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New Traveling Wave Solutions of the Higher Dimensional Nonlinear Evolution Equation by the Improved $\left(\frac{G'}{G}\right)$-expansion Method

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Abstract: In this article, we investigate the nonlinear evolution equation, namely, the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation by applying the improved (G'/G)-expansion method to construct some new traveling wave solutions. The obtained solutions are expressed in terms of the hyperbolic, the trigonometric and the rational functions including solitons and periodic solutions. The attained solutions become some special functions when the arbitrary constants taken particular values. It is important to mention that some of our solutions are in good harmony with the existing results which certifies our other solutions.

Key words: The improved (G'/G)-expansion method · The modified KdV-Zakharov-Kuznetsev equation · Exact solutions · Nonlinear partial differential equations

Mathematics Subject Classification: 35K99 · 35P99 · 35P05

INTRODUCTION

The investigation of travelling wave solutions for the nonlinear evolution equations (NLEEs) plays crucial roles in many scientific and engineering areas, such as, plasma physics, chemical physics, optical fibres, solid state physics, fluid mechanics, chemistry and many others. In recent years, various methods have been presented to obtain traveling wave solutions of the NLEEs, for example, the Hirota's bilinear transformation method [1], the homotopy perturbation method [2-5], the tanh-function method [6], the Jacobi elliptic function expansion method [7], the inverse scattering method [8], the F-expansion method [9], the projective Riccati equation method [10, 11], the tanh-coth method [12], the variational iteration method [13-21], the He's polynomials method [22-24], the first integral method [25], the Cole-Hopf transformation method [26], the Exp-function method [27-30], the direct algebraic method [31, 32] and others [33-37].

Recently, Wang *et al.* [38] introduced a method called the (G'/G)-expansion method and obtain traveling wave solutions for the four well established nonlinear evolution equations. Then, many researchers used this method to solve many nonlinear partial differential equations. For example, Naher *et al.* [39] obtained abundant traveling wave solutions of the Caudrey-Dodd-Gibbon equation by using the method. Abazari [40] concerned about the same method for constructing exact solutions for three nonlinear evolution equations. Feng *et al.* [41] applied this method to seek solutions of the Kolmogorov-Petrovskii-Piskunov equation. Neirameh and Alibeigi [42] constructed the traveling wave solutions of the (3+1)-dimensional Kadomtsev-Petviashvili equation via the (G'/G)-expansion method. Roozi and Mahmeiani [43] studied the (2+1)-dimensional Kadomtsev-Petviashvili equation to construct exact solutions via the same method. Lately, the method has been extended for searching exact solutions by different authors, such as, Hayek [44] expand the method called extended (G'/G)-expansion method to construct exact solutions of the KdV Burgers equations with power-law nonlinearity, Guo and Zhou [45] enlarge the method to obtain analytical solutions for some NLEEs and so on.

More lately, Zhang *et al.* [46] generalized the method, called the improved (G'/G)-expansion method for solving nonlinear evolution equations. Afterwards, many authors used the method for obtaining exact traveling wave solutions of the nonlinear PDEs. For instance, Zhao *et al.* [47] applied the method to the variant Boussinesq equations, Hamad *et al.* [48] implemented the method to the higher dimensional potential YTSF equation, Nofel *et al.* [49] related the method for searching traveling wave solutions of the fifth-order KdV equation and so on.

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In this article, we apply the improved (G'/G)-expansion method to obtain new traveling wave solutions for the nonlinear evolution equation, namely, the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation.

Description of the Improved (G'/G)-expansion Method: Suppose the general nonlinear partial differential equation:

$$Q\begin{pmatrix} u, u_t, u_x, u_y, u_z, u_{tt}, u_{xt}, u_{xx}, u_{xy}, \\ u_{yy}, u_{yt}, u_{zz}, u_{zt}, u_{zx}, u_{zy}, \ldots \end{pmatrix} = 0, \qquad (1)$$

Where $u = u(x, y, z, t)$ is an unknown function, Q is a polynomial in $u(x, y, z, t)$ and the highest order derivatives and the nonlinear terms are involved in its partial derivatives.

The main steps of the improved (G'/G)-expansion method [46] are as follows:

Step 1: Consider the traveling wave variable:

$$u(x,y,z,t) = v(\eta), \quad \eta = x + y + z - Vt, \qquad (2)$$

Where V is the wave speed. Now using Eq. (2), Eq. (1) is transformed into an ordinary differential equation for $v(\eta)$.

$$F(v, v', v'', v''', \ldots) = 0, \qquad (3)$$

Where the superscripts stand for the ordinary derivatives with respect to η.

Step 2: If possible, integrate Eq. (3) term by term one or more times, yields constant(s) of integration. For simplicity, the integral constant(s) may be zero.

Step 3: We assume, the wave solution of Eq. (3) can be expressed in the form [46]:

$$v(\eta) = \sum_{j=-n}^{n} a_j \left(\frac{G'}{G}\right)^j \qquad (4)$$

with $G = G(\eta)$ satisfies the second order linear ordinary differential equation (ODE):

$$G'' + \lambda G' + \mu G = 0 \qquad (5)$$

Where $a_j (j = 0, \pm 1, \pm 2, \ldots, \pm n)$, λ and μ are constants.

Step 4: To determine the integer n, substituting Eq. (4) along with Eq. (5) into Eq. (3) and then consider homogeneous balance between the highest order derivatives and the highest order nonlinear terms appearing in Eq. (3).

Step 5: Substitute Eq. (4) and Eq. (5) into Eq. (3) with the value of n obtained in Step 4. Equating the coefficients of $(G'/G)^r$, $(r = 0, \pm 1, \pm 2, \ldots)$, then setting each coefficient to zero, yields a set of algebraic equations for:

$$a_j (j = 0, \pm 1, \pm 2, \ldots, \pm n), V, \lambda \text{ and } \mu \qquad (6)$$

Step 6: Solve the system of algebraic equations with the aid of Maple 13 and we obtain values for $a_j (j = 0, \pm 1, \pm 2, \ldots, \pm n)$, V, λ and μ. Then, substitute obtained values in Eq. (4) along with Eq. (5) with the value of n, we obtain the traveling wave solutions of Eq. (1).

Applications of the Method: In this section, the method is used to construct some new traveling wave solutions for the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation which is very important nonlinear evolution equation in applied sciences. The obtained solutions and the solutions obtained in previous literature have been compared and discussed in this section. Furthermore, the obtained solutions are demonstrated in graphs using the commercial software Maple.

The (3+1)-Dimensional Modified Kdv-Zakharov-Kuznetsev Equation: In this subsection, we consider the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation [50]:

$$u_t + \beta u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \qquad (7)$$

Where β is a nonzero constant parameter.

Now, we use the wave transformation Eq. (2) into Eq. (7), which yields:

$$-V v' + \beta v^2 v' + 3 v''' = 0, \qquad (8)$$

Where the superscripts stand for the derivatives with respect to η.

Eq. (8) is integrable, so that, integrating once with respect to η yields:

$$-V v + \frac{1}{3} \beta v^3 + 3 v'' + K = 0, \qquad (9)$$

Where K is an integral constant that could be determined later.

Taking the homogeneous balance between v^3 and v'' in Eq. (9), we obtain $n = 1$ Therefore, the solution of Eq. (9) is the form as:

$$v(\eta) = \sum_{j=-1}^{1} a_j \left(\frac{G'}{G}\right)^j, \tag{10}$$

Where $a_j(j = 0,\pm1)$ are all constants to be determined.

Substituting Eq. (10) together with Eq. (5) into Eq. (9), the left-hand side of Eq. (9) is converted into a polynomial of $(G'/G)^j$, ($j = 0,\pm1,\pm2,...$) According to Step 5, collecting all terms with the same power of (G'/G) and setting each coefficient of the resulted polynomial to zero, yields a set of algebraic equations (which are omitted to display, for simplicity) for $a_j(j = 0,\pm1,\pm2,...,\pm n)$ V,K,λ and μ.

Solving the system of obtained algebraic equations with the aid of algebraic software Maple, we obtain three different types of solutions.

Case1:

$$a_{-1} = 0,\quad a_0 = \pm\frac{3\lambda i}{\sqrt{2\beta}},\quad a_1 = \pm\frac{6i}{\sqrt{2\beta}},\quad K = 0,\quad V = -\frac{3}{2}\lambda^2 + 6\mu, \tag{11}$$

where λ μ are free parameters and $\beta\neq 0$.

Case 2:

$$a_{-1} = a_{-1},\ a_0 = \pm\frac{3\lambda i}{\sqrt{2\beta}},\ a_1 = 0,\ K = 0,\ V = -\frac{3}{2}\lambda^2 \pm i\,a_{-1}\sqrt{2\beta},\ \mu = \frac{\pm i\,a_{-1}\sqrt{2\beta}}{6} \tag{12}$$

where a_{-1}, λ are free parameters and $\beta \neq 0$.

Case 3:

$$a_{-1} = \pm\frac{6\mu i}{\sqrt{2\beta}},\quad a_0 = \pm\frac{3\lambda i}{\sqrt{2\beta}},\quad a_1 = \pm\frac{6i}{\sqrt{2\beta}},\quad K = \pm\frac{36\lambda\mu i}{\sqrt{2\beta}},\quad V = -\frac{3}{2}\lambda^2 - 12\mu, \tag{13}$$

where λ μ are free parameters and $\beta \neq 0$.

Case 1: Substituting Eq. (11) together with the general solution Eq. (5) into Eq. (10), we obtain three types of travelling wave solutions of Eq. (9):

Hyperbolic Function Solutions: When $\lambda^2 - 4\mu>0$, we obtain

$$v(\eta) = \pm\frac{3i\sqrt{\lambda^2 - 4\mu}}{\sqrt{2\beta}}\left(\frac{A\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\eta + B\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\eta}{A\cosh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\eta + B\sinh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\,\eta}\right) \tag{14}$$

where $\eta = x + y + z - \left(\frac{-3}{2}\lambda^2 + 6\mu\right)t$, A and B are arbitrary constants.

Various known results can be rediscovered, if A and B are taken as special values.

For Example:

- If $A = 0$ but $B \neq 0$ we obtain,

$$v(\eta) = \frac{\pm 3i\sqrt{\lambda^2 - 4\mu}}{\sqrt{2\beta}}\coth\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\;\eta. \tag{15}$$

- If $B = 0$ but $A \neq 0$ we obtain,

$$v(\eta) = \frac{\pm 3i\sqrt{\lambda^2 - 4\mu}}{\sqrt{2\beta}}\tanh\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\;\eta. \tag{16}$$

- If $A \neq 0$, $A > B$, we obtain,

$$v(\eta) = \frac{\pm 3i\sqrt{\lambda^2 - 4\mu}}{\sqrt{2\beta}} \tanh\left(\frac{1}{2}\sqrt{\lambda^2 - 4\mu}\;\eta + \eta_0\right), \tag{17}$$

where $\eta_0 = \tanh^{-1}\frac{B}{A}$, $\eta = x + y + z - \left(\frac{-3}{2}\lambda^2 + 6\mu\right)t$.

Trigonometric Function Solutions:

when $\lambda^2 - 4\mu < 0$, we obtain,

$$v(\eta) = \pm\frac{3i\sqrt{4\mu - \lambda^2}}{\sqrt{2\beta}}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\eta + B\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\eta}{A\cos\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\eta + B\sin\frac{1}{2}\sqrt{4\mu - \lambda^2}\,\eta}\right) \tag{18}$$

where $\eta = x + y + z - \left(\frac{-3}{2}\lambda^2 + 6\mu\right)t$, A and B are arbitrary constants.

Various known results can be rediscovered, If A and B are taken as special values.

For Example:

- If $A = 0$ but $B \neq 0$, we obtain,

$$v(\eta) = \frac{\pm 3i\sqrt{4\mu - \lambda^2}}{\sqrt{2\beta}} \cot\frac{1}{2}\sqrt{4\mu - \lambda^2}\;\eta. \tag{19}$$

- If $B = 0$ but $A \neq 0$, we obtain,

$$v(\eta) = \frac{\pm 3i\sqrt{4\mu - \lambda^2}}{\sqrt{2\beta}} \tan\frac{1}{2}\sqrt{4\mu - \lambda^2}\;\eta. \tag{20}$$

- If $A \neq 0$, $A > B$, we obtain,

$$v(\eta) = \frac{\mp 3i\sqrt{4\mu - 4\lambda^2}}{\sqrt{2\beta}} \tan\left(\frac{1}{2}\sqrt{4\mu - \lambda^2}\;\eta - \eta_0\right), \tag{21}$$

where $\eta_0 = \tan^{-1}\frac{B}{A}$.

Rational Function Solution: When $\lambda^2 - 4\mu = 0$ we obtain

$$v(\eta) = \pm\frac{6i}{\sqrt{2\beta}}\left(\frac{B}{A + B\eta}\right), \tag{22}$$

where $\eta = x + y + z - \left(\frac{-3}{2}\lambda^2 + 6\mu\right)t$, A and B are arbitrary constants.

Case 2: Substituting Eq. (12) together with the general solution Eq. (5) into Eq. (10), we obtain three types of travelling wave solutions of Eq. (9):

Hyperbolic Function Solutions: When $\lambda^2 - 4\mu = 0$, we obtain

$$v(\eta)=a_{-1}\left(\frac{-\lambda}{2}+\frac{\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)}{\left(A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)}\right)^{-1}\pm\frac{3\lambda i}{\sqrt{2\beta}}, \tag{23}$$

Where $\eta = x+y+z-\left(\frac{-3}{2}\lambda^2 \pm ia_{-1}\sqrt{2\beta}\right)t$, $\mu = \frac{\pm ia_{-1}\sqrt{2\beta}}{6}$, A and B are arbitrary constants.

Various known results can be rediscovered, if A and B are taken as special values.

For Example:

- If $A = 0$ but $B \neq 0$, we obtain,

$$v(\eta)=a_{-1}\left(\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)^{-1}\pm\frac{3\lambda i}{\sqrt{2\beta}}. \tag{24}$$

- If $B = 0$ but $A \neq 0$ we obtain,

$$v(\eta)=a_{-1}\left(-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)^{-1}\pm\frac{3\lambda i}{\sqrt{2\beta}}. \tag{25}$$

- If $A \neq 0$, $A>B$, we obtain,

$$v(\eta)=a_{-1}\left(-\frac{\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}}{2}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+\eta_0\right)\right)^{-1}\pm\frac{3\lambda i}{\sqrt{2\beta}}. \tag{26}$$

where $\eta_0 = \tanh^{-1}\frac{B}{A}$, $\eta = x+y+z-\left(\frac{-3}{2}\lambda^2 \pm ia_{-1}\sqrt{2\beta}\right)t$, $\mu = \frac{\pm ia_{-1}\sqrt{2\beta}}{6}$.

Trigonometric Function Solutions: When $\lambda^2 - 4\mu < 0$, we obtain,

$$v(\eta)=a_{-1}\left(\frac{-\lambda}{2}+\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta}\right)^{-1}\pm\frac{3\lambda i}{\sqrt{2\beta}}, \tag{27}$$

Where $\eta = x+y+z-\left(\frac{-3}{2}\lambda^2 \pm ia_{-1}\sqrt{2\beta}\right)t$, $\mu = \frac{\pm ia_{-1}\sqrt{2\beta}}{6}$, A and B are arbitrary constants.

Various known results can be rediscovered, if A and B are taken as special values.

Rational Function Solution: When $\lambda^2 - 4\mu = 0$, we obtain

$$v(\eta)=a_{-1}\left(-\frac{\lambda}{2}+\frac{B}{A+B\eta}\right)^{-1}\pm\frac{3\lambda i}{\sqrt{2\beta}}, \tag{28}$$

Where $\eta=x+y+z-\left(\frac{-3}{2}\lambda^2\pm ia_{-1}\sqrt{2\beta}\right)t,\ \ \mu=\frac{\pm ia_{-1}\sqrt{2\beta}}{6},\ A$ and B are arbitrary constants.

Case 3: Substituting Eq. (13) together with the general solution Eq. (5) into Eq. (10), we obtain three types of travelling wave solutions of Eq. (9):

Hyperbolic Function Solutions: When $\lambda^2-4\mu>0$, we obtain

$$\begin{aligned}v(\eta)=&\pm\frac{6\mu i}{\sqrt{2\beta}}\left(\frac{-\lambda}{2}+\frac{\sqrt{\lambda^2-4\mu}\left(A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)}{2\left(A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)}\right)^{-1}\\&\pm\frac{3i}{\sqrt{2\beta}}\left(-\lambda+\frac{\sqrt{\lambda^2-4\mu}\left(A\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+B\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)}{\left(A\cosh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta+B\sinh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\eta\right)}\right)\pm\frac{3\lambda i}{\sqrt{2\beta}}\end{aligned} \tag{29}$$

Where $\eta=x+y+z-\left(\frac{-3}{2}\lambda^2-12\mu\right)t,\ A$ and B are arbitrary constants.

Various known results can be rediscovered, if A and B are taken as special values.

Trigonometric Function Solutions: When $\lambda^2-4\mu<0$, we obtain

$$\begin{aligned}v(\eta)=&\pm\frac{6\mu i}{\sqrt{2\beta}}\left(\frac{-\lambda}{2}+\frac{1}{2}\sqrt{4\mu-\lambda^2}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta}\right)\right)^{-1}\\&\pm\frac{3i}{\sqrt{2\beta}}\sqrt{4\mu-\lambda^2}\left(\frac{-A\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta+B\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta}{A\cos\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta+B\sin\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\eta}\right)\pm\frac{3\lambda i}{\sqrt{2\beta}},\end{aligned} \tag{30}$$

Where $\eta=x+y+z-\left(\frac{-3}{2}\lambda^2-12\mu\right)t,\ A$ and B are arbitrary constants.

Various known results can be rediscovered, if A and B are taken as special values.

For Example:

- If $A=0$ but $B\neq0$ we obtain

$$v(\eta)=\frac{\pm3i}{\sqrt{2\beta}}\left(2\mu\left(\frac{-\lambda}{2}+\frac{\sqrt{4\mu-\lambda^2}}{2}\cot\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)^{-1}+\sqrt{4\mu-\lambda^2}\cot\frac{\sqrt{4\mu-\lambda^2}}{2}\eta+\lambda\right). \tag{31}$$

- If $B=0$ but $A\neq0$ we obtain

$$v(\eta)=\frac{\pm 3i}{\sqrt{2\beta}}\left(2\mu\left(\frac{-\lambda}{2}-\frac{\sqrt{4\mu-\lambda^2}}{2}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)^{-1}-\sqrt{4\mu-\lambda^2}\tan\frac{\sqrt{4\mu-\lambda^2}}{2}\eta+\lambda\right). \tag{32}$$

- If $A \neq 0, A > B$, we obtain

$$v(\eta)=\frac{\pm 3\mu i}{\sqrt{2\beta}}\left(-\lambda-\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta-\eta_0\right)\right)^{-1} \pm\frac{3i}{\sqrt{2\beta}}\sqrt{4\mu-\lambda^2}\tan\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta-\eta_0\right)\pm\frac{3\lambda i}{\sqrt{2\beta}}, \tag{33}$$

Where $\eta_0=\tan^{-1}\frac{B}{A}$.

Rational Function Solution: When $\lambda^2-4\mu=0$ we obtain

$$v(\eta)=\pm\frac{3i}{\sqrt{2\beta}}\left(2\mu\left(-\frac{\lambda}{2}+\frac{B}{A+B\eta}\right)^{-1}+\frac{2B}{A+B\eta}\right), \tag{34}$$

Where $\eta=x+y+z-\left(\frac{-3}{2}\lambda^2-12\mu\right)t,\ A$ and B are arbitrary constants.

DISCUSSIONS

Many researchers obtained traveling wave solutions for the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation by using different methods. For example, Naher *et al.* [29] applied the Exp-function method for constructing traveling wave solutions. Zayed [50] investigated the equation via the basic (G'/G)-expansion method. Xu [51] utilized an elliptic equation method to solve this equation. But, to the best of our knowledge, the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation is not studied by using the improved (G'/G)-expansion method.

Table 1: Comparison between Zayed [50] solutions and our solutions

Zayed [50] solutions	Our solutions
i. If $A \neq 0$ and $B=0$, solution Eq. (3.37) becomes: $u(\xi)=\pm 3i\sqrt{\frac{\lambda^2-4\mu}{2\alpha}}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi.$	i. If $\beta=\alpha$ and $\eta=\xi$, solution Eq. (15)becomes: $u(\xi)=\pm 3i\sqrt{\frac{\lambda^2-4\mu}{2\alpha}}\coth\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi.$
ii.If $A=0$ and $B \neq 0$, solution Eq. (3.37) becomes: $u(\xi)=\pm 3i\sqrt{\frac{\lambda^2-4\mu}{2\alpha}}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi.$	ii. If $\beta=\alpha$ and $\eta=\xi$, solution Eq. (16)becomes: $u(\xi)=\pm 3i\sqrt{\frac{\lambda^2-4\mu}{2\alpha}}\tanh\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi.$
iii. If $A \neq 0$ and $B > A$, solution Eq. (3.37) becomes: $u(\xi)=\pm 3i\sqrt{\frac{\lambda^2-4\mu}{2\alpha}}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+\xi_0\right).$	iii. If $\beta=\alpha$ and $\eta=\xi$, solution Eq. (17)becomes: $u(\xi)=\pm 3i\sqrt{\frac{\lambda^2-4\mu}{2\alpha}}\tanh\left(\frac{1}{2}\sqrt{\lambda^2-4\mu}\,\xi+\xi_0\right).$
iv. If $A \neq 0$ and $B=0$, solution Eq. (3.38) becomes: $u(\xi)=\pm 3i\sqrt{\frac{4\mu-\lambda^2}{2\alpha}}\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi.$	iv. If $\beta=\alpha$ and $\eta=\xi$, solution Eq. (19)becomes: $u(\xi)=\pm 3i\sqrt{\frac{4\mu-\lambda^2}{2\alpha}}\cot\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi.$
v. If $A=0$ and $B \neq 0$, solution Eq. (3.38) becomes: $u(\xi)=\pm 3i\sqrt{\frac{4\mu-\lambda^2}{2\alpha}}\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi.$	v. If $\beta=\alpha$ and $\eta=\xi$, solution Eq. (19)becomes: $u(\xi)=\pm 3i\sqrt{\frac{4\mu-\lambda^2}{2\alpha}}\tan\frac{1}{2}\sqrt{4\mu-\lambda^2}\,\xi.$
vi. If $\lambda^2-4\mu=0$ solution Eq. (3.39) is: $u(\xi)=\pm\frac{6i}{\sqrt{2\alpha}}\left(\frac{B}{A+B\xi}\right).$	vi. If $\beta=\alpha$ and $\eta=\xi$, solution Eq. (22)becomes: $u(\xi)=\pm\frac{6i}{\sqrt{2\alpha}}\left(\frac{B}{A+B\xi}\right).$

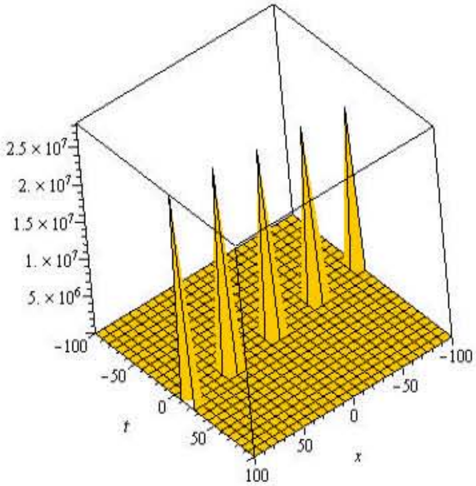

Fig. 1: Soliton solution for $\beta = 1$, $\lambda = 4$, $\mu = 5$

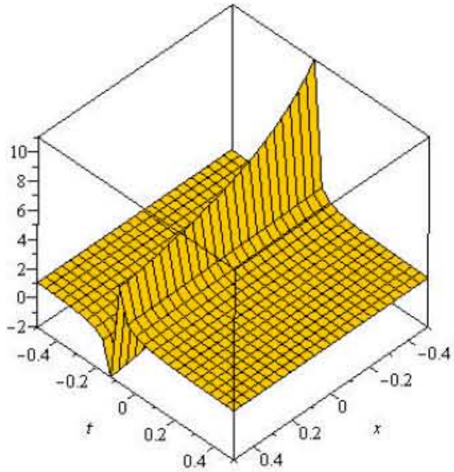

Fig. 4: Periodic solution for $\beta = 0.75$, $\lambda = 4$, $\mu = 4$, $A = 7$, $B = 1$

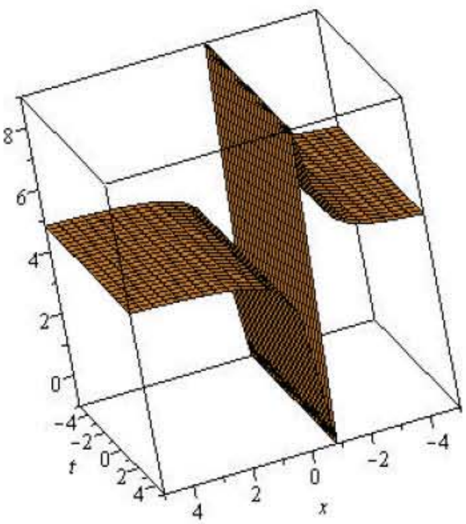

Fig. 2: Periodic solution for $\beta = 0.5$, $\lambda = 2$, $\mu = 1$, $a_{-1} = 1$, $A = 2$, $B = 1$

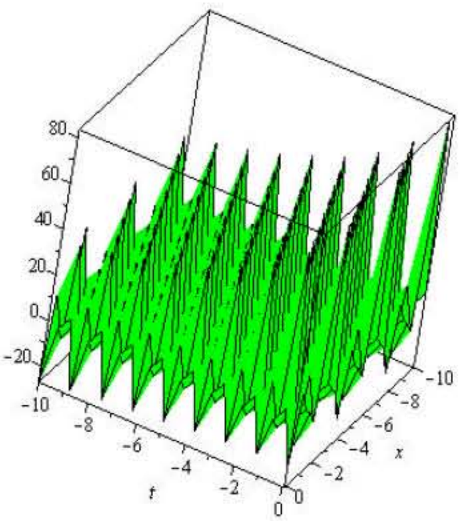

Fig. 5: Periodic solution for $\beta = 2$, $\lambda = 5$, $\mu = 9$, $\eta_0 = 5$

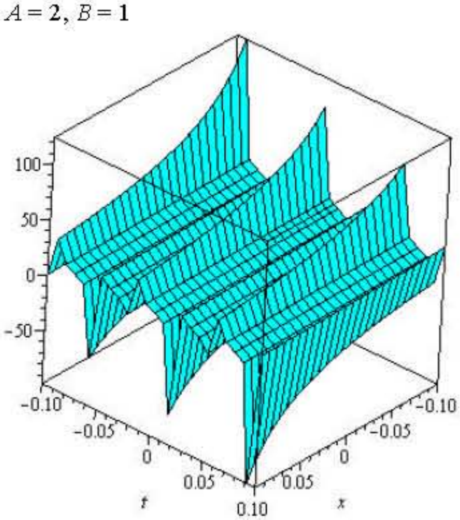

Fig. 3: Periodic solution for $\beta = 0.5$, $\lambda = 4$, $\mu = 5$

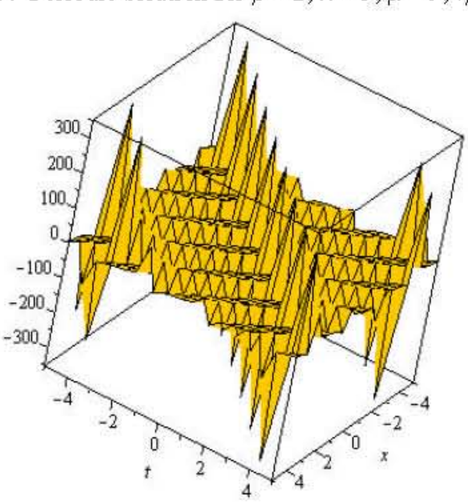

Fig. 6: Periodic solution for $\beta = 0.5$, $\lambda = 3$, $\mu = 4$

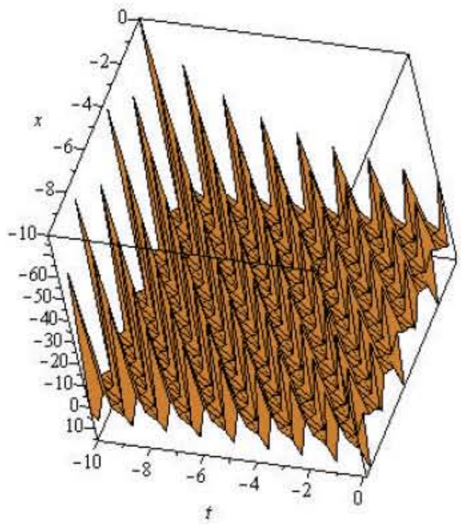

Fig. 7: Periodic solution for $\beta = 5$, $\lambda = 7$, $\mu = 15$, $\eta_0 = 2$

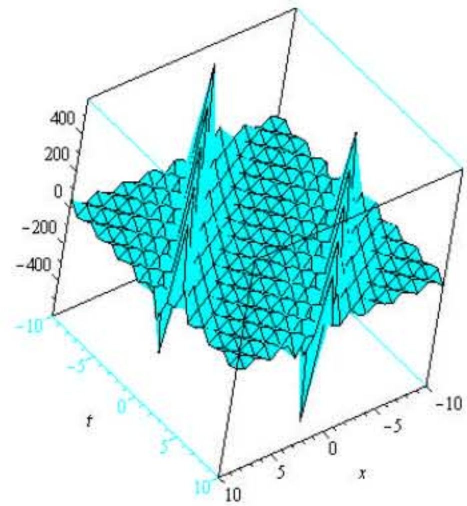

Fig. 8: Periodic solution for $\beta = 0.25$, $\lambda = 3$, $\mu = 4$

Beside this table, we obtain further new exact traveling wave solution Eqs. (24), (25), (26), (27), (28), (30), (31), (32), (33) and (34) in this article, which have not been reported in the previous literature. When some arbitrary values considered for these obtained new solutions are taken as some special functions.

Graphical Representations of the Solutions: The graphical illustrations of the solutions are depicted in the figures 1 to 8 with the aid of commercial software Maple.

CONCLUSIONS

Some new exact traveling wave solutions of the (3+1)-dimensional modified KdV-Zakharov-Kuznetsev equation are constructed in this article by applying the improved(G'/G)-expansion method. The obtained solutions are presented in terms of the hyperbolic, the trigonometric and the rational functions. It is significant to disclose that some of our obtained solutions are in good agreement with the published results and some are new. Also, the solutions show that the application of the method is trustworthy, straightforward and gives many solutions. We hope this method can be more effectively used to solve many nonlinear partial differential equations in applied mathematics, engineering sciences and mathematical physics.

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