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A Comparative Study of Numerical Methods for Solving the Riccati Equation

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Abstract: In this paper, a Laplace transform decomposition algorithm (LTDA) is proposed to solve the Riccati equation. Comparisons are made among the homotopy perturbation method (HPM), the Adomian decomposition method (ADM) and the proposed method. It is shown that the homotopy perturbation method with a specific convex homotopy is equivalent to the Adomian decomposition method and the Laplace transform decomposition algorithm for solving the Riccati equation.

Key words: Laplace transform decomposition algorithm · Adomian's decomposition method · Homotopy perturbation method · Riccati equations

INTRODUCTION

Sometimes using several numerical methods to solve a problem may give similar results. It is noticeable that applying different numerical methods to solve a problem may provide just the same results. For example, using the ADM and the successive approximation method for linear integral equations [1], the ADM and the power series method for differential equations [2] and the ADM and the Jacobi iterative method for system of linear equation [3], we will get just the same results.

Since the beginning of the 1980s, the ADM has been applied to a wide class of functional equations [4, 5]. The ADM has been demonstrated to provide accurate and computable solutions for a wide class of linear or nonlinear equations involving differential operators by representing nonlinear terms in Adomian's polynomials [6, 7]. This procedure requires neither linearization nor perturbation.

The LTDA is an approach based on the ADM, which is to be considered an effective method in solving many problems for it provides, in general, a rapidly convergent series solution. Since the use of the Laplace transform replaces differentiation with simple algebraic operations on the transform, the algebraic equation is then solvable by decomposition. The LTDA approximates the exact solution with a high degree of accuracy using only few terms of the iterative scheme [8].

Numerical Methods for the Riccati Equation: Many authors have applied this method to solve Bratu's equation [9], the Duffing equation [10], the Klein-Gordon equation [11], some nonlinear differential equations [11], the nonlinear coupled partial differential equation [13] and integro-differential equations [14]. It is shown that this algorithm solves the Riccati equation too.

Until recently, the application of the HPM in nonlinear problems has been developed by scientists and engineers because this method continuously deforms a simple problem that is easy to solve into the difficult problem under study. Most perturbation methods assume an existing small parameter but most nonlinear problems have no small parameter whatsoever. Many new methods have been proposed to eliminate the small parameter [15, 16]. The homotopy theory becomes a powerful mathematical tool when it is successfully coupled with perturbation theory [17, 18].

The Riccati equations are one of the most important classes of nonlinear differential equations and play a significant role in many fields of applied science [19]. The importance of the Riccati equation usually arises in optimal control problems. Several authors have proposed different methods to solve the Riccati equation. For example, Bulut and Evans [20] applied the decomposition method for solving the Riccati equation. El-Tawil *et al* [21] applied the multistage Adomian's decomposition method for solving the Riccati differential equation and compared

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the result with the standard ADM. S. Abbasbandy [22] applied the HPM to solve the Riccati equation and compared the obtained results for this equation. Also, he [23] used the iterated homotopy perturbation method to solve this equation.

In this paper, we consider the Riccati equation

$$\frac{dy(t)}{dt} = 2y(t) - y^2(t) + 1;\ \ y(0) = 0, \tag{1}$$

and apply three numerical methods including the ADM, the LTDA and the HPM for solving it. We show that the HPM with a specific convex homotopy for solving the Riccati equation is equivalent to the ADM and the LTDA.

Numerical Methods for the Riccati Equation

Adm for the Riccati Equation: Consider the Riccati equation (1). Denoting $\frac{d}{dt}$ by G, we have G^{-1} as an integration from 0 to t. Operating with G^{-1} and using the initial condition, we obtain

$$y(t) = \int_0^t dt + 2\int_0^t y(t)dt - \int_0^t y^2(t)dt. \tag{2}$$

To apply the ADM to (2), let $y(t) = \sum_{n=0}^{\infty} y_n(t)$ and $y^2(t) = \sum_{n=0}^{\infty} A_n(t)$, whsere the $A_n(t)$'s are the Adomian polynomials depending on $y_0(t), y_1(t), \ldots, y_n(t)$. Upon substitution, (2) can be written as

$$\sum_{i=0}^{\infty} y_i(t) = t + 2\int_0^t \sum_{i=0}^{\infty} y_i(t)dt - \int_0^t \sum_{i=0}^{\infty} A_i(t)dt. \tag{3}$$

Based on the recursion scheme of the ADM, we define

$$y_o(t) = t \tag{4}$$

Numericnal Methods for the Riccati Equation:

$$y_{n+1} = 2\int_0^t y_n(t)\,dt - \int_0^t A_n(t)\,dt \text{ for } n \geq 0.$$

Note that the Adomian polynomials $A_n(t)$´s for the quadratic nonlinearty are written as [24, 25]

$$\begin{aligned} &A_0(t) = y_0^2(t),\\ &A_1(t) = 2y_0(t)y_1(t),\\ &A_2(t) = y_1^2(t) + 2y_0(t)y_2(t),\\ &A_3(t) = 2y_1(t)y_2(t) + 2y_0(t)y_3(t),\\ &\vdots \end{aligned} \tag{6}$$

The solution components $y_n(t)$ from (5) can be calculated as

$$y_1(t) = 2\int_0^t y_0(t)\,dt - \int_0^t A_0(t)\,dt = t^2 - \frac{t^3}{3}, \tag{7}$$

$$y_2(t) = 2\int_0^t y_1(t)\,dt - \int_0^t A_1(t)\,dt = \frac{2}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5. \tag{8}$$

Similarly, the solution components $y_n(t)$ are calculated for n = 3, 4, . . . but are not listed for brevity. It is clear that better approximations can be obtained by evaluating more components of the decomposition series solution $y(t)$. We note here that the convergence question of this technique has been formally proved and justified by [26, 27, 28].

LTDA for the Riccati Equation: Here the LTDA is applied to find the solution of (1). The method consists of applying the Laplace transformation (denoted throughout this paper by L) to both sides of (1), where

$$L[y'(t)] = L[1] + 2L[y(t)] - L[y^2(t)]. \tag{9}$$

Applying the formulas of the respective Laplace transforms, we obtain

$$sL[y(t)] - y(0) = L[1] + 2L[y(t)] - L[y^2(t)]. \tag{10}$$

Using the initial condition $y(0) = 0$, we have

$$L[y(t)] = \frac{1}{s}L[1] + \frac{2}{s}L[y(t)] - \frac{1}{s}L[y^2(t)]. \tag{11}$$

The Laplace transform decomposition technique consists of representing $y(t)$ by $\sum_{n=0}^{\infty} y_n(t)$ and $y^2(t)$ by $\sum_{n=0}^{\infty} A_n(t)$, where the $A_n(t)$'s are the same as those in subsection 2.1. Therefore, Equation (11) becomes

$$L[\sum_{n=0}^{\infty} y_n(t)] = \frac{1}{s}L[1] + \frac{2}{s}L[\sum_{n=0}^{\infty} y_n(t)] - \frac{1}{s}L[\sum_{n=0}^{\infty} A_n(t)] \tag{12}$$

Numerical Methods for the Riccati Equation: Matching both sides of (12), the following iterative algorithm is obtained

$$L[y_0(t)] = \frac{1}{s}L[1] = \frac{1}{s^2} \tag{13}$$

and

$$L[y_{n+1}(t)] = \frac{2}{s}L[y_n(t)] - \frac{1}{s}L[A_n(t)] \text{ for } n \geq 0 \tag{14}$$

Applying the inverse Laplace transform to Eq. (13), we obtain $y_o(t) = 0$. We now find the first few iterates of the recursive scheme. For $n = 0$, Eq. (14), becomes

$$L[y_1(t)] = \frac{2}{s}L[y_0(t)] - \frac{1}{s}L[A_0(t)] \tag{15}$$

Substituting $y_o(t)$ and $A_o(t)$ into (15) and applying the inverse Laplace transform, we have

$$y_1(t) = t^2 - \frac{t^3}{3} \tag{16}$$

For $n = 1$, Eq. (14) converts to

$$L[y_2(t)] = \frac{2}{s}L[y_1(t)] - \frac{1}{s}L[A_1(t)] \tag{17}$$

Putting (16) and (6) into Eq. (17) and applying the inverse Laplace transform yields

$$y_2(t) = \frac{2}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5 \tag{18}$$

The other terms $y_3(t)$, $y_4(t)$ are obtained recursively in a similar way by using (14) and applying the inverse Laplace transform. Notice that components of $y(t)$obtained from the LTDA method are just the same as those of the ADM. Therefore, these two methods are equivalent for solving the Ricatti equation and so the convergence of the LTDA method is the same as the ADM, which was mentioned in previous subsection.

HPM for the Riccati Equation: In order to apply the HPM for the Riccati equation, we can use two different kinds of convex homotopy. S. Abbabandy in [22] applied the HPM with one of the two kinds of convex homotopy to solve the Riccati Equation (1). In the following, we explain it briefly. Consider Equation (1) in the form

$$L(y(t)) + N(y(t)) = f(r) \tag{19}$$

Where $L(y(t)) = \frac{dy(t)}{dt} - 2y(t)$, $N(y(t)) = y^2(t)$ and $f(r)$ =1.

The initial approximation of (1) has the form $v_0(t) = t$. The convex homotopy that was considered is

$$\begin{aligned} H(y(t),p) = L(y(t)) - L(v_0(t)) + \\ pL(v_0(t)) + p[N(y(t)) - f(r)] = 0 \end{aligned} \tag{20}$$

Numerical Methods for the Riccati Equation: Based on the structure of the HPM, the solution of (20) can be expressed as a series in p

$$y(t) = y_0(t) + py_1(t) + p^2y_2(t) + p^3y_3(t) + \dots \tag{21}$$

when $p^{-}1$, (20) corresponds to (19) and (21) becomes the approximate solution of (19), i.e.

$$v(t) = \lim_{p\to 1} y(t) = y_0(t) + y_1(t) + y_2(t) + \dots \tag{22}$$

By substituting (21) in (20) and equating the terms with identical powers of p, we have

$$\begin{aligned} &p^0 : L(y_0(t)) - L(v_0(t)) = 0, \\ &p^1 : L(y_1(t)) + L(v_0(t)) + y_0^2(t) - 1 = 0, \\ &p^2 : L(y_2(t)) + 2y_0(t)y_1(t) = 0. \\ &\vdots \end{aligned} \tag{23}$$

For simplicity we always set $y_o(t)$, = $v_o(t)$ = 1. Accordingly, we have

$$y_1(t) = \frac{1}{4}(-1 + e^{2t} - 2t + 2t^2) \tag{24}$$

$$y_2(t) = \frac{1}{4}(t^2 - e^{2t}t^2 + 2t^3), \tag{25}$$

$$\vdots$$

Expanding $v_n(t) = \sum_{i=0}^{n} y_i(t)$, by using the Taylor expansion about t = 0 gives

$$\begin{aligned} &v_0(t) = t, \\ &v_1(t) = t + t^2 + \frac{1}{3}t^3 + \frac{1}{6}t^4 + \dots \\ &v_2(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{3}t^4 - \frac{7}{15}t^5 + \dots, \end{aligned}$$

In a similar manner, the components $y_n(t)$ are calculated for $n = 3,4,5$

Now, we consider the HPM with the other convex homotopy to solve the Riccati equation (1). Towards this end, denoting $\frac{d}{dt}$ by G, we have G^{-1} as an integration from 0 to 1 and so we have

$$Gy(t) = 2y(t) - y^2(t) + \tag{26}$$

Operating with G^{-1} and using the initial condition $y(0) = 0$, we have.

$$y(t) = 2G^{-1}(y(t)) - G^{-1}(y^2(t)) + G^{-1}(1)1 \tag{27}$$

Consider (27) as

$$L(t) = y(t) - 2G^{-1}(y(t)) + G^{-1}(y^2(t)) - G^{-1}(1) \tag{28}$$

CONCLUSION

As a possible remedy, we can define a homotopy $H(t,p)$ by $H(t,0) = F(t)$ and $H(t,1) = L(t)$ Then we choose a convex homotopy by

$$H(t,p) = (1\text{-}p)F(t) + pL(t) = 0 \quad (29)$$

where

$$F(t) = y(t)\text{-}G^{-1}(1) \quad (30)$$

So that we can continuously trace an implicitly defined curve from a starting point $H(y_o 0)$ to a solution function $H(y,1)$ The embedding parameter p monotonically increases from zero to one as the trivial problem $F(t) = 0$ is continuously deformed to the original problem $L(t) = 0$. The embedding parameter $p\in[0,1]$ can be considered as an expanding parameter. So, by substituting (28) and (30) in (29), we obtain

$$\begin{aligned} &H(t,p) = y(t) - G^{-1}(t) \\ &-2pG^{-1}(y(t)) + pG^{-1}(y^2(t)) = 0. \end{aligned} \quad (31)$$

Rewriting (31) as

$$y(t) = G^{-1}(1) + 2pG^{-1}(y(t)) - pG^{-1}(y^2(t)) \quad (32)$$

and considering $y(t) = \sum_{j=0}^{\infty} p^j\, y_j(t),$, we obtain

$$\begin{aligned} &p^0 y_0(t) + p^1 y_1(t) + p^2 y_2(t) + \ldots \\ &= 2p\int_0^t (p^0 y_0(t) + p^1 y_1(t) + p^2 y_2(t) + \ldots)dt \\ &-p\int_0^t (p^0 y_0(t) + p^1 y_1(t) + p^2 y_2(t) + \ldots)^2 dt \end{aligned} \quad (33)$$

By equating the terms with identical powers of p, we have

$$p^0 : y_0(t) = \int_0^t dt = t \quad (34)$$

$$p^1 : y_1(t) = 2\int_0^t y_0(t)dt - \int_0^t A_0 dt = t^2 - \frac{t^3}{3}, \quad (35)$$

$$p^2 : y_2(t) = 2\int_0^t y_1(t)dt - \int_0^t A_1 dt = \frac{2}{3}t^3 - \frac{2}{3}t^4 + \frac{2}{15}t^5 \quad (36)$$

The other terms y_n for n = 3,4,5 can be obtained recursively in a similar way but for brevity are not listed here.

CONCLUSION

In this work, we applied the powerful and efficient ADM, LTDA and HPM for solving the Riccati equation. This study showed that solution components $y_0(t)$, $y_1(t)$ and $y_2(t)$obtained by LTDA are just the same terms obtained by the ADM and the HPM. This equality for other terms takes place too. It was shown that the HPM with a specific convex homotopy for solving the Riccati equation was equivalent to the ADM and the LTDA.

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