

## Solutions of a Subclass of Singular Second Order Differential Equations of Lane-Emden and Emden-Fowler Type

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**Abstract:** In this paper, we discuss an efficient method for obtaining the solutions of a subclass of singular differential equations of Lane-Emden and Emden-Fowler type with imposed boundary (or initial) conditions. Exact solutions of such type of problems are determined by substituting a suitable polynomial of undetermined coefficients into the given boundary (or initial) value problem. The proposed method is simple and illustrated by trustworthy examples justifying its efficiency and reliability.

**Key word:** Singular boundary value problems • He's polynomial method • Analytic solution

### INTRODUCTION

The singular boundary value differential equations occur in many disciplines e.g. Mathematics, Physics, Engineering. They are used to model physical phenomena in astrophysics, electro-hydrodynamics, to name a few. The general form of a singular boundary value differential equation is

$$\frac{1}{p(x)}y''(x) + \frac{1}{q(x)}y'(x) + \frac{1}{r(x)}y(x) = f(x) \quad 0 < x \leq 1 \quad (1)$$

### Subject to the Boundary (Or Initial) Conditions

$$y(0) = A_1, y(1) = B_1 \quad \text{or} \quad y(0) = A_2, y'(0) = B_2$$

where  $p, q, r, f$  are analytical functions of  $x$  defined over  $[0,1]$  and  $A_1, A_2, B_1, B_2$  are real numbers.

The existence and uniqueness about solutions of the differentiation equations are discussed in [4]. Determining a solution of such type of problems is very important as they have wide applications in scientific and engineering disciplines such as boundary layer theory, flow networks of biology, control and optimization, etc. Our aim is to always confine to a method that could ease the difficulty of finding a solution to the problem is fundamental in nature. Various universal phenomena like temperature variation of a self gravitating star, thermal behaviour of a spherical cloud of a gas under mutual attraction of its molecules, isothermal process in gas spheres, kinetics of combustion, reactants concentration in chemical reactor

[9], when modeled give rise to a general class of singular second order differential equation of Lane-Emden and Emden-Fowler type of the problem

$$y''(x) + \frac{k}{x}y'(x) + f(x, y) = g(x) \quad 0 < x \leq 1, k \geq 0 \quad (2)$$

### With Conditions

$$y(0) = A_1, y'(0) = 0 \quad (3)$$

$$y(0) = A, y(1) = B \quad (4)$$

where  $A_1, A, B$  are real constant,  $f(x, y)$  is continuous and real valued and  $g(x) \in C[0,1]$ .

The differential equation (2) along with (3) constitutes an initial value problem and along with (4) it is a boundary value problem. Moreover, if  $f(x, y)$  becomes purely a function of  $y$  in  $R$  in some appropriate domain, then (2) is specifically recognized as Lane-Emden type of the problem [1,2,5,7,10,11,12,21]. Several methods like B-splines [8,19], Homotopy method [3,11,12,15,16,22], Lie group analysis, Power series and variational iteration methods [13,14,17] have been applied to solve such differential equations. We consider an interesting case when  $g(x)$  is merely a polynomial function. Exact solution to such problem may be tried. In order to find the solution of (2) in the neighbourhood of a singular point  $x=0$ , (2) is modified as follows.

$$y''(x) + \frac{k}{x}y'(x) + f(x, y) = g(x) \quad 0 < x \leq 1 \quad (5)$$

and

$$xy''(x) + ky'(x) + xf(x, y) = xg(x) \quad x = 0 \quad (6)$$

**Method of Solution:** Restrict the general class of differential equation (2) to a subclass of differential equation as

$$y''(x) + \frac{k}{x}y'(x) + ap_1(x)y^m(x) = p_2(x) \quad 0 < x \leq 1, m \in I^+$$

and

$$xy''(x) + ky'(x) + axp_1(x)y^m(x) = xp_2(x) \quad x = 0$$

Here  $a$  is a real number,  $p_1(x)$  and  $p_2(x)$  are polynomials of appropriate degree such that degree of  $p_1(x)$  is less than the degree of  $p_2(x)$ . Solutions of such types of problems exist and are unique [20]. For finding the required solution of these types of problems, a polynomial  $h(x)$  with undetermined coefficients satisfying the given conditions is assumed. The undetermined coefficients are determined using the given conditions in such a way that the polynomial becomes the solution of the given problem. The degree of the solution polynomial  $h(x)$  is given by the degree of  $p_2(x)$  minus the degree of  $p_1(x)$  divided by  $m$ . In next section, we apply this method for obtaining solutions of the problems taken from literature.

**Numerical Illustration:** In this section, we apply our method for finding the solution of the problems of type (2) along with the prescribed conditions.

**Example:**

**Problem 1:** Consider the inhomogeneous Lane-Emden equation [18]

$$y''(x) + \frac{8}{x}y'(x) + xy(x) = x^5 - x^4 + 44x^2 - 30x \quad x \geq 0 \quad (7)$$

**Subject to Initial Conditions:**

$$y(0)=0, \quad y'(0)=0 \quad (8)$$

or

$$y(0)=0, \quad y(1)=0 \quad (9)$$

**Solution:** In (7) non-homogeneous function is of degree 5 and non-derivative term is  $x(y(x))$ . Therefore, we must have the solution polynomial satisfying the condition (8) as

$$y(x) = x^4 + bx^3 + cx^2 \quad (10)$$

Putting (9) in (7) and then comparing the coefficients of respective powers of  $x$  on both sides give  $a = -1$ ,  $b = 0$ . Using the values of  $a$  and  $b$ , the solution is

$$y(x) = x^4 - x^3$$

Similarly by Taking

$$y(x) = x^4 + ax^3 + bx^2 + cx$$

as solution polynomial and applying the conditions (9) we get the solution of (7) as

$$y(x) = x^4 - x^3$$

**Problem 2:** [6, 2] Consider the following problem

$$y''(x) + \frac{2}{x}y'(x) + y(x) = 6 + 12x + x^2 + x^3 \quad x \geq 0 \quad (11)$$

$$\text{Subject to } y(0)=0, y'(0)=0 \quad (12)$$

**Solution:** Since  $y(0)$  and  $y'(0)$  both are zero, the solution should be of the form

$$y(x) = x^3 + ax^2 + bx \quad (13)$$

By substituting (13) in (11) and comparing the coefficients of  $x^3$  and  $x^2$  on both sides, we get  $a=1$  and  $b=0$ .

Thus the solution is  $y(x) = x^3 + x^2$

**Problem 3:** [19] Consider the following problem

$$y''(x) + \frac{1}{x}y'(x) + y(x) = \frac{5}{4} + \frac{x^2}{16} \quad (14)$$

$$\text{Subject to } y'(0)=0, y(1)=17/16 \quad (15)$$

**Solution:** Let  $y(x)$  be the solution of the following form of (14)

$$y(x) = ax^2 + bx + c$$

Then  $y'(0)$  gives  $b=0$ . Then the solution is given by

$$y(x) = ax^2 + c \quad (16)$$

Putting (16) in (14), we get  $a=1/16$  and  $c=1$ . Thus, the solution is given by

$$y(x) = \frac{x^2}{16} + 1$$

**Problem 4:** [19] Consider the following problem

$$-y''(x) - \frac{2}{x}y'(x) + (1-x^2)y(x) = x^4 - 2x^2 + 7 \quad (17)$$

$$\text{Subject to } y'(0)=0, y(1)=0 \quad (18)$$

**Solution:** Since the right hand side of the (17) is a polynomial of degree four, left hand side must be a polynomial. When the solution of (17) is taken as polynomial, function,  $(1-x^2)y(x)$  must be polynomial of degree four. Thus  $y(x)$  must be a polynomial of degree two. Let  $y(x)$  be the solution of (17) given below

$$y(x) = ax^2 + bx + c$$

Putting  $y'(0)$  gives  $b=0$ . Thus, we have

$$y(x) = ax^2 + c \quad (19)$$

Putting (19) in (17), we get  $a=-1$  and  $c=1$ . Thus, the solution is given by

$$y(x) = -x^2 + 1.$$

**Problem 5:** [6] Consider the following non-linear differential equation of Lane-Emden type

$$y''(x) + \frac{2}{x}y'(x) + y^3(x) = x^6 + 6 \quad (20)$$

$$\text{subject to } y'(0)=0, y(1)=1 \quad (21)$$

**Solution:** Since the non-homogeneous term is of degree 6 and nonlinear term is of degree 3 in (20), the solution should be a polynomial of degree 2. Left and right hand sides of (20) are of same degree when  $y(x)$  is substituted as a polynomial in (20). Let  $y(x)$  be the solution of (20) given below

$$y(x) = x^2 + bx + c$$

Putting  $y'(0)$  gives  $b=0$ . Thus, we have

$$y(x) = x^2 + c \quad (22)$$

Putting (22) in (20), we get  $c=0$ .

Thus, the solution is given by

$$y(x) = x^2$$

**Problem 6:** [23] Consider another inhomogeneous differential equation

$$y''(x) + \frac{4}{x}y'(x) + y^2(x) = x^6 + 4x^3 + 18x + 4 \quad (23)$$

$$\text{subject to } y(0)=2, y'(0)=0 \quad (24)$$

**Solution:** Since the degree of inhomogeneous term is 6 and that of the nonlinear term is 2, the solution of (23) must be a polynomial of degree 3. Let  $y(x)$  be the solution of (23) given below

$$y(x) = x^3 + ax^2 + bx + c$$

Putting  $y'(0)$  gives  $b=0$ . Thus, we have

$$y(x) = x^3 + ax^2 + c \quad (25)$$

We have

$$y'(x) = 3x^2 + 2ax \quad (26)$$

$$y''(x) = 6x + 2a \quad (27)$$

Using (25)-(27) in (23) gives  $a=0$  and  $c=2$ . Thus, the solution is given by

$$y(x) = x^3 + 2$$

## CONCLUSION

In this paper, it is shown that by proper and suitable use of polynomial function as a solution of the problem, we can get the exact solution of the singular differential equation of second order when inhomogeneous term is a polynomial function (of any degree). The method discussed in this paper very simple and appealing enough as one can get exact is solution at preliminary stage.

## REFERENCES

1. Chowdhary, M.S.H. and I. Hasim, 2009. Solutions of Emden-Flower Equations by Homotopy perturbation method, Nonlinear Analysis Real world Appl. 10: 1004-115.

2. Erturk, V.S., 2007. Differential Transformation Method for Solving Differential Equations of Lane-Emden type, *Mathematical and Computational Applications*. 12(3): 135-139.
3. Fereidoon, A., M.R. Davoudabadi, H. Yaghoobi and D.D. Ganji, 2010. Application of Homotopy Perturbation Method and Differential Transformation Method to determine Displacement of a Damped System with Nonlinear Spring, *World Appl. Sci. J.* 9(6): 681-688.
4. Ford, W.F. and J.A. Pennline, 2009. Singular non-linear two-point boundary value problems: Existence and uniqueness, *Nonlinear Analysis: Theory, Methods & Applications*, 71: 1059-1072.
5. Gupta, V.G. and P. Sharma, 2009. Solving singular initial value problems of Emden-Flower and Lane-Emden type, *Appl. Mathematics and Comp.*, 1(4): 206-212.
6. Hasan, Y.Q. and L.M. Zhu, 2008. Modified Adomian decomposition Method for singular initial value Problems in the second-order ordinary Differential equations, *Surveys in Mathematics and its Applications*, 3: 183-193.
7. Jafari, M., M.M. Hosseini and S.T. Mohyud-Din, 2010. Solution of singular boundary value problems of Emden-flower type by the variational iteration method, *World Appl. Sci. J.*, 10(2): 154-160.
8. Jalilian, R., 2009. Convergence Analysis of Spline Solutions for Special Nonlinear Two-Order Boundary Value Problems, *World Applied Sciences Journal 7 (Special Issue of Applied Math)* 7: 19-24.
9. Kumar, M. and N. Singh, 2010. Modified Adomian Decomposition Method and computer implementation for solving singular boundary value problems arising in various physical problems, *Computers & Chemical Engineering*, 34(11): 1750-1760.
10. Motsa, S.S. and P. Sibanda, 2010. A new algorithm for solving singular IVPs of Lane-Emden type, *Latest Trends On Applied Mathematics, Simulation, Modelling*, ISBN: 978-960-474-210-3, 176-180.
11. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. He's homotopy perturbation method for Solving second-order singular problems using He's polynomials, *World Applied Sciences J.*, 6(6): 769-775.
12. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Solving second-order singular problems using He's polynomials. *World Appl. Sci. J.* 6(6): 769-775.
13. Mohyud-Din, S.T., 2009. Variational Iteration method for Evolution Equations, *World Appl. Sci. J.*, 7: 103-108.
14. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Modified Variational Iteration Method for Solving Sine-Gordon Equations, *World Appl. Sci. J.*, 6(7): 999-1004.
15. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Variational iteration method for Flierl-Petviashvili iteration method for solving higher-order nonlinear equations using He's polynomials and boundary value problems using He's polynomials and Pade' approximants, *World Appl. Sci. J.*, 6(9): 1298-1303.
16. Neyrameh, A., H. Neyrameh, M. Ebrahimi and A. Roozi, 2010. Analytic Solution Diffusivity Equation in Radial Form, *World Appl. Sci. J.*, 10(7): 764-768.
17. Noor, M.A. and S.T. Mohyud-Din, 2008. Modified Variational Iteration Method for Goursat and Laplace Problems, *World Applied Sci. J.*, 4(4): 487-498.
18. Parand, K., M. Dehghan, A.R. Rezaei and S.M. Ghaderi, 2010. An approximation algorithm for the solution of the nonlinear Lane-Emden type equations arising in astrophysics using Hermite functions collocation method, *Computer Physics Communications*, 181(6): 1096-1108.
19. Ravikanth, A.S.V. and K. Aruna, 2010. He's variational iteration method for treating nonlinear singular boundary value problems. *Computers and Mathematics with Applications* 60(3): 821-829.
20. Russell, R.D. and L.F. Shampine, 1975. Numerical methods for singular boundary value problem, *SIAM J. Numerical Analysis*, 12: 13-36.
21. Shawagfeh, N.T., 1993. Nonperturbative approximate solution for Lane-Emden equation. *J. Math. Phys.*, 34: 43-64.
22. Yildirim, A. and A. Kelleci, 2009. Numerical Simulation of the Jaulent-miodek Equation by He's Homotopy Perturbation Method, *World Applied Sci. J.*, 7. (Special Issue for Applied Math) 7: 84-89.
23. Wazwaz, A.M., 2006. A modified decomposition method for analytic treatment of differential equations, *Applied Mathematics and Computation*, 173: 165-176.