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Spectral Galerkin Method For Solving Dynamic Stochastic Games

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Abstract: In this paper we propose a spectral Galerkin method for solving the simultaneous functional equations that occur in dynamic stochastic games. The linear-quadratic (LQ) approximation can be used as a numerical method for solving these equations. The spectral Galerkin method is a special case of the so-called weighted residual methods, commonly used in computational physics for solving partial differential equations. This method leads to very simple solutions with minimal computational effort.

Key words: Spectral galerkin method · Dynamic stochastic games · Linear-quadratic · Weighted residual method

INTRODUCTION

The literature about dynamic game theory applied to capital accumulation and income distribution in the tradition of the Lancaster model grew extensively during the eighties. For example, the original result that its equilibrium was inefficient rested on the solution concept that it was used (feedback Nash equilibrium). However, this inefficiency changes if we allow for different solution concepts, such as the open-loop Stackelberg solution in which one player acts as a leader and announces his whole course of action in advance [1] or the feedback Stackelberg solution in which the leadership property holds recursively in each period of time [2]. The use of versions of the Folk theorem designed for dynamic games, finally, allows for the emergence of efficient equilibria, such as the ones developed by [3, 4].

A well-known continuous solution methodology for dynamic games is Markov Perfect equilibrium (MPE). A Markov perfect equilibrium (MPE) is defined as a profile of Markov strategies that yields Nash equilibrium in every proper sub-game [5, 6]. Few studies have provided empirical estimates of equilibrium values, in part, because solving dynamic games using either a feedback solution or a Markov perfect solution is a challenging task. Ligon and Narain [7] describe three solution methods for obtaining MPE. They first discuss the classical approach, in which the Euler equation is used to derive MPE.

Consider a stationary discrete-time infinite horizon game in which two firms produce two goods that are close substitutes. The goods are infinitely perishable, which eliminates a possibility of carryover. As an illustration, one may think of two bakeries across the street from each other, one bakes only donuts, the other bakes only bagels. While each consumer prefers either bagels or donuts, one good can substitute for the other if the other good has a very high price. In addition, "yesterday's" donuts or bagels have no appeal to an average consumer. First, consider a deterministic version of the model. In every period t, firm i decides on an amount of good q_i to produce, a price p_i at which to sell it and an amount x_i of new investment in the firm's capital. The demand $d_i = D_i(p_1, p_2)$ for each good is a function of both prices. Since no inventories are allowed, each firm sells s_i - $\min\{d_i, q_i\}$ i.e. any undemanded good is disposed of and any unsatisfied demand results only in lost potential profits. The cost of production $C_i(q_i, k_i)$ depends on the quantity produced and the firm's capital stock. In our formulation, we will think of k as a natural but flexible production capacity, below which quantities can be produced at a fixed cost and above which quantities can be produced, but only at increasingly higher costs. Investment adds to the following period's level of capital, $k'_i = (1-\xi)\,k_i + x_i$, where ξ is the capital depreciation rate, but also adds additional cost $h_i(x_i)$ to the current period's net income. Thus, firm's current period profit can be presented as:

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$$\pi_i\,(p_1, p_2, q_1, q_2, x_i, k_i = p_i\, s_i - C_i\,(q_i, k_i) - h_i\,(x_i),$$
$$s_i = \min\{D_i\,(p_1, p_2), q_i\}$$
$$s_i = \min\{D_i\,(p_1, p_2), q_i\} \quad (1)$$

Each firm chooses its price, production and investment $\{p_i, q_i, x_i)$ so as to maximize the present value of current and future profits discounted at a rate β, taking the other firm's policy as given. If $V_i\,(k_1, k_2)$ denotes the value of firm i given capital stocks k_1 and k_2, then, by Bellman's principle of optimality [8]:

$$V_i(k_1,k_2) = \max\{p_i s_i - C_i(q_i,k_i) - h_i(x_i) + \beta V_i(k_1',k_2')\}$$
$$k_i' = (1-\xi)k_i + x_i \quad (2)$$

This pair of functional equations must be solved in order to find optimal strategies of each player. Before proceeding with a discussion of numerical solution methods, we discuss the properties of the model and compare it to those exhibited by the model in Judd [9, 10].

Galerkin Method: In the last ten years, spectral methods have witnessed a growing interest in collocation methods relying upon Lagrangian bases as well as in variational formulations of Galerkin type using Legendre polynomials [11, 12]. In particular, in the context of the Galerkin method, Jie Shen [13] introduced a new basis of Legendre polynomials to solve problems in two dimensions by diagonalization. Shen's basis has the interesting property of being orthogonal in the energy norm (i.e., the L2 norm of the first derivative of the variable), so that the diagonalization has to be performed on the mass matrix which has a very simple pentadiagonal profile. In Shen's algorithm, the spectral decomposition is performed only in one spatial direction and the algorithm has been extended also to deal with a spectral representation based on Chebyshev polynomials [14]. As a matter of fact, the mass diagonalization for the Legendre approximation can be applied in both spatial directions. In mathematics, in the area of numerical analysis, Galerkin methods are a class of methods for converting a continuous operator problem (such as a differential equation) to a discrete problem.

Let $w(x)$ denotes a non-negative, integrable, real-valued function over the interval γ. We define:

$$L_w^2(\gamma) = \{v : \gamma \to \mathrm{R} : |v|_w < \infty\}$$

Where

$$|v|_w = \left(\int_0^\infty |v(x)|^2\, w(x)\mathrm{d}x\right)^{\frac{1}{2}},$$

is the norm induced by the inner product of the space $L_w^2(\gamma)$ and $< u,v >_w = \int_0^\infty u(x)v(x)w(x)\mathrm{d}x.$ (3)

For applying Galerkin method, the unknown function $u(x)$ can be approximated as:

$$u(x) = \sum_{j=1}^{n} c_j \phi_j(x) \quad (6)$$

Where the unknown coefficients c_j are called the spectral coefficients and ϕ_j are the basic functions that are orthogonal, which it means

$$< \phi_n\,(x), \phi_m\,(x) > = \delta_{nm} \quad (7)$$

Where δ_{nm} is the Kronecker delta function. The unknown coefficients c_j in Eq. (1) are determined by orthogonalizing the residual $u(x)$ with respect to the functions $\{\phi_j\}_{j=1}^m$.

This yields the discrete system

$$< u(x), \phi_j > = 0, j = 1,2,..., m \quad (8)$$

Eq. (2) gives m nonlinear algebraic equations which can be solved for the unknown coefficients c_j and consequently, unknown function $u(x)$ can be calculated.

Solving Model: The infinite horizon, autonomous dynamics is frequently used to analyze the standard models of capital accumulation. In the economic growth problems, a social planner is confronted with the following optimization question:

$$V(k_0) = \max_{\{k_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t F(k_t, k_{t+1}) \quad (9)$$

Subject to:

$$k_{t+1} \in \Gamma\,(k_t)\; t = 0,1,2,... \quad (10)$$
$$k_0 \in K, \beta \in (0,1)$$

Here all capital stocks k_t, t = 0,1,2,... belong to a given convex set K, which is called the state space. The mapping $F(\cdot,\cdot)$ is called the return function and Γ is called the technological correspondence and β is the time-preference discount factor. Dynamic programming approach starts with the value function $V(k_0)$, defined in Eq. (3). Because the objective possesses the recursive property, the value function must satisfy the following functional equation:

$$V(k_0) = \max_{k_1 \in \Gamma(k_0)} [F(k_0,k_1) + \beta V(k_1)]. \quad (11)$$

Actually, the principle of optimality in dynamic programming has the formal representation:

$$V(k_t) = \max_{k_{t+1} \in \Gamma(k_t)} [F(k_t, k_{t+1}) + \beta V(k_{t+1})] \quad t = 0,1,...$$

Which is the Bellman equation. The solution of the single-agent dynamic optimization problem is characterized by Bellman's equation:

$$V(k_t) = \max_{x_t \in X(k_t)} \{f(k_t, x_t) + \beta E_{k_t} V(g(k_t, x_t, \varepsilon_t)\}, \ k_t \in K \quad (4)$$

The evolution of state variable over time can be represented by:

$$g(k_t, x_t, \varepsilon_t) = k_{t+1} \quad (14)$$

Where g $(k_t, x_t, \varepsilon_t)$ is the transition equation function and ε_t is an exogenous random shock. Taking into account this statement of the problem, an appropriate solution method exists in the literature, called stochastic dynamic Nash game with perfect information (PSDNG) [15]. The above equation is a functional equation. Its unknowns, the value function $V(.)$ and the optimal policy $x_t^*(.)$, are both functions defined on the state space K. Except in very special cases, Bellman's functional equation lacks an analytic closed form solution and can only be solved approximately using computational methods. A variety of methods are available for computing approximate solutions to Bellman equations, including linear-quadratic approximation and space discretization. However, in most applications, particularly stochastic models with bounded decisions, these methods either provide unacceptably poor approximations or are computationally inefficient [16]. To compute an approximate solution of the unknown value function V using Galerkin methods, the value function approximate should be written as a linear combination of n known basis functions $\phi_1, \phi_2, ..., \phi_n$ on k with undetermined coefficients as

$$V(k_t) = \sum_{j=1}^{n} c_j \phi_j(k_t) \quad (15)$$

Where c_j is the basis function coefficient. Then, the value function on the left hand side of Eq. (4) and the second term on the right-hand side are replaced by Eq. (5).

The unknown coefficients c_j in Eq. (5) are determined by orthogonalizing the residual $V(k_t) - f(k_t, x_t) + \beta E_{kt} V(g(k_t, x_t, \varepsilon_t)$ with respect to the functions $\{\phi_j\}_{j=1}^{m}$. This yields the discrete system

$$< V(k_t) - f(k_t, x_t) + \beta E_{k_t} V(g(k_t, x_t, \varepsilon_t), \phi_j >= 0, j = 1,2,...,m \quad (16)$$

The weighted inner product <,> is taken to be

$$< g(x), f(x) >= \int_K g(x) f(x) w(x) dx \quad (17)$$

Here, $w(x)$ plays the role of a weight function which is chosen depending on the boundary conditions, the domain and the differential equation. Eq. (6) gives m nonlinear algebraic equations which can be solved for the unknown coefficients c_j by using the well known Newton's method and consequently, $V(k_t)$ given in Eq. (4) can be calculated.

CONCLUSIONS

The focus of this paper is on the development of models that may be used to predict investment, production and price trajectories associated with alternative economic scenarios that may unfold. However, these trajectories depend upon the behavior of the players.

Spectral Galerkin solution method (developed in the physical sciences) is used to solve the model of dynamic stochastic games. Proposed method is a generally useful technique that is flexible, accurate and numerically efficient.

The choice of basis functions is related to the curvature of the value function. The larger the number of basis functions, the greater the computational burden, so the researcher will want to experiment with various basis functions schemes and dimensions of the problem to render it computationally efficient.

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