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Convergence of Ninth Spline Function to the Solution of a System of Initial Value Problems

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Abstract: The aim of this paper is to investigate the performance of the ninth degree spline method for solving the system of ordinary differential equations and to estimate the numerical solution in the whole interval. By considering the maximum absolute errors in the solution at grid points for different choices of step size, we conclude that ninth spline produces the accurate results in comparison with other methods.

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INTRODUCTION

Consider the system of *n*-equations

$$\frac{dA}{dt} = f(t, u_1, u_2, u_3, \ldots, u_n) \qquad A(t_0) = \eta \tag{1.1}$$

Where $A = [u_1, u_2, u_3, \ldots, u_n]^T$, $f = [f_1, f_2, f_3, \ldots, f_n]^T$, $\eta = [\eta_1, \eta_2, \eta_3, \ldots, \eta_n]$.

Many problems in applied sciences and engineering are modeled as system of differential equations such as spring-mass systems, bending of beams, chemical reactions and so forth can be formulated in terms of differential equations. Since the system of differential equations has wide application in scientific research, therefore faster and accurate numerical solutions to this problem is very importance [1-3].

There are several methods that can be used to solve the system of differential equations numerically. It had been proposed by [4, 5] solved some order initial value problems, a numerical methods for solving system of ordinary differential equations has been proposed by [6].

The basic motivation of this paper is discussed convergence analysis of the ninth spline method for solutions system of differential equations. In this paper, we will use the function, third, fifth and seventh boundary conditions, to constructed the ninth spline with two initial conditions. Section 2 is devoted to the description of the method; existence and uniqueness of the method are obtained which required in proving the convergence analysis of the presented method in section 3. Finally, in section 4, numerical evidences are included to show the practical applicability and the absolute errors are superiority.

Description of the Method: we present a ninth spline interpolation for one dimensional and given sufficiently smooth function $f(x)$ defined on I = [a, b] and Δ_n: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$, denote the uniform partition of I with knots $x_i = a + ih$, where $i = 0,1,2,\ldots,n$ – and $h = \frac{b-a}{n}$ is the distance of each subintervals and denoted the ninth spline by $S_\Delta(x)$ in [a, b] as:

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$$s_0(x) = y_0 + (x-x_0)y_0' + \frac{(x-x_0)^2}{2}y_0'' + \frac{(x-x_0)^3}{6}y_0''' + \frac{(x-x_0)^4}{24}a_{0,4} + \frac{(x-x_0)^5}{120}y_0^{(5)} + (x-x_0)^6 a_{0,6} + \frac{(x-x_0)^7}{5040}y_0^{(7)} + (x-x_0)^8 a_{0,8} + (x-x_0)^9 a_{0,9} \quad (2.1)$$

On the subintervals $[x_0, x_1]$ where $a_{0,j}, j = 5,6,7$ are unknowns to be determined.

Let us examine subintervals $[x_i,x_{i+1}]$ i=1,2,…, n-2. By taking into account the interpolating conditions, we can write the expression, for $S_i(x)$ in the following form [4, 5].

$$s_i(x) = y_i + (x-x_0)a_{i,1} + (x-x_i)^2 a_{i,2} + \frac{(x-x_i)^3}{6}y_i''' + (x-x_i)^4 a_{i,4} + \frac{(x-x_i)^5}{120}y_i^{(5)} + (x-x_i)^6 a_{i,6} + \frac{(x-x_i)^7}{5040}y_i^{(7)} + (x-x_i)^8 a_{i,8} + (x-x_i)^9 a_{i,9}, \quad (2.2)$$

Where $a_{i,j}$, $i = 1(1)(n-1)$, j = 1,2,4,6,7,8,9. are unknown values to be determine.

Convergence Analysis: In this section, we investigate the convergence analysis of the method for ninth degree spline function which are developed by [4, 7, 8, 9]. The equation (2.1) yields the two initial conditions $S'(x_0) = y'(x_0)$ and $S''(x_0) = y''(x_0)$ and applied the boundary conditions $S^{(r)}(x_i) = y^{(r)}(x_i)$, r = 0,3,5,7, for i = 0, 1,...,n in equation (2.2), we obtained the following:

$$a_{0,4} = \frac{18}{13h^4}[y_1 - y_0] - \frac{18}{13h^3}y_0' - \frac{9}{13h^2}y_0'' - \frac{1}{312h}[5y_1''' + 67y_0'''] + \frac{h}{9360}[7y_1^{(5)} - 40y_0^{(5)}] - \frac{h^3}{157248}[4y_1^{(7)} - 7y_0^{(7)}],$$

$$a_{0,6} = -\frac{6}{13h^6}[y_1 - y_0] + \frac{6}{13h^5}y' + \frac{3}{13h^4}y_0'' + \frac{1}{52h^3}[y_1''' + 3y_0'''] - \frac{1}{9360h}[11y_1^{(5)} + 43y_0^{(5)}] + \frac{h}{786240}[37y_1^{(7)} - 133y_0^{(7)}],$$

$$a_{0,8} = \frac{9}{91h^8}[y_1 - y_0] - \frac{9}{91h^7}y' - \frac{9}{182h^6}y_0'' - \frac{3}{728h^5}[y_1''' + 3y_0'''] + \frac{1}{36400h^3}[20y_1^{(5)} + 25y_0^{(5)}] - \frac{1}{1834560h}[64y_1^{(7)} + 161y_0^{(7)}],$$

and

$$a_{0,9} = -\frac{2}{91h^9}[y_1 - y_0] + \frac{2}{91h^8}y' + \frac{1}{91h^7}y_0'' + \frac{1}{1092h^6}[y_1''' + 3y_0'''] - \frac{1}{163800h^4}[20y_1^{(5)} + 25y_0^{(5)}] + \frac{1}{5503680h^2}[73y_1^{(7)} + 77y_0^{(7)}].$$

Substituting these values of $a_{0,4}$, $a_{0,6}$, $a_{0,8}$ and $a_{0,9}$ in equation (2.1), we obtain:

$$a_{1,2} = \frac{306}{91h}[y_1 - y_0] - \frac{215}{91}y_0' - \frac{62h}{91}y_0'' + \frac{h^2}{1092}[29y_0''' - 95y_0'''] - \frac{h^4}{32760}[25y_1^{(5)} - 37y_0^{(5)}] + \frac{h^6}{5503680}[115y_1^{(7)} - 133y_0^{(7)}],$$

and

$$2a_{1,2} = \frac{612}{91h^2}[y_1 - y_0] - \frac{612}{91h}y_0' - \frac{612}{91}y_0'' + \frac{h}{91}[20y_0''' - 31y_0'''] - \frac{h^3}{10920}[47y_1^{(5)} - 55y_0^{(5)}] + \frac{h^5}{305760}[33y_1^{(7)} - 35y_0^{(7)}].$$

We shall find the coefficients of $S_i(x)$ for i = 1,2,3,....n – 1. From equation (2.2) we have:

$$a_{i,4} = \frac{18}{13h^4}[y_{i+1} - y_i] - \frac{18}{13h^3}a_{i,1} - \frac{18}{13h^2}a_{i,2} - \frac{1}{312h}[5y_{i+1}''' + 67y_i'''] + \frac{h}{9360}[7y_{i+1}^{(5)} - 40y_i^{(5)}] - \frac{h^3}{157248}[4y_{i+1}^{(7)} - 7y_i^{(7)}],$$

$$a_{i,6} = -\frac{6}{13h^6}[y_{i+1} - y_i] + \frac{6}{13h^5}a_{i,1} + \frac{6}{13h^4}a_{i,2} + \frac{1}{52h^3}[y_{i+1}''' + 3y_i'''] - \frac{1}{9360h}[11y_{i+1}^{(5)} + 43y_i^{(5)}] + \frac{h}{786240}[37y_{i+1}^{(7)} - 133y_i^{(7)}],$$

$$a_{i,8} = \frac{9}{91h^8}[y_{i+1} - y_i] - \frac{9}{91h^7}a_{i,1} - \frac{9}{91h^6}a_{i,2} - \frac{3}{728h^5}[y_{i+1}''' + 3y_i'''] + \frac{1}{36400h^3}[20y_{i+1}^{(5)} + 25y_i^{(5)}] - \frac{1}{1834560h}[64y_{i+1}^{(7)} + 161y_i^{(7)}],$$

and

$$a_{i,9} = -\frac{2}{91h^9}[y_{i+1} - y_i] + \frac{2}{91h^8}a_{i,1} + \frac{2}{91h^7}a_{i,2} + \frac{1}{1092h^6}[y_{i+1}''' + 3y_i'''] - \frac{1}{163800h^4}[20y_{i+1}^{(5)} + 25y_i^{(5)}] + \frac{1}{5503680h^2}[73y_{i+1}^{(7)} + 77y_i^{(7)}].$$

Substituting the values of $a_{i,4}$, $a_{i,6}$, $a_{i,8}$ and $a_{i,9}$ in the equation (2.2), we obtain the following relation, for $i = 1,2,...,n-1$.

$$a_{i+1,1} = \frac{306}{91h}[y_{i+1} - y_i] - \frac{215}{91}a_{i,1} - \frac{124h}{91}a_{i,2} + \frac{h^2}{1092}[29y'''_{i+1} - 95y'''_i] - \frac{h^4}{32760}[25y^{(5)}_{i+1} - 37y^{(5)}_i] + \frac{h^6}{5503680}[115y^{(7)}_{i+1} - 133y^{(7)}_i],$$

and

$$2a_{i+1,2} = \frac{612}{91h^2}[y_{i+1} - y_i] - \frac{612}{91h}a_{i,1} - \frac{430}{91}a_{i,2} + \frac{h}{91}[20y'''_{i+1} - 31y'''_i] - \frac{h^3}{10920}[47y^{(5)}_{i+1} - 55y^{(5)}_i] + \frac{h^5}{305760}[33y^{(7)}_{i+1} - 35y^{(7)}_i].$$

Now the coefficient matrix of the above system of equations can be found, in the unknowns $a_{i,1}$, $a_{i,2}$, $a_{i+1,1}$ and $a_{i+1,2}$, $i = 1, 2, \ldots, n-1$ which is a non-singular matrix and hence all the coefficients are determined uniquely.

Theorem 3.1: Let $y(x)$ be the exact solution of the system (1.1) and we assume y_i, $i = 1,2,...n-1$, be the numerical solution of (1.1) and $S(x)$ be a unique ninth spline function which is a solution of the problem (2.1). Then for $x \in [x_0,x_1]$ we have:

$$\left\| S_0^{(9-r)}(x) - y^{(9-r)}(x) \right\| \le \begin{cases} \frac{2}{17199}h^r w_9(f;h), r=9, & \frac{16}{22929}h^r w_9(f;h), r=8 \\ \frac{133}{35062}h^r w_9(f;h), r=7, & \frac{407}{21840}h^r w_9(f;h), r=6 \\ \frac{103}{1260}h^r w_9(f;h), r=5, & \frac{85}{273}h^r w_9(f;h), r=4 \\ \frac{2143}{2184}h^r w_9(f;h), r=3, & \frac{431}{182}h^r w_9(f;h), r=2 \\ \frac{326}{91}h^r w_9(f;h), r=1, & \frac{216}{91}h^r w_9(f;h), r=0. \end{cases}$$

Where $W_9(f;h)$ denotes the modules of continuity of $y^{(9)}$.

Proof: Let $x \in [x_0,x_1]$ from equation (2.1) and using Taylor's expansion formula we get

$$S_{0_i}^{(9)}(x) - y^{(9)}(x) = 362880a_{0,9} - y_i^{(9)}(x) \tag{3.1}$$

Using (2.1) and $a_{0,9}$, we obtain

$$\left|S_0^{(9)}(x) - y^{(9)}(x)\right| = \left|362880a_{0,9} - y^{(9)}(x)\right| \le \frac{216}{91}w_9(f;h) \tag{3.2}$$

From (2.1) taking the eight derivatives and subtracting the function, we obtain $S_0^{(8)}(x) - y^{(8)}(x) = 40320a_{0,8} + 362880h\,a_{0,9} - y^{(8)}(x)$, using Taylor's series expansion on $y^{(9)}(x)$ about $x = x_1$, we get.

$$\left|S_0^{(8)}(x) - y^{(8)}(x)\right| = \left|40320a_{0,8} + 362880(x-x_1)a_{0,9} - y^{(8)}(x_0) + (x-x_0)y^{(9)}(\alpha_1)\right| \le \frac{346}{91}hw_9(h,f), \text{ where } x_0 < a_1 < x_1. \tag{3.3}$$

Also, from (2.1) taking the seventh derivatives, we get

$$S_0^{(7)}(x) - y^{(7)}(x) = y_0^{(7)} + 40320a_{0,8} + 181440h^2 a_{0,9} - y^{(7)}(x),$$

Hence

$$\left|S_0^{(7)}(x) - y^{(7)}(x)\right| \le \frac{431}{182}h^2 w_9(f;h), \tag{3.4}$$

and by taking the sixth derivatives with using Taylor's series expansion on $y^{(6)}(x)$ about $x = x_1$, we get:

$$\left|S_0^{(6)}(x) - y^{(6)}(x)\right| \le \frac{2143}{2184}h^3\left|y^{(9)}(\beta_1) - y^{(9)}(\beta_2)\right| \le \frac{2143}{2184}h^3 w_9(f;h) \tag{3.4}$$

Where $x_0 < \beta_1, \beta_2 < x_1$.
From equations (2.1) and substituting the coefficients, we get:

$$\left|S_0^{(5)}(x) - y^{(5)}(x)\right| \le \frac{85}{273} h^4 w_9(f;h). \quad (3.5)$$

Using Taylor's series expansion, by the same way we can find the following:

$$\left|S_0^{(4)}(x) - y^{(4)}(x)\right| \le \frac{103}{1260} h^5 w_9(f;h),$$

$$\left|S_0^{(3)}(x) - y^{(3)}(x)\right| \le \frac{407}{21840} h^6 w_9(f;h),$$

$$\left|S_0'(x) - y'(x)\right| \le \frac{16}{22929} h^8 w_9(f;h)$$

and

$$\left|S_0(x) - y(x)\right| \le \frac{2}{17199} h^9 w_9(f;h)$$

Lemma 3.1: Let $y(x)$ be the exact solution of the system of (1.1) and we assume y_i, $i = 1,2,...n-1$ then $|e_{i,1}| \le c_i h^8 w_9 (f;h)$ for $i= 1, \ldots, n-1$

Where

$$e_{i,1} = a_{i,1} - y_i', \quad (3.6)$$

and c_i depend on the numbers of intervals.

Proof: For $y(x) \in C^9$ [0,1] then using Taylor's expansion formula, we have:

$$y(x) = y(x_i) + (x - x_i)y'(x_i) + \frac{(x-x_i)^2}{2} y''(x_i) + ... + \frac{(x-x_i)^9}{5040} y^{(9)}(\theta_i),$$

Where $x_i < a_i < x_{i+1}$ and similar expressions for the derivatives of $y(x)$ can be used.
Now if i=1 then from equation of $a_{1,1}$ and using (3.6) we obtain

$$e_{1,1} - y_1 = \frac{17h^8}{1834560} y^{(9)}(\alpha_1) + \frac{29h^8}{786240} y^{(9}(\alpha_2) - \frac{5h^8}{157248} y^{(9)}(\alpha_3) + \frac{23h^8}{2201472} y^{(9)}(\alpha_5) - \frac{h^8}{40320} y^{(9)}(\alpha_6) \quad (3.7)$$

Where $x_0 < a_1, a_2, a_3, a_4, a_5, a_6, < x_1$.
from equation (3.7) we get:

$$\left|e_{1,1}\right| \le \frac{89\,h^8}{1572480} w_9(f;h) \text{ so } c_1 = \frac{89}{1572480}.$$

Also, if i=2 then from equation of $a_{2,1}$ and using (3.6) we obtain

$$\left|e_{2,1}\right| \le \frac{64062}{143095680} h^8\, w_9(f;h) \text{ so } c_2 = \frac{64062}{143095680}.$$

By the same way as in the above, we see that the inequality $|e_{i,1}| \le c_i h^8 w_9 (f;h)$ for $i= 1, \ldots, n-1$ holds.
This completes the proof of the Lemma 3.1

Lemma 3.2: Let $y(x)$ be the exact solution of the system of (1.1) and we assume y_i, $i = 1,2,...n-1$, then $|e_{i,2}| \le c'_i h^7 w_9 (f; h)$ for $i= 1, \ldots, n-1$

Where

$$e_{i,2} = 2a_{i,2} - y_i''. \quad (3.8)$$

and c_i' depend on the numbers of intervals.

Proof: For $y(x) \in C^9$ [0,1] then using Taylor's expansion formula and similar expressions for the derivatives of y (x) can be used.
Now if i=1 then from equations $\alpha_{1,2}$ and using (3.8) we obtain

$$e_{1,2} = 2a_{1,2} - y_1'' = \frac{17h^7}{917280} y^{(9)}(\beta_1) + \frac{h^7}{3276} y^{(9}(\beta_2) - \frac{47h^7}{262080} y^{(9)}(\beta_3) + \frac{11h^7}{203840} y^{(9)}(\beta_5) - \frac{h^7}{5040} y^{(9)}(\beta_6), \quad (3.9)$$

Where $x_0 < \beta_1, \beta_2, \beta_3, \beta_3, \beta_4, \beta_5, \beta_6, < x_1$.
From equations (3.8) we get:

$$\left|e_{1,2}\right| \le \frac{11}{29120} h^7 \, w_9(f;h) \quad \text{so} \quad c_1' = \frac{11}{29120}.$$

Also, if i=2 then $\alpha_{1,2}$ and using (3.8), become

$$\left|e_{2,2}\right| = \left|2\,a_{1,2} - y_2''\right| \le \frac{182772}{143095680} h^7 \, w_9(f;h), \quad \text{so} \quad c_2' = \frac{182772}{143095680}.$$

By the same way as in the above, we can see that the inequality $\left|e_{i,2}\right| \le c_i' \, h^7 \, w_9(f;h)$ for i= 1, …, n-1 holds.

This completes the proof of the Lemma 3.2.

Theorem 3.2: Let $S(x)$ be a unique spline function of ninth degree where $y(x) \in C^9$ [0,1] the solution of (2.1). Then for $x \in [x_i, x_{i+1}]$; i=1, 2,…, n-1, the following error bounds are holds:

$$\left\| S_i^{(r)}(x) - y^{(r)}(x) \right\| \le \begin{cases} \frac{h^9}{34398}(34398\,c_i + 17199\,c_i' + 4)\,w_9(f;h), r = 0, \\ \frac{h^8}{22929}(22929\,c_i + 22929\,c_i' + 16)\,w_9(f;h), r = 1, \\ \frac{h^7}{35062}(35062\,c_i' + 133)\,w_9(f;h), r = 2, \ \frac{407\,i}{21840} h^6 w_9(f;h), r = 3, \\ \frac{103\,i}{1260} h^5 w_9(f;h), r = 4, \frac{85}{273} h^4 w_9(f;h), r = 5, \ \frac{346i}{91} h^3 w_9(f;h), r = 6, \\ \frac{431\,i}{182} h^2 w_9(f;h), r = 7, \ \frac{346\,i}{91} h\,w_9(f;h), r = 8, \ \frac{261\,i}{91} w_9(f;h), r = 9. \end{cases}$$

Proof: Let $x \in [x_i, x_{i+1}]$ where i=1, 2, …, n-1, then from equation (2.2) and using Taylor's expansion formula we get

$$S_i^{(9)}(x) - y^{(9)}(x) = 362880 a_{0,9} - y_i^{(9)}(x) \quad (3.10)$$

Using (2.1) and $\alpha_{0,9}$, we obtain

$$\left|S_i^{(9)}(x) - y^{(9)}(x)\right| = \left|362880 a_{i,9} - y^{(9)}(x)\right| \le \frac{216\,i}{91} w_9(f;h). \quad (3.11)$$

From (2.1) taking the eight derivatives and subtracting the function $y^{(8)}(x)$, we obtain

$$S_i^{(8)}(x) - y^{(8)}(x) = 40320 a_{i,8} + 362880 h\,a_{i,9} - y^{(8)}(x),$$

Using Taylor's series expansion on $y^{(8)}(x)$ about $x = x_i$, we get

$$\left|S_i^{(8)}(x) - y^{(8)}(x)\right| = \left|40320 a_{i,8} + 362880(x - x_i)a_{i,9} - y^{(8)}(x_i) + (x - x_i)y^{(9)}(\alpha_{i,1})\right| \ \le \frac{346\,i}{91} h w_9(h,f), \quad (3.12)$$

Where $x_i < \alpha_{i,1} < x_i$.
Also, from (2.2) taking the seventh derivatives, we get $S_i^{(7)}(x) - y^{(7)}(x) = y_i^{(7)} + 40320a_{i,8} + 181440h^2 a_{i,9} - y^{(7)}(x)$ ·
Also, from (2.2) taking the seventh derivatives, we get

$$\left|S_i^{(7)}(x) - y^{(7)}(x)\right| \le \frac{431i}{182} h^2 w_9(f;h), \tag{3.13}$$

Where $x_i < \alpha_{i,2} < x_i$.
And by taking the sixth derivatives with using Taylor's series expansion on $y^{(6)}(x)$ about $x = x_i$, we get

$$\left|S_i^{(6)}(x) - y^{(6)}(x)\right| \le \frac{2143}{2184} h^3 \left|y^{(9)}(\alpha_{i,3}) - y^{(9)}(\alpha_{i,4})\right| \le \frac{2143i}{2184} h^3 w_9(f;h), \tag{3.14}$$

Where $x_i < \alpha_{i,3}$, $\alpha_{i,4} < xi_{+1}$ and also

$$\left|S_i^{(5)}(x) - y^{(5)}(x)\right| \le \frac{85i}{273} h^4 w_9(f;h), \tag{3.15}$$

Where $x_i < \alpha_{i,3}$, $\alpha_{i,4}$ $\alpha_{i,5} < xi_{+1}$.
Using Taylor's series expansion, by the same way we can find the following:

$$\left|S_i^{(4)}(x) - y^{(4)}(x)\right| \le \frac{103i}{1260} h^5 w_9(f;h),$$
$$\left|S_i^{(3)}(x) - y^{(3)}(x)\right| \le \frac{407i}{21840} h^6 w_9(f;h).$$

Now using the above two lemmas (3.1) and (3.2), we obtain

$$\left|S_i''(x) - y''(x)\right| \le \left|2a_{i,2} - y_i''\right| + \frac{133}{35062} h^7 w_9(f;h) \le \frac{h^7}{35062}(35062c_i' + 133)w_9(f;h),$$
$$\left|S_i'(x) - y'(x)\right| \le a_{i,1} - y_i' + h(2a_{i,2} - y_i'') + \frac{h^8}{22929} w_9(f;h) \le h^8 c_i w_9(f;h) + h^8 c_i' w_9(f;h) + \frac{h^8}{22929} w_9(f;h)$$
$$\le \frac{h^8}{22929}(22929c_i + 22929c_i' + 16)w_9(f;h),$$

and

$$\left|S_i(x) - y(x)\right| \le h^9 (a_{i,1} - y_i')w_9(f;h) + \frac{h^9}{2}(2a_{i,2} - y_i'')w_9(f;h) + \frac{2h^9}{17199} w_9(f;h) \le \frac{h^9}{34398}(34398c_i + 17199c_i' + 4)w_9(f;h).$$

This proves Theorem 4.2 for $x \in [x_i, x_{i+1}, x_{i+1}]$, $i = 1,2,...,n-1$

Numerical Illustrations: In this section, three numerical examples are presented and those problems are referred in [7] and [3]. The problems are tested to the efficiency of the development solutions and to demonstrate its convergence computationally. The problems have been solved using our method with different values of step size h; it's tabulated in Tables 1, 2 and 3. These show that our results are more accurate.

Problem 1: [3]
Consider the system
$\dot{y}_1 = y_2$
$\dot{y}_2 = y_3$
$\dot{y}_3 = \cos(t) - 2y_3 - y_2 + y_1$
$y_1(0) = 0,\ y_2(0) = 1,\ y_3(0) = 2.$

Table 4.1: Absolute maximum error for $S(x)$ and its derivative with different values of tolerance for the problem 1:

TOL	FS	AMAXE$^{(0)}$	AMAXE$^{(1)}$	AMAXE$^{(2)}$	AMAXE$^{(3)}$	TIME (ms)
10^{-1}	0	2.0935×10^{-5}	1.94×10^{-2}	13.95×10^{-2}	6.735×10^{-1}	0.03509
10^{-2}	0	6.214×10^{-8}	1.93×10^{-4}	16.5×10^{-2}	2.025×10^{-1}	0.034659
10^{-3}	0	4.0985×10^{-8}	2.1793×10^{-6}	17×10^{-3}	1.489×10^{-1}	0.047356
TOL	FS	AMAXE$^{(4)}$	AMAXE$^{(5)}$	AMAXE$^{(6)}$	AMAXE$^{(7)}$	TIME (ms)
10^{-1}	0	4.2072×10^{0}	7.4365×10^{0}	18.12×10^{0}	9.2218×10^{0}	0.03509
10^{-2}	0	4.9219×10^{0}	8.3941×10^{0}	19.31×10^{0}	9.5228×10^{0}	0.034659
10^{-3}	0	4.9979×10^{0}	8.4941×10^{0}	13.829×10^{0}	9.5245×10^{0}	0.047356

Table 4.2: Absolute maximum error for $S(x)$ and its derivative with different values of tolerance for the problem 2:

TOL	FS	AMAXE$^{(0)}$	AMAXE$^{(1)}$	AMAXE$^{(2)}$	AMAXE$^{(3)}$	TIME (ms)
10^{-1}	0	1.7234×10^{-7}	15×10^{-2}	13.18×10^{-2}	1.07×10^{-2}	0.089503
10^{-2}	0	1.6722×10^{-12}	1.5×10^{-3}	11.7×10^{-4}	1.0067×10^{-4}	0.160630
10^{-3}	0	1.648×10^{-17}	15×10^{-4}	12×10^{-6}	1×10^{-6}	0.056985
TOL	FS	AMAXE$^{(4)}$	AMAXE$^{(5)}$	AMAXE$^{(6)}$	AMAXE$^{(7)}$	TIME (ms)
10^{-1}	0	2.21×10^{-1}	2.431×10^{-1}	4.6385×10^{0}	6.7533×10^{0}	0.089503
10^{-2}	0	2.02×10^{-0}	2.04×10^{0}	4.0611×10^{0}	6.0795×10^{0}	0.160630
10^{-3}	0	2×10^{-3}	2.004×10^{0}	86.706×10^{0}	6.008×10^{0}	0.056985

Table 4.3: Absolute maximum error for $S(x)$ and its derivative with different values of tolerance for the problem3:

TOL	FS	AMAXE$^{(0)}$	AMAXE$^{(1)}$	AMAXE$^{(2)}$	AMAXE$^{(3)}$	TIME (ms)
10^{-1}	0	1.3889×10^{-9}	5×10^{-3}	4.25×10^{-2}	1.6667×10^{-4}	0.05605
10^{-2}	0	1.1102×10^{-15}	5×10^{-5}	3.9×10^{-3}	1.6667×10^{-7}	0.184103
10^{-3}	0	0	5×10^{-7}	3.85×10^{-4}	1.6667×10^{-10}	0.056230
TOL	FS	AMAXE$^{(4)}$	AMAXE$^{(5)}$	AMAXE$^{(6)}$	AMAXE$^{(7)}$	TIME (ms)
10^{-1}	0	5×10^{-3}	1×10^{-1}	1×10^{0}	52×10^{-4}	0.05605
10^{-2}	0	5.057×10^{-5}	1×10^{-2}	9.433×10^{-1}	5.0167×10^{-5}	0.184103
10^{-3}	0	1.4×10^{-3}	1×10^{-3}	1.4229×10^{4}	5.0017×10^{-7}	0.056230

Problem 2: [10]
Consider the system

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = 2y_2 - y_1$$
$$y_1(0) = 0,\ y_2(0) = 1, \qquad [0,\ 50]$$

The exact solution is $y_1(x) = e^x$, $y_2(x) = (1+x(e^x))$.

Problem 3: [10]
Consider the system

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = y_1$$
$$y_1(0) = 1,\ y_2(0) = 1, \qquad [0,\ 100]$$

The exact solution is $y_1(x) = e^x$, $y_2(x) = e^x$.
The following notations will indicate in the tables:

TOL Tolerance
FS Total failure steps
AMAXE Absolute of the maximum error with respect to derivatives
and

TIME(ms) The execution time taken in microseconds.
The absolute of maximum error with respect to derivatives defined as [1]:

$$\text{AMAXE}^{(j)} = \max_{1\le i\le n}\left\|e_i^{(j)}\right\| = \max_{1\le i\le n}\left\|s^{(j)}(x_i) - y^{(j)}(x_i)\right\| \quad \text{where } j$$

be order of derivatives on whole intervals and $y(x_i)$ is the exact solution.

CONCLUSION

The approximate solutions of the system of differential equations by using ninth spline interpolation show that our method is better in the sense of accuracy and applicability. These have been verified by maximum absolute errors given in the tables it changes with respect to the step size of the tolerance and various problems. Some properties of spline are obtained which are required in proving the uniqueness, existences and convergence analysis of the present method.

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