World Applied Sciences Journal 16 (7): 898-902, 2012
ISSN 1818-4952

On Exact Solutions of Higher Order Boundary Value Problems

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Abstract: In this paper, we use an efficient technique which is called the Exp-function method for finding appropriate solutions of fifth and sixth-order nonlinear boundary value problems. This technique has been extensively used for finding the soliton and traveling wave solutions of the nonlinear partial differential equations but we obtain the exact solution of a class of nonlinear boundary value problems. The proposed technique is quite efficient and is practically well suited for use in these problems. Several examples are given to verify the reliability and efficiency of the suggested method.

Key words: Exp-function method · Nonlinear problems · Boundary value problems · Error estimates

INTRODUCTION

In this paper, we consider the general nth-order boundary value problems of the type

$$u^{(n)}(x) = f(x, u) \tag{1}$$

Subject to boundary conditions

$$u(a)=A_0,\quad u''(a)=A_2,\quad u^{(iv)}(a)=A_4,\ldots,u^{(n-2)}(a)=A_{n-2},$$
$$u'(b)=A_1,\quad u'''(b)=A_3,\quad u^{(v)}(a)=A_5,\ldots,u^{(n-1)}(b)=A_{n-1},$$

Where f is continuous function on given interval $[a,b]$ and A_i, $i = 0,1,2,\ldots n-1$ are finite constants.

The fifth-order boundary value problems arise in the mathematical modeling of the viscoelastic flows and other branches of mathematical, physical and engineering sciences, see [2-10] and references therein. The sixth-order boundary value problems are known to arise in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by sixth-order boundary-value problems, see [3-10, 12, 13, 23-26, 28] and references therein. Glatzmaier also notice that dynamo action in some stars may be modeled by such equations, see [13]. Moreover, when an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in [2-10], when this instability is of ordinary convection than the governing ordinary differential equation is of sixth-order, see [8-10] and the references therein. Several techniques have been developed by various authors for finding solutions of higher-order boundary value problems, see [1-26] and the references therein. Inspired and motivated by the ongoing research in this area, we have used an efficient analytic technique which is called the Exp-function method [1, 11, 14, 16-22, 27, 29] for solving higher order boundary value problems. In the present study, we employed the Exp-function method on fifth and sixth-order boundary value problems. The results are very encouraging and reveal the complete reliability of the proposed algorithm.

Exp-Function Method: Consider the general nth-order boundary value problem as given in (1)

$$u^{(n)}(x) = f(x, u),$$

Subject to boundary conditions

$$u(a)=A_0,\quad u''(a)=A_2,\quad u^{(iv)}(a)=A_4,\ldots,u^{(n-2)}(a)=A_{n-2},$$
$$u'(b)=A_1,\quad u'''(b)=A_3,\quad u^{(v)}(a)=A_5,\ldots,u^{(n-1)}(b)=A_{n-1},$$

By using the idea of the Exp-function method for nonlinear partial differential equations [1, 11, 14, 16-22, 27, 29], we assume that the solution of nonlinear ordinary differential equations an be expressed in the following

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form

$$u(x)=\frac{\sum_{n=-c}^{d}a_n\exp[nx]}{\sum_{m=-p}^{q}b_m\exp[mx]},\qquad(2)$$

Where p,q,c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants,

The equation (2) can be written as the following alternate and useful form

$$u(x)=\frac{a_c\exp[cx]+\ldots+a_{-d}\exp[-dx]}{b_p\exp[px]+\ldots+b_{-q}\exp[-qx]}.\qquad(3)$$

Numerical Applications: In this section, we apply the Exp-function method [1, 11, 14, 16-22, 27, 29] reviewed in section 2 for solving the nonlinear fifth and sixth-order boundary value problems. Numerical results are very encouraging showing the complete reliability and efficiency of the proposed method.

Example 3.1: [7, 10].Consider the following nonlinear boundary value problem of fifth-order

$$u^{(v)}(x)=e^{-x}u^2(x),\qquad(4)$$

With boundary conditions

$u(0)=u'(0)=u''(0)=1,\ u(1)=u'(1)=e.$

The exact solution for this problem is

$$u(x)=e^x$$

By using the Exp-function method for ordinary differential equaitons, the trial solution of the equation (4) which has also mentioned earlier in equation (3) can be expressed as follows,

$$u(x)=\frac{a_c\exp[cx]+\ldots+a_{-d}\exp[-dx]}{b_p\exp[px]+\ldots+b_{-q}\exp[-qx]}.$$

To determine the value of c and p, we balance the linear term of highest order of equation (4) with the highest order nonlinear term.

$$u^{(v)}=\frac{c_1\exp[(31p+c)x]+\cdots}{c_2\exp[32px]+\cdots}\qquad(5)$$

and

$$u^2=\frac{c_3\exp[2cx]+\cdots}{c_4\exp[2px]+\cdots}=\frac{c_3\exp[(30p+2c)x]+\cdots}{c_4\exp[32px]+\cdots},\qquad(6)$$

Where c_i are determined coefficients only for simplicity; balancing the highest order of Exp-function in (5) and (6), we have

$$31p+c=30p+2c,$$

Which in turn gives

$$p=c$$

To determine the value of d and q, we balance the linear term of lowest order of equation (4) with the lowest order non-linear term

$$u^{(v)}=\frac{\cdots+d_1\exp[(-31q-d)x]}{\cdots+d_2\exp[-32qx]},\qquad(7)$$

and

$$u^2=\frac{\cdots+d_3\exp[-2dx]}{\cdots+d_4\exp[-2qx]}=\frac{\cdots+d_3\exp[(-30q-2d)x]}{\cdots+d_4\exp[-32qx]},\qquad(8)$$

Where d_i are determined coefficients only for simplicity.

Now, balancing the lowest order of Exp-function in (7) and (8), we have

$$-31q-d=-30q-2d,$$

Which in turn gives

$$q=d$$

Case 3.1.1: $p=c=1$ and $q=d=1$.
Equation (3) reduces to

$$u(x)=\frac{a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_1\exp[x]+b_0+b_{-1}\exp[-x]}.\qquad(9)$$

Substituting (9) into (4), we have

$$\begin{aligned}&\frac{1}{A}\Big[C_5\exp(5x)+C_4\exp(4x)+C_3\exp(3x)+C_2\exp(2x)+C_1\exp(x)+C_0\\&+C_{-1}\exp(-x)+C_{-2}\exp(-2x)+C_{-3}\exp(-3x)+C_{-4}\exp(-4x)\\&+C_{-5}\exp(-5x)+C_{-6}\exp(-6x)+C_{-7}\exp(-7x)\Big]=0,\end{aligned}\qquad(10)$$

Where $A=(b_1\exp(x)+b_0+b_{-1}\exp(-x))^6$ $C_i(i=-7,-6,-4,\ldots,4,5)$ are constants obtained by Maple 7.

Equating the coefficients of $\exp(nx)$ to be zero, we obtain

$$\{C_{-7}=0, C_{-6}=0, C_{-5}=0, C_{-4}=0, C_{-3}=0, C_{-2}=0,\\ C_{-1}=0,\ C_0=0, C_1=0, C_2=0,\ C_3=0, C_4=0,\ C_5=0.\} \tag{11}$$

Solution of (11) will yield

$$\{a_{-1}=0, b_1=0, a_0=0,\quad b_0=a_1,\ b_{-1}=0,\ a_1=a_1.\} \tag{12}$$

Consequently, the following exact solution is obtained.

$$u(x) = e^x.$$

Case 3.1.2: $p = c = 2$ and $q = d = 1$.
Equation (3) reduces to

$$u(x)=\frac{a_2\exp[2x]+a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_2\exp[2x]+b_1\exp[x]+b_0+b_{-1}\exp[-x]}. \tag{13}$$

Proceeding as before, we obtain

$$\{a_{-2}=0,\quad a_{-1}=b_{-2},\quad a_0=a_0,\quad a_1=0,\quad a_2=0,\\ b_{-2}=b_{-2},\quad b_{-1}=a_0,\quad b_0=0,\quad b_1=0,\quad b_2=0.\} \tag{14}$$

Using values of unknowns from (14) into trial solution (13), we have

$$u(x)=\frac{a_0+b_{-2}e^{(-x)}}{a_0e^{(-x)}+b_{-2}e^{(-2x)}},$$
$$=\frac{a_0+b_{-2}e^{(-x)}}{e^{(-x)}(a_0+b_{-2}e^{(-x)})},\qquad where\ \ a_0+b_{-2}e^{(-x)}\neq 0$$

Consequently, the exact solution is obtained as

$$u(x) = e^x.$$

Example 3.2: [3-6, 8-9, 17, 23-26, 28]. Consider the following nonlinear boundary value problem of sixth-order.

$$u^{(vi)}(x) = e^x u^2(x),\ 0< x < 1 \tag{15}$$

With boundary conditions

$u(0) = 1$, $u''(0) = -1$, $u''''(0) = 1$; $u(1) = e^{-1}$, $u'(1) = -e^{-1}$, $u''(1) = e^{-1}$

The exact solution for this problem is

$$u(x) = e^{-x}.$$

The trial solution of the given boundary value problem can be expressed in the following form

$$u(x)=\frac{a_c\exp[cx]+\ldots+a_{-d}\exp[-dx]}{b_p\exp[px]+\ldots+b_{-q}\exp[-qx]}.$$

Proceeding as before, we obtain

$$p = c,\ q = d$$

Case 3.2.1: $p = c = 1$ and $q = d = 1$.
The trial solution has the following form

$$u(x)=\frac{a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_1\exp[x]+b_0+b_{-1}\exp[-x]}.$$

Using the trial solution into (15), we have

$$\frac{1}{A}\big[C_8\exp(8x)+C_7\exp(7x)+C_6\exp(6x)+C_5\exp(5x)+C_4\exp(4x)+C_3\exp(3x)\\ +C_2\exp(2x)+C_1\exp(x)+C_0+C_{-1}\exp(-x)+C_{-2}\exp(-2x)+C_{-3}\exp(-3x)\\ +C_{-4}\exp(-4x)+C_{-5}\exp(-5x)+C_{-6}\exp(-6x)\big]=0, \tag{16}$$

Where $A = (b_1 \exp(x) + b_0 + b_{-1} \exp(-x))^7$, $C_i (i = -6, -5, -4, \ldots\ldots, 6,7,8)$ are constants obtained by Maple 7.
Equating the coefficients of $\exp(nx)$ to be zero from (16), we obtain

$$\{C_{-6}=0,\ C_{-5}=0,\ C_{-4}=0, C_{-3}=0, C_{-2}=0, C_{-1}=0, C_0=0,\\ C_1=0,\quad C_2=0,\ C_3=0,\ C_4=0,\quad C_5=0,\quad C_5=0,\quad C_5=0,\\ C_5=0.\} \tag{17}$$

Solution of (17) will yield

$$\{a_{-1}=0,\quad b_1=a_0,\quad a_0=a_0,\quad b_0=0,\quad b_{-1}=0, a_1=0.\} \tag{18}$$

By using the values of unknown from (18) into trial solution, the following exact solution is obtained

$$u(x) = e^{-x}.$$

Case 3.2.2: $p = c = 2$ and $q = d = 2$.
Equation (3) reduces to

$$u(x)=\frac{a_2\exp[2x]+a_1\exp[x]+a_0+a_{-1}\exp[-x]+a_2\exp[-2x]}{b_2\exp[2x]+b_1\exp[x]+b_0+b_{-1}\exp[-x]+b_2\exp[-2x]}, \tag{19}$$

There are some free parameters in Equation (19), we set $b_1 = b_{-1}$ for simplicity, the trial-function (19) is simplified as follows:

$$u(x) = \frac{a_2 \exp[2x] + a_1 \exp[x] + a_0 + a_{-1} \exp[-x] + a_2 \exp[-2x]}{b_2 \exp[2x] + b_0 + b_2 \exp[-2x]}, \quad (20)$$

Proceeding as before, we obtain

$$\{a_{-2} = 0,\ a_{-1} = a_{-1},\ a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ b_{-2} = 0,\ b_0 = a_{-1},\ b_2 = 0.\} \quad (21)$$

By using values of unknown from (21) into trail solution (20), we have

$$u(x) = \frac{a_{-1} e^{(-x)}}{a_{-1}}, \text{where } a_{-1} \neq 0$$

Consequently, the exact solution is obtained as follows

$$u(x) = e^{-x}$$

CONCLUSION

In this paper, we applied the Exp-function method developed by He and Wu for solving the fifth and sixth-order boundary value problems. This technique have been extensively used for finding the soliton compacton and traveling wave solutions of the nonlinear partial differential equations but we obtain the exact solutions of a class of nonlinear boundary value problems. The results are very promising and encouraging. Hence, we conclude that the proposed Exp-function method may be used as an alternative for solving the nonlinear boundary value problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result, the size reduction amounts to the improvement of performance of approach. Hence, we conclude that Exp-function method may be considered as an alternative for solving nonlinear boundary value problems.

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