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An New Inverse Technique for Estimating Heat Capacity

Abdolsedeh Neisy

Department of Mathematics and Statistics, Faculty of Economics,
Allameh Tabataba'ee University, Tehran, Iran

Abstract: This paper considers the problem of estimation of heat capacity in a one-dimensional heat conduction problem from temperature measurement in the domain. This is a typical inverse heat conduction problem (IHCP). The corresponding direct heat conduction problem (DHCP) will be solved by an application of the Finite-Difference approximation and analytical method and the heat capacity to be estimated by using Inverse Technique. Finally, we compare the numerical results with the analytic solution, numerically and graphicly.

AMS(MOS) Subject Classification: 35R30

Key words: Inverse heat conduction problem · Direct problem · Diffusion coefficient · Overposed data

INTRODUCTION

The inverse heat convection problems involve the determinations of the thermal properties, boundary condition, energy-generation rate, or thermophysical properties, from the knowledge of the temperature measurements taken in the domain.

Inverse problems are encountered in various branches of science and engineering. Aerospace and chemical engineers, mathematicians, astrophysicists, statisticians and specialists of many other disciplines all are interested in inverse problems.

This paper considers the problem of estimation of heat-capacity in a one-dimensional heat conduction problem from temperature measurement on the slab and it deals with the method for determining heat-capacity, which is based on the solution of the inverse problem of the identification of unknown heat-capacity parameters.

The problem of parameter identification is solved by nonlinear least-square method. The solution of this inverse problem requires a finite set of temperature measurements taken inside the slab and assumes that the heat-capacity belongs to set polynomial. The effectiveness of the inverse problem's solution is substantially dependent on related direct problem's solution.

Using Finite-Difference approximation, the direct heat conduction problem has been transferred to inhomogeneous initial value problem (IVP) and then an analytical solution presented for IVP [1, 2].

The Direct Problem: Consider a direct problem of a one-dimensional heat conduction problem of the following form:

$$\begin{aligned}
&c(x)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial}(k(x)\frac{\partial T(x,t)}{\partial x}) + \gamma(x,t), && 0<x<1, \quad 0<t<t_f \\
&T(x,0) = f(x), && 0 \le x \le 1 \\
&-k(0)\frac{\partial T}{\partial x}(0,t) = p(t), && 0<t<t_f \\
&-k(1)\frac{\partial T}{\partial x}(1,t) = q(t), && 0<t<t_f \qquad (1)
\end{aligned}$$

Where $k(x)\frac{\partial T(x,t)}{\partial x}$ is the heat flux, $k(x)$ is the thermal conductivity, $c(x)$ is the heat capacity and γ, f, p, q are known functions. Then the problem is concerned with the determination of temperature distribution $T(x,t)$ in the interior the slab as a function of position and time. we shall refer to such traditional problems as the DHCP's.

The following method is used to solve this direct problem [3]:

Corresponding Author: Abdolsedeh Neisy, Department of Mathematics and Statistics, Faculty of Economics, Allameh Tabataba'ee University, Tehran, Iran.

The central-difference approximations of finite difference represents the derivatives as:

$$k(x_i)\frac{\partial T}{\partial x}(x_i,t)=\frac{k_iT_{i+\frac{1}{2}}(t)-k_iT_{i-\frac{1}{2}}(t)}{\Delta x},$$

$$\frac{\partial}{\partial x}(k(x_i)\frac{\partial T(x_i,t)}{\partial x})=\frac{1}{(\Delta x)^2}[k_{i-\frac{1}{2}}T_{i-1}(t)-(k_{i-\frac{1}{2}}+k_{i+\frac{1}{2}})$$
$$T_i(t)+k_{i+\frac{1}{2}}T_{i+1}(t)],i=1,2,...,M \tag{2}$$

Where Δx is the increment in the x spatial coordinates, i is the i th grid along the x coordinate, $M\Delta x=1$ and $T_i(t)$ is the temperature at the grid point i.

The boundary conditions at $x = 0$ and $x = 1$ are transformed as follows:

$$-k_0(T_1-T_0)=(\Delta x)p(t) \tag{3}$$
$$-k_M(T_{M+1}-T_M)=(\Delta x)\,q\,(t), \tag{4}$$

and be initial condition at $t = 0$, is $T_i(0) = f(x_i) = f_i$.

Then the problem (1) can be expressed as the following recursive forms:

$$c_i\frac{dT_i(t)}{dt}=\frac{1}{(\Delta x)^2}[k_{i-\frac{1}{2}}T_{i-1}(t)-(k_{i-\frac{1}{2}}+k_{i+\frac{1}{2}})T_i(t)+k_{i+\frac{1}{2}}T_{i+1}(t)]+\gamma_i(t)$$

$$T_i(0)=f_i,\quad i=1,2,...,M \tag{5}$$

The above recursive forms can be expressed in the following matrix equation:

$$C\frac{dT}{dt}=AT+F,\qquad T(0)=f, \tag{6}$$

Where

$$A=\frac{1}{(\Delta x)^2}\begin{pmatrix} -k_{\frac{3}{2}} & k_{\frac{3}{2}} & \cdots & 0 & 0 \\ k_{\frac{3}{2}} & -(k_{\frac{3}{2}}+k_{\frac{5}{2}}) & k_{\frac{5}{2}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & k_{M-\frac{1}{2}} & -k_{M-\frac{1}{2}} \end{pmatrix}$$

$$T=\begin{pmatrix} T_1 \\ \vdots \\ T_M \end{pmatrix},\qquad \frac{\partial T}{\partial t}=\begin{pmatrix} \frac{\partial T_1}{\partial t} \\ \vdots \\ \frac{\partial T_M}{\partial t} \end{pmatrix},\qquad f=\begin{pmatrix} f_1 \\ \vdots \\ f_M \end{pmatrix}$$

$$F(t)=\begin{pmatrix} \gamma_1(t)+\frac{k_{\frac{1}{2}}}{(\Delta x)\,k_0}p(t) \\ \vdots \\ \gamma_M(t)+\frac{k_{M+\frac{1}{2}}}{(\Delta x)\,k_M}q(t) \end{pmatrix},\ C=\begin{pmatrix} c_1 & \cdots & c_M \end{pmatrix}$$

the solution of the inhomogeneous initial value problem (6) may be found as following steps:

Step 1: Find the fundamental matrix.

$$\phi(t)=e^{t(D^{-1}A)}=I+t(D^{-1}A)+\frac{1}{2!}t^2(D^{-1}A)^2+\cdots+\frac{1}{n!}t^n(D^{-1}A)^n+\cdots$$

of the homogeneous system $\frac{dT}{dt}=D^{-1}AT$, by using the Laplace transform.

Step 2: Compute, the integral

$$\int_0^t e^{-\theta D^{-1}A}D^{-1}F(\theta)d\theta=\int_0^t\phi(-\theta)D^{-1}F(\theta)d\theta$$

Step 3: The solution of initial value problem is

$$T(t)=e^{tD^{-1}A}f+\int_0^t e^{(t-\tau)D^{-1}A}D^{-1}F(\tau)d\tau$$
$$=\phi(t)f+\int_0^t\phi(t-\tau)D^{-1}F(\tau)d\tau,$$

Where

$$D^{-1}=\begin{pmatrix} \frac{1}{c_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{c_2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{1}{c_M} \end{pmatrix}$$

The Inverse Problem and Inverse Technique: The mathematical formulation of the inverse problem is similar to that of the direct problem given by equations (1) except the heat-capacity $c(x)$ is unknown function, but everything else in equations (1) is known. To determine $c(x)$ from boundary and initial data, we need additional temperature measurements taken at some spatial positions and time.

Now we consider M sensors placed in the slab such that M-2 are placed at the locations $x_2, x_3, ..., x_{M-1}$ inside the slab and the remaining two are placed at the boundary $x_1 = 0$ and $x_M = 1$. Temperature measurements at these points, at different times t_j ($j = 1,...,N$) are given by:

$$Y_{m,j} = T(x_m, t_j),\ m = 1,2,...,M,\ j = 1,...,N. \tag{7}$$

Thus a total of MN temperature measurements are available.

The $c(x)$ is obtained as the solution of the minimization problem of the least-squares norm $\|T^c - Y\|^2$, where $T^c = T(x_m, t_j; c)$ is a solution of (1) for any given $c(x)$.

In order to achieve a unique solution problem, the unknown function $c(x)$ is parameterised by assuming that the $c(x)$ is taken as a set of polynomials [2, 4, 5].

$$c(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) + ... + c_M(x - x_1)(x - x_2)...(x - x_{M-1}) \tag{8}$$

and the least-squares norm in discretised form is

$$E(C) = \sum_{m=1}^{m=M}\sum_{j=1}^{j=N}[T^c_{m,j}(C) - Y_{m,j}]^2 = \sum_{m=1}^{m=M}[T^c_{m,1}(C) - Y_{m,1}]^2 + ... + [T^c_{m,N}(C) - Y_{m,N}]^2 \tag{9}$$

$$= [T^c_{1,1}(C) - Y_{1,1}]^2 + ... + [T^c_{1,N}(C) - Y_{1,N}]^2$$

$$+ ... + [T^c_{M,1}(C) - Y_{M,1}]^2 + ... + [T^c_{M,N}(C) - Y_{M,N}]^2 = \sum_{j=1}^{j=MN}[\hat{T}^c_j(C) - \hat{Y}_j]^2,$$

Where $T^c_{m,j}(C)$ is the estimated temperature obtained from the solution of the direct problem (1) by using the estimated values of the unknown parameters.

Equation (9) is minimized by differentiating it with respect to each of the unknown parameters c_i ($i = 1,2,..,M$) and then setting the resulting expression equal to zero.

$$\frac{\partial E(C)}{\partial c_i} = 2\sum_{j=1}^{j=MN}\frac{\partial \hat{T}^c_j(C)}{\partial c_i}[\hat{T}^c_j(C) - \hat{Y}_j],\ \ i = 1,...,M \tag{10}$$

Equations (10) is written in the matrix form as

$$\frac{\partial E}{\partial C} = 2\frac{\partial \hat{T}}{\partial C}(\hat{Y} - \hat{T}) = 0,$$

Where

$$\frac{\partial \hat{T}}{\partial C} = \begin{pmatrix} \frac{\partial \hat{T}^c_1}{\partial c_1} & \cdots & \frac{\partial \hat{T}^c_1}{\partial c_M} \\ \vdots & \vdots & \vdots \\ \frac{\partial \hat{T}^c_{MN}}{\partial c_1} & \cdots & \frac{\partial \hat{T}^c_{MN}}{\partial c_M} \end{pmatrix}$$

$$\hat{T} = \left(\hat{T}^c_1\ \ ....\ \ \hat{T}^c_{MN}\right)^T, \quad \hat{Y} = \left(\hat{Y}_1\ \ ....\ \ \hat{Y}_{MN}\right)^T$$

This is a nonlinear system and an iterative solution method is required. A common approach is a technique given by

$$C^{k+1} = C^k + (J^T J + \mu_k I)^{-1} J^T(\hat{Y} - \hat{T}), k = 1,2,... \tag{11}$$

Where the μ_k's are damping parameters and J is the jacobian matrix defined by $J = \frac{\partial \hat{T}}{\partial C}$.

Clearly, for $\mu_k = 0$ equation (11) reduces to Newton's method.

The Solution Algorithm Is as Follows:
Suppose C^k at the k th iteration are available.

Step 1: Solve the direct problem with our method by using the estimated values of the parameters $C^k = (c_1, c_2, ..., c_M)^T$ at the k th iterative and compute $\hat{T}$.

Step 2: Solve the direct problem M more times, each time perturbing only one of the parameters by a small amount and compute

$$\hat{T}(c_1 + \Delta c_1, c_2, ..., c_M),\ \hat{T}(c_1, c_2 + \Delta c_2, c_3, ..., c_M),\ ....,\ \hat{T}(c_1, c_2, ..., c_M + \Delta c_M)$$

Step 3: Compute the coefficients matrix $\frac{\partial T}{\partial C}$ for each parameter. For example, with respect to c_1 we have

$$\frac{\partial \hat{T}_i}{\partial c_1} = \frac{\hat{T}(c_1 + \Delta c_1, c_2, ..., c_M) - \hat{T}(c_1, c_2, ..., c_M)}{\Delta c_1}$$

for $i = 1,2,.., MN$ and determine the matrix J.

Step 4: Compute $(J^T J + \mu_k I)^{-1} J^T(\hat{Y} - \hat{T})$

Step 5: Compute C^{k+1} according to equation (11)

A Numerical Example: We consider a test specimen of the final measurement time $t_f = 301$, temporal temperature readings are taken with sensors at ten locations (*i.e.,* $x_0 = 0, x_1 = 0.10, x_2 = 0.20, ..., x_{10} = 1$) over a period of $0 < t < 301$.

$$\gamma(x,t) = e^{-\pi^2 t}(-9.8696 - 10.1657x - 19.6976x^2)cos(\pi x)$$

$f(x) = cos(\pi x)$, $p(t) = 0$, $q(t) = 0$.

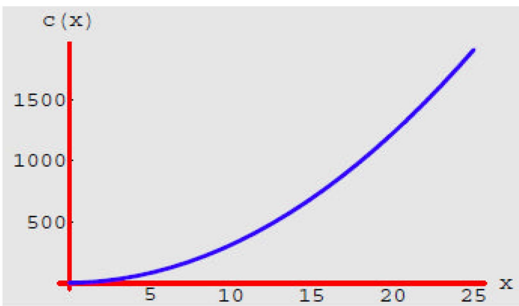

Fig. 1: Exact heat capacity $c(t)$

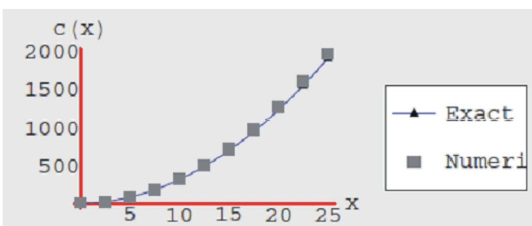

Fig. 2: Numerical heat capacity $c(t)$ approximated by Inverse technique

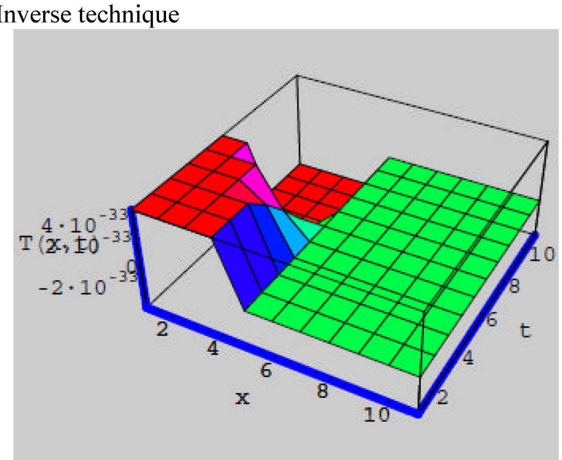

Fig. 3: Numerical Temperature Distribution $T(x,t)$

To simulate the measured temperature containing measurement error Err are introduced to the exact temperature as $Y = T_{exact} + Err$, where the exact T_{Exact} temperature is determined from the solution of the direct problem by using the exact values of the Heat Capacity, where Value of error (Err) lying in the range $-0.00025 < Err < 0.00025$ [4].

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