma 3(2), it remains to show $f\langle\boxdot\rangle(g \boxplus h) \supseteq f\langle\boxdot\rangle g \boxplus f\langle\boxdot\rangle h$. Note first that $g \boxplus h \supseteq g$ and $g \boxplus h \supseteq h$, and so $f\langle\boxdot\rangle(g \boxplus h) \supseteq f\langle\boxdot\rangle g$ and $f\langle\boxdot\rangle(g \boxplus h) \supseteq f\langle\boxdot\rangle h$ by Lemma 4. In addition, it follows from Theorem 13 that $f\langle\boxdot\rangle(g \boxplus h) \in RI^T$. Hence $f\langle\boxdot\rangle(g \boxplus h) = f\langle\boxdot\rangle(g \boxplus h) \boxplus f\langle\boxdot\rangle(g \boxplus h) \supseteq f\langle\boxdot\rangle g \boxplus f\langle\boxdot\rangle h$. Therefore, $f\langle\boxdot\rangle(g \boxplus h) = f\langle\boxdot\rangle g \boxplus f\langle\boxdot\rangle h$. $\square$

Theorem 15. *$(RI^T, \boxplus, \langle\boxdot\rangle)$ is an additively idempotent hemiring with zero element $0_{T_{1_R}}$. In addition, if T is commutative, then so is RI^T.*

Proof. The desired result follows from the following facts.

(1) $(RI^T, \boxplus)$ is a commutative idempotent monoid with identity element $0_{T_{1_R}}$ by Lemmas 2, 10 and Theorem 13.

(2) $(RI^T, \boxdot)$ is a semigroup by Lemma 2 and Theorem 13.

(3) "$\boxdot$" distributes over "$\boxplus$" from either side by Lemma 3.14. $\square$

Lemma 16. *Let $f, g \in R^T$ and $h \in RI^T$. Then $f \boxdot g \subseteq h$ if and only if $f\langle\boxdot\rangle g \subseteq h$.*

Proof. It is obvious that $f\langle\boxdot\rangle g \subseteq h$ implies $f \boxdot g \subseteq h$ by Lemma 2. Now assume that $f \boxdot g \subseteq h$. Let $x \in T$. If there does not exist $u_i, v_i \in T$ such that $x \preceq u_1 \odot v_1 \oplus \cdots \oplus u_m \odot v_m$, then

$$(f\langle\boxdot\rangle g)(x) = 0_R \leq h(x).$$

Otherwise, we have

$$\begin{aligned}(f\langle\boxdot\rangle g)(x) &= \bigvee_{x \preceq \sum_{i=1}^{m} u_i \odot v_i} \prod_{i}^{m} f(u_i) g(v_i)\\ &\leq \bigvee_{x \preceq \sum_{i=1}^{m} u_i \odot v_i} \prod_{i}^{m} (f \boxdot g)(u_i \odot v_i)\\ &\leq \bigvee_{x \preceq \sum_{i=1}^{m} u_i \odot v_i} \prod_{i}^{m} h(u_i \odot v_i) \leq h(x).\end{aligned}$$

It follows $f\langle\boxdot\rangle g \subseteq h$. $\square$

Theorem 17. *$(RI^T, \boxplus, \cap, \subseteq)$ is a complete bounded lattice with the minimal element $0_{T_{1_R}}$ and maximal element χ_T. In addition, $f\langle\boxdot\rangle g \subseteq f \cap g$ for all $f, g \in RI^T$.*

Proof. Let $f, g \in RI^T$. It follows from Theorem 13 that $f \wedge g = f \cap g \in RI^T$ and $f \boxplus g \in RI^T$. We now show that $f \vee g = f \boxplus g$. It is clear that $f \subseteq f \boxplus g$ and $g \subseteq f \boxplus g$. Now, let $g \in RI^T$ be such that $f \subseteq h$ and $g \subseteq h$. Then, $f \boxplus g \subseteq h \boxplus g \subseteq g$. Hence $f \vee g = f \boxplus g$. This implies that $(RI^T, \boxplus, \cap, \subseteq)$ is a lattice. And it is easy to see that RI^T is complete and that $0_{T_{1_R}}$ and χ_T are the minimal element and maximal element of RI^T, respectively. Next, we show that $f\langle\boxdot\rangle g \subseteq f \cap g$. From Lemma 11, $f \boxdot g \subseteq f \boxdot \chi_T \cap \chi_T \boxdot g \subseteq f \cap g$. Hence it follows from Lemma 16 that $f\langle\boxdot\rangle g \subseteq f \cap g$. $\square$

From Lemma 4, Theorems 15 and 17, we have the following result.

Theorem 18. *$(RI^T, \boxplus, \langle\boxdot\rangle, \subseteq, 0_{T_{1_R}}, \chi_T)$ is an additively idempotent bounded CLO-hemiring.*

THE EMBEDDING OF SOME ORDERED HEMIRINGS

Given $f \in RI^T$, we call the least R-valued ideal of T, containing f, the R-valued ideal of T generated by f, denoted by $\langle f\rangle$. By Theorem 13, the R-valued ideal $\langle f\rangle$ of T is actually the intersection $\bigcap\{g \in RI^T | f \subseteq g\}$. From Theorems 15 and 17, it is easy to check that $f\langle\boxdot\rangle g = \langle f \boxdot g\rangle$ for all $f, g \in R^T$.

Lemma 19. *Let $x \in T$ and $r \in R \setminus \{0_R\}$. Then $\langle x_r\rangle$ is the R-valued ideal generated by x_r, where $\langle x_r\rangle$ is defined by*

$$\langle x_r\rangle(y) = \begin{cases} 1_R & \text{if } y = 0_T,\\ r & \text{if } y \in I(x)\backslash\{0_T\},\\ 0_R & \text{otherwise}.\end{cases}$$

for all $y \in T$.

Proof. It is easy to see that $\langle x_r \rangle$ is an R-valued ideal of T. Now let $f \in RI^T$ be such that $x_r \subseteq f$ and $y \in T$. We show that $\langle x_r \rangle \subseteq f$. The verification is as follows.

Case 1 $y \notin I(x)$. Then $\langle x_r \rangle(y) = 0_R \leq f(y)$.

Case 2 $y = 0_T$. Then $f(y) = 1_R = \langle x_r \rangle(y)$.

Case 3 $y \in I(x)\backslash\{0_T\}$. Then there exist $m \in \{0, 1, 2, \cdots\}$ and $a, b, c, d \in T$ such that $y \preceq mx \oplus a \odot x \oplus x \odot b \oplus c \odot x \odot d$, and so

$$\begin{aligned} f(y) \geq& f(x)f(a \odot x)f(x \odot b)f(c \odot x \odot d) \\ \geq& f(x)f(x)f(x)f(x) = f(x) \\ \geq& r = \langle x_r \rangle(y). \end{aligned}$$

Thus, in any case, we have $\langle x_r \rangle \subseteq f$. This completes the proof. $\square$

Lemma 20. *Let $x, y \in T$ and $r \in R\backslash\{0_R\}$. If T is commutative, then*

(1) $\langle x_r \rangle \boxplus \langle y_r \rangle = \langle (x \oplus y)_r \rangle$.
(2) $\langle x_r \rangle \langle \boxdot \rangle \langle y_r \rangle = \langle (x \odot y)_r \rangle$.

Proof. (1) Let $z \in T$. We consider the following cases.

Case 1. $z \notin I(x \oplus y)$. Then $\langle (x \oplus y)_r \rangle(z) = 0_R$ and $a \notin I(x)$ or $b \notin I(y)$ for $z \preceq a \oplus b$. Otherwise, if $a \notin I(x)$ and $b \notin I(y)$, then there exist $m, n \in \{0, 1, \cdots, \}$ and $c, d \in T$ such that $a \preceq mx \oplus c \odot x$ and $b \preceq nx \oplus d \odot x$ and so $z \preceq a \oplus b \preceq mx \oplus c \odot x \oplus nx \oplus d \odot x = (m+n)x \oplus (c \oplus d) \odot x$, i.e., $z \in I(x \oplus y)$, a contradiction. Hence $\langle x_r \rangle(a) = 0_R$ or $\langle y_r \rangle(b) = 0_R$ and so

$$(\langle x_r \rangle \boxplus \langle y_r \rangle)(z) = \bigvee_{z \preceq a \oplus b} \langle x_r \rangle(a) \langle y_r \rangle(b) = 0_R.$$

Case 2. $z = 0_T$. Then $\langle (x \oplus y)_r \rangle(z) = 1_R$ and $(\langle x_r \rangle \boxplus \langle y_r \rangle)(z) = \bigvee_{z \preceq a \oplus b} \langle x_r \rangle(a) \langle y_r \rangle(b) \geq \langle x_r \rangle(0_T) \langle y_r \rangle(0_T) = 1_R$.

Case 3. $z \in I(x \oplus y)\backslash\{0_T\}$. Then $\langle (x \oplus y)_r \rangle(z) = r$ and there exist $m \in \{0, 1, \cdots, \}$ and $a \in T$ such that $z \preceq m(x \oplus y) \oplus a \odot (x \oplus y) = (mx \oplus a \odot x) \oplus (my \oplus a \odot y)$. Hence $(\langle x_r \rangle \boxplus \langle y_r \rangle)(z) = \bigvee_{z \preceq a \oplus b} \langle x_r \rangle(a) \langle y_r \rangle(b) \geq \langle x_r \rangle(mx \oplus a \odot x) \langle y_r \rangle(my \oplus a \odot y) = r$. It is obvious that $(\langle x_r \rangle \boxplus \langle y_r \rangle)(z) \neq 1_R$, otherwise $z = 0_T$. Hence $(\langle x_r \rangle \boxplus \langle y_r \rangle)(z) = r$.

Thus, in any case, we have $(\langle x_r \rangle \boxplus \langle y_r \rangle)(z) = \langle (x \oplus y)_r \rangle(z)$. It follows $\langle x_r \rangle \boxplus \langle y_r \rangle = \langle (x \oplus y)_r \rangle$.

(2) It is obvious that $\langle x_r \rangle \langle \boxdot \rangle \langle y_r \rangle = \langle x_r \odot y_r \rangle = \langle (x \odot y)_r \rangle$. $\square$

Theorem 21. *If T is a commutative positive totally ordered hemiring satisfying the following condition: for any $x, y \in T$, $x \neq y$ implies $I(x) \neq I(y)$, then $(T, \oplus, \odot, \preceq)$ is embedded in $(RI^T, \boxplus, \langle \boxdot \rangle, \subseteq)$.*

Proof. Let $r \in R\backslash\{0_R\}$. We consider the mapping $\varphi : T \to RI^T \mid x \mapsto \langle x_r \rangle$. Then we have:

(1) The mapping φ is well defined. The verification is as follows. Let $x, y_1, y_2 \in T$ be such that $y_1 = y_2$. Then

$$\langle x_r \rangle(y_1) = \langle x_r \rangle(y_2) = \begin{cases} 1_R & \text{if } y_1 = 0_T, \\ r & \text{if } y_1 \in I(x)\backslash\{0_T\}, \\ 0_R & \text{otherwise.} \end{cases}$$

Let $x_1, x_2, y \in T$ be such that $x_1 = x_2$. Then

$$\langle x_{1r} \rangle(y) = \langle x_{2r} \rangle(y) = \begin{cases} 1_R & \text{if } y = 0_T, \\ r & \text{if } y \in I(x_1)\backslash\{0_T\}, \\ 0_R & \text{otherwise.} \end{cases}$$

(2) φ is a homomorphism. The verification is as follows.

(a) φ is isotone. Let $x, y \in T$ be such that $x \preceq y$. For any $z \in T$, we consider the following cases.

Case 1. $z = 0_T$. Then $\langle x_r \rangle(z) = 1_R = \langle y_r \rangle(z)$.

Case 2. $z \notin I(x)$. Then $\langle x_r \rangle(z) = 0_R \leq \langle y_r \rangle(z)$.

Case 3. $z \in I(x)\backslash\{0_T\}$. Then there exist $m \in \{0, 1, 2, \cdots\}$ and $a \in T$ such that $z \preceq mx \oplus a \odot x \preceq my \oplus a \odot y$, and so $\langle x_r \rangle(z) = r = \langle y_r \rangle(z)$.

Thus, in any case, we have $\langle x_r \rangle \subseteq \langle y_r \rangle$, i.e., $\varphi(x) \subseteq \varphi(y)$.

(b) $\varphi(x \oplus y) = \varphi(x) \boxplus \varphi(y)$ and $\varphi(x \odot y) = \varphi(x) \langle \boxdot \rangle \varphi(y)$. This is straightforward from Lemma 20.

(3)φ is reverse isotone. Let $x, y \in T$ be such that $\varphi(x) \subseteq \varphi(y)$, i.e., $\langle x_r \rangle \subseteq \langle y_r \rangle$. Suppose if possible, $x \npreceq y$, then $x \succ y$ since $\preceq$ is a totally order on T. Analogous to the proof of (a), we have $\varphi(x) \supset \varphi(y)$ since $I(x) \neq I(y)$, a contradiction. Hence $x \preceq y$. $\square$

Example 22. Define on the set $T = \{0, x, y\}$ a totally ordered relation by $0 \prec x \prec y$, and an addition operation " $\oplus$ " and a multiplication operation " $\odot$ " by the tables:

$\oplus$	0	x	y
0	0	x	y
x	x	x	y
y	y	y	y

and

$\odot$	0	x	y
0	0	0	0
x	0	x	x
y	0	x	x

Then $(T, \oplus, \odot, \preceq)$ is commutative positive totally ordered hemiring satisfying the following condition: for any $a, b \in T$, $a \neq b$ implies $I(a) \neq I(b)$ and can be embedded a commutative additively idempotent bounded CLO-hemiring.

CONCLUSIONS

In this paper, we introduced the concept of an R-valued ideal of an ordered hemiring and provided some of its properties. We further showed that the set of all R-valued ideals of a positive ordered hemiring forms an additively idempotent bounded CLO-hemiring and that some ordered hemirings can be embedded in it. Our future work on this topic will focus on studying other R-valued ideals of hemirings (semirings) such as R-valued k-ideals and R-valued h-ideals, and applying the results to the other algebraic structures.

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REFERENCES

1. Zadeh, L.A. 1965. Fuzzy sets, Inform and control, 8: 338-353.

2. Ahsan, J., Saifullah, K., Farid Khan, M. 1993. Fuzzy semirings, Fuzzy Sets Syst., 60: 309-320.

3. Akram, M. 2011. A new structure of fuzzy Lie algebras, World Applied Sciences Journal, 14(12):1879-1887.

4. Akram, M. 2011. Bipolar fuzzy $\mathcal{L}$-Lie algebras, World Applied Sciences Journal, 14(12):1908-1913.

5. Baik, S.I., Kim, H.S. 2000. On fuzzy k-ideals in semirings, Kangweon-Kyunki Math. J., 8: 147-154.

6. Dudek, W.A., Shabir, M., Irfan Ali, M. 2009. (α, β)-fuzzy ideals of hemirings, Comput. Math. Appl., 58: 310-321.

7. Dutta, T.K., Biswas, B.K. 1994. Fuzzy prime ideals of a semiring, Bull. Malaysian Math. Soc, 17: 9-16.

8. Dutta, T.K., Biswas, B.K. 1995. Fuzzy k-ideals of semirings, Bull. Cal. Math. Soc, 87: 91-96.

9. Jun, Y.B., $\ddot{O}zt\ddot{u}$rk, M.A., Song, S.Z. 2004. On fuzzy h-ideals in hemirings, Inform. Sci, 162: 211-226.

10. Kim, C.B. 2000. Isomorphism theorems and fuzzy k-ideals of k-semirings, Fuzzy Sets Syst, 112: 333-342.

11. Ma, X.L., Zhan, J.M. 2009. Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings, Inform. Sci, 179: 1249-1268.

12. Yin, Y.Q., Li, H.X. 2008. The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings, Inform. Sci, 178: 3451-3464.

13. Yin, Y.Q., Huang, X.K., Xu, D., Li, F. 2009. The characterization of h-semisimple hemirings, Int. J. Fuzzy Syst., 11: 116-122.

14. Yin, Y.Q., Zhan, J.M., Huang, X.K. 2011. A new way to fuzzy ideals of hemirings, Iran. J. Fuzzy Syst., 8(5): 81-101.

15. Yin, Y.Q., Jun, Y.B., Zhan, J.M. 2011. Vague soft hemirings, Comput. Math. Appl., 62: 199-213.

16. Zhan, J.M., Dudek, W.A. 2007. Fuzzy h-ideals of hemirings, Inform. Sci, 177: 876-886.

17. Menger, K. 1942. Statistical metrics, Proc Natl Acad Sci USA, 28(12): 535-537.

18. Menger, K. 1951. Probabilistic theories of relations, Proc. Nat. Acad. Sci. USA, 37(3): 178-80.

19. Butnariu, D., Klement, E.P. 1993. Triangular norm-based measures and games with fuzzy coalitions, Kluwer, Drodrecht.

20. Goguen, J.A. 1967. L-fuzzy sets, J. Math. Anal. Appl., 18: 145-174.

21. Jun, Y.B., Neggers, J., Kim, H.S. 1996. Normal L-fuzzy ideals in semirings, Fuzzy Sets Syst, 82: 383-386.

22. Zhan, J.M., Tan, Z. 2003. T-fuzzy k-ideals of semirings, Sci. Math. Japon, 58: 597-601.

23. Glazek, K. 2002. A Guide to the Literature on Semirings and their Applications in Mathematics and Information Sciences: With Complete Bibliography, Kluwer Acad. Publ., Dordrecht.

24. Golan, J.S., 1992. The theory of semirings with applications in mathematics and theoretical computer science, Academic Press, Inc. Duluth, MN, USA.

25. Golan, J.S. 1999. Power algebras over semirings with applications in mathematics and computer science, Kluwer Academic Publishers, Boston, London.

26. Jonathan, S., Golan, J.S. 1999. Semirings and their Applications, Kluwer, Dodrecht.