

Solvability of the Boundary Value Problem Coordinated with the Anisotropic Helmholtz-Schrodinger Equation in Case of $k_+ = k_-$

¹S.A. Hosseini Matikolai and ²Roshanak Lotfekar

¹Department of Mathematics and Mechanics, Yerevan State University, Yerevan, Armenia

²Islamic Azad University Ilam branch, Ilam, Iran

Abstract: In this paper, we have shown solvability of the boundary value problem connected with the Anisotropic Helmholtz-Schrodinger equation with the boundary condition of the first and second type in the case $k_+ = k_-$. And find the solution of this equation with given initial condition. In general, necessary and sufficient conditions for the correctness of the problem in the Sobolev space are presented as well as explicit formulas for a factorization of the Fourier symbol matrix of the one-medium problem.¹

Key words: Helmholtz-Schrodinger equation • Factorization of matrix-function • Boundary value problem • Wiener-Hopf equation.

INTRODUCTION

Various physical problem in diffraction theory lead us to study modification of the Sommerfeld half-plane governed by two proper elliptic partial differential equation is complementary R^3 half-space Ω^\pm and allow different boundary or transmission condition on two half-planes, which together form the common boundary of Ω^\pm [1].

Investigated a certain class of diffraction problems leading to simultaneous 2×2 systems of Wiener-Hopf equations.

First the classical Wiener-Hopf technique, represented by Noble [2]. This type of Problems studied by A.J.Sommerfeld for the wave diffraction on the interface of two media [3, 1], were investigated in the isotropic case [4, 1] and studied the problem of finding a function μ in a suitable space with satisfies [1].

In this Paper, Consider the Case of Anisotropic Helmholtz-schrodinger Equation:

$$\begin{cases} \Delta u + (k_+^2 + 2\beta_+^2 \operatorname{sech}^2(\beta_+ y)) u = 0 & \text{in } \Omega^+ \\ \Delta u + (k_-^2 + 2\beta_-^2 \operatorname{sech}^2(\beta_- y)) u = 0 & \text{in } \Omega^- \end{cases} \quad (1)$$

Where $k_+ = k_-$ and get the solution of the boundary value problem and then prove solvability of this.

Convention: As a rule, upper or lower indices \pm are related to the half-spaces Ω^\pm except for some standard notation R_\pm and $H^{\pm \frac{1}{2}}$.

Solvability of Boundary Problem in the Case $k_+ = k_-$: Consider the following anisotropic Helmholtz-Schrodinger equation

$$\begin{cases} \Delta u + (k^2 + 2\beta_+^2 \operatorname{sech}^2(\beta_+ y)) u = 0, & \text{in } \Omega^+, \\ \Delta u + (k^2 + 2\beta_-^2 \operatorname{sech}^2(\beta_- y)) u = 0, & \text{in } \Omega^-. \end{cases} \quad (2)$$

Let $\Omega^\pm = \{(x, y) \in R^2 : y > 0 \text{ (} y < 0 \text{)}\}$ where $\operatorname{Re}(k) > 0$, $\operatorname{Im}(k) > 0$ and $H^{1/2}(\Omega^+)$, $H^{-1/2}(\Omega^+)$, are the corresponding Sobolev spaces [5].

This equation is the particular case of the equation (1). We suppose that the following boundary conditions are fulfilled

$$\begin{cases} \begin{cases} a_0 u(x, +0) + b_0 u(x, -0) = h_0(x) \\ a_1 \frac{\partial u(x, +0)}{\partial y} + b_1 \frac{\partial u(x, -0)}{\partial y} = h_1(x) \end{cases} & \text{in } R^+ \\ \begin{cases} c_0 u(x, +0) + d_0 u(x, -0) = p_0(x) \\ c_1 \frac{\partial u(x, +0)}{\partial y} + d_1 \frac{\partial u(x, -0)}{\partial y} = p_1(x) \end{cases} & \text{in } R^- \end{cases} \quad (3)$$

where $h_0 \in H^{1/2}(R^+)$, $h_1 \in H^{1/2}(R^+)$, $P_0 \in H^{1/2}(R^-)$, $P_1 \in H^{1/2}(R^-)$ and $a_0, a_1, b_0, b_1, c_0, c_1, d_0, d_1$ are the complex constants. For finding of the boundary value problem (2) in $L^2(R^2)$, with same boundary conditions (3) and $a_1, d_1 = b_1, c_1$.

apply Fourier integral transform to the solution $\mu \in L^2(R^2)$, over the variable x ne derives the following system of ordinary differential equations

$$\begin{cases} \frac{d^2 \hat{u}}{dy^2} + (\kappa^2(\lambda) + 2\beta_+^2 \sec h^2(\beta_+ y)) \hat{u} = 0, \text{ for } y > 0 \\ \frac{d^2 \hat{u}}{dy^2} + (\kappa^2(\lambda) + 2\beta_-^2 \sec h^2(\beta_- y)) \hat{u} = 0, \text{ for } y < 0 \end{cases} \quad (4)$$

Then $\hat{u} \in L^2(R^2)$ and one considers that $\text{Im}(k) > 0$ we denote $\gamma(\lambda) = \sqrt{\lambda^2 - k^2} = i\kappa(\lambda)$

It follows that the general solution of the system of ordinary differential equations (4) in the $L^2(R^2)$ -space has the following form:

$$\hat{u}(\lambda, y) = \begin{cases} a(\lambda) \frac{i\kappa(\lambda) - \beta_+ \tanh(\beta_+ y)}{i\kappa(\lambda)} e^{i\kappa(\lambda)y}, & \text{for } y > 0 \\ b(\lambda) \frac{i\kappa(\lambda) + \beta_- \tanh(\beta_- y)}{i\kappa(\lambda)} e^{-i\kappa(\lambda)y}, & \text{for } y < 0. \end{cases} \quad (5)$$

Let $\chi_{\pm}(y) = 1/2(1 \pm \text{sgny})$ and

So

$$\frac{d\hat{u}(\lambda, y)}{dy} = \begin{cases} a(\lambda) \left[(i\kappa(\lambda) - \beta_+ \tanh(\beta_+ y)) - \frac{\beta_+^2}{i\kappa(\lambda) \cosh^2(\beta_+ y)} \right] e^{i\kappa(\lambda)y}, & \text{for } y > 0 \\ -b(\lambda) \left[(i\kappa(\lambda) - \beta_- \tanh(\beta_- y)) - \frac{\beta_-^2}{i\kappa(\lambda) \cosh^2(\beta_- y)} \right] e^{-i\kappa(\lambda)y}, & \text{for } y < 0 \end{cases} \quad (10)$$

Using boundary conditions (3) and taking into account eqs.(4), (9) one derives

$$\begin{cases} a_0 a(\lambda) + b_0 b(\lambda) = u_-(\lambda) + \hat{h}_0(\lambda) \\ -a_1 \left[\kappa^2(\lambda) + \beta_+^2 \right] a(\lambda) - b_1 \left[\kappa^2(\lambda) + \beta_-^2 \right] b(\lambda) \\ \frac{\quad}{i\kappa(\lambda)} + \frac{\quad}{i\kappa(\lambda)} = w_-(\lambda) + \hat{h}_1(\lambda) \end{cases} \quad (11)$$

where

$$\hat{h}_0(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 h_0(x) e^{i\lambda x} dx, \quad \hat{h}_1(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 h_1(x) e^{i\lambda x} dx$$

Assume that the determinant $\Delta(\lambda)$ of system (11) is not zero, i.e.

$$\begin{cases} \hat{u}_+(\lambda, y) = \frac{\chi_+(y)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{i\lambda x} dx \\ \hat{u}_-(\lambda, y) = \frac{\chi_-(y)}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, y) e^{i\lambda x} dx. \end{cases} \quad (6)$$

Then from eq.(4) it follows that

$$\hat{u}(\lambda, y) = \hat{u}_+(\lambda, y) + \hat{u}_-(\lambda, y) \quad (7)$$

we introduce functions

$$\begin{cases} u_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 (a_0 u(x, +0) + b_0 u(x, -0) - h_0(x)) e^{i\lambda x} dx \\ w_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 (a_1 \frac{\partial u(x, +0)}{\partial y} + b_1 \frac{\partial u(x, -0)}{\partial y} - h_1(x)) e^{i\lambda x} dx, \end{cases} \quad (8)$$

similarly

$$\begin{cases} u_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (c_0 u(x, +0) + d_0 u(x, -0) - p_0(x)) e^{i\lambda x} dx \\ w_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (c_1 \frac{\partial u(x, +0)}{\partial y} + d_1 \frac{\partial u(x, -0)}{\partial y} - p_1(x)) e^{i\lambda x} dx. \end{cases} \quad (9)$$

$$\Delta(\lambda) = a_0 b_1 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} + a_1 b_0 \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} = a_0 b_1 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)} + a_1 b_0 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} \neq 0 \quad (12)$$

In view of eq(11) :

$$\begin{cases} a(\lambda) = \frac{1}{\Delta(\lambda)} \left\{ b_1 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} (u_-(\lambda) + \hat{h}_0(\lambda)) - b_0 (w_-(\lambda) + \hat{h}_1(\lambda)) \right\} \\ b(\lambda) = \frac{1}{\Delta(\lambda)} \left\{ a_1 \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} (u_-(\lambda) + \hat{h}_0(\lambda)) + a_0 (w_-(\lambda) + \hat{h}_1(\lambda)) \right\}, \end{cases} \quad (13)$$

then, taking into account that

$$\begin{cases} u_+(\lambda) = c_0 a(\lambda) + d_0 b(\lambda) - \hat{p}_0(\lambda) \\ w_+(\lambda) = \frac{-c_1 [\kappa^2(\lambda) + \beta_+^2] a(\lambda)}{i\kappa(\lambda)} + \frac{d_1 [\kappa^2(\lambda) + \beta_-^2] b(\lambda)}{i\kappa(\lambda)} - \hat{p}_1(\lambda) \end{cases} \quad (14)$$

where

$$\hat{p}_0(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty p_0(x) e^{i\lambda x} dx, \quad \hat{p}_1(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^\infty p_1(x) e^{i\lambda x} dx$$

which derives the following boundary problem of Riman-Hilbert with respect to the

$$\vec{u}_+(\lambda) = \begin{pmatrix} u_+(\lambda) \\ w_+(\lambda) \end{pmatrix}, \quad \vec{u}_-(\lambda) = \begin{pmatrix} u_-(\lambda) \\ w_-(\lambda) \end{pmatrix}, \quad (15)$$

Vector founctions, which are analytical functions in the upper and lower semiplanes respectively in the vector notations this problem takes the following form:

$$\vec{u}_+(\lambda) = L(\lambda) \vec{u}_-(\lambda) + \vec{m}(\lambda) \quad (16)$$

where the matrix function $L(\lambda)$ is:

$$L(\lambda) = \frac{1}{\Delta(\lambda)} \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) \end{pmatrix} \quad (17)$$

with

$$\begin{aligned} A_{11}(\lambda) &= a_1 d_0 \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} + b_1 c_0 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} = a_1 d_0 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} + b_1 c_0 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)}, \\ A_{12}(\lambda) &= a_0 d_0 - b_0 c_0, \\ A_{21}(\lambda) &= (a_1 d_1 - b_1 c_1) \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} + b_1 c_0 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} = (a_1 d_1 - b_1 c_1) \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} \times \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)}, \\ A_{22}(\lambda) &= b_0 c_1 \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} + a_0 d_1 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} = b_0 c_1 \frac{\gamma^2(\lambda) - \beta_+^2}{\gamma(\lambda)} + a_0 d_1 \frac{\gamma^2(\lambda) - \beta_-^2}{\gamma(\lambda)}. \end{aligned}$$

The coordinates of the vector-function $\vec{m}(\lambda)$

$$\vec{m}(\lambda) = \begin{pmatrix} m_1(\lambda) \\ m_2(\lambda) \end{pmatrix} \quad (18)$$

have the following form

$$m_1(\lambda) = \frac{\hat{h}_0(\lambda)}{\Delta(\lambda)} \left\{ a_1 d_0 \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} + b_1 c_0 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} \right\} + \frac{a_0 d_0 - b_0 c_0}{\Delta(\lambda)} \hat{h}_1(\lambda) - \hat{p}_0(\lambda),$$

$$m_2(\lambda) = \frac{\hat{h}_1(\lambda)}{\Delta(\lambda)} \left\{ b_0 c_1 \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} + a_0 d_1 \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} \right\} + \frac{a_1 d_1 - b_1 c_1}{\Delta(\lambda)} \hat{h}_0(\lambda) \frac{\kappa^2(\lambda) + \beta_+^2}{i\kappa(\lambda)} \times \frac{\kappa^2(\lambda) + \beta_-^2}{i\kappa(\lambda)} - \hat{p}_1(\lambda)$$

and the matrix function

$$L(\lambda) = \begin{pmatrix} \frac{a_1 d_0 (\gamma^2(\lambda) - \beta_+^2) + b_1 c_0 (\gamma^2(\lambda) - \beta_-^2)}{a_1 b_0 (\gamma^2(\lambda) - \beta_+^2) + a_0 b_1 (\gamma^2(\lambda) - \beta_-^2)}; & \frac{(a_0 d_0 - b_0 c_0) \gamma(\lambda)}{a_1 b_0 (\gamma^2(\lambda) - \beta_+^2) + a_0 b_1 (\gamma^2(\lambda) - \beta_-^2)} \\ 0; & \frac{b_0 c_1 (\gamma^2(\lambda) - \beta_+^2) + a_0 d_1 (\gamma^2(\lambda) - \beta_-^2)}{a_1 b_0 (\gamma^2(\lambda) - \beta_+^2) + a_0 b_1 (\gamma^2(\lambda) - \beta_-^2)} \end{pmatrix}. \quad (19)$$

Consider the matrices $J_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$, $(i=0,1)$, generated from the coefficients of the boundary conditions (2). It is

easy to verify that without loss of the generality non-degenerate cases (i.e. $\Delta(\lambda) \neq 0$, $\det(L(\lambda)) \neq 0$) are possible only in the following cases

$$\begin{aligned} 1) J_0 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, & 2) J_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \\ 3) J_0 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, & 4) J_0 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \\ 5) J_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} v & 1 \\ v & 1 \end{pmatrix}, & 6) J_0 &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} v & 1 \\ v & 1 \end{pmatrix}, \\ 7) J_0 &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} v & 1 \\ v & 1 \end{pmatrix}, & 8) J_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, J_1 = \begin{pmatrix} v & 1 \\ v & 1 \end{pmatrix}, \\ 9) J_0 &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} v & 1 \\ v & 1 \end{pmatrix}, & 10) J_0 &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, J_1 = \begin{pmatrix} v & 1 \\ v & 1 \end{pmatrix}. \end{aligned}$$

Where $v \neq 0,1$ is a constant.

In all cases (1-10) the matrix function $L(\lambda)$ can be presented in the form

$$L(\lambda) = \begin{pmatrix} \frac{\alpha \gamma^2(\lambda) - \chi}{\delta \gamma^2(\lambda) - \rho}; & \frac{\varepsilon \gamma(\lambda)}{\delta \gamma^2(\lambda) - \rho} \\ 0; & 1 \end{pmatrix} \quad (20)$$

Therefor we have the following cases

$$\begin{aligned} 1) a &= \delta = 1, \chi = \rho = \beta_+^2, \varepsilon = 1, \\ 2) a &= \delta = 1, \chi = \rho = \beta_+^2, \varepsilon = -1, \\ 3) a &= \delta = 1, \chi = \rho = \beta_-^2, \varepsilon = -1, \\ 4) a &= \delta = 1, \chi = \rho = \beta_-^2, \varepsilon = 1, \\ 5) a &= v, \delta = 1, \chi = \beta_+^2, \rho = \beta_-^2, \varepsilon = -1, \\ 6) a &= v, \delta = v + 1, \chi = \beta_+^2, \rho = \beta_-^2 + v\beta_+^2, \varepsilon = 1, \\ 7) a &= 1, \delta = v + 1, \chi = \beta_-^2, \rho = \beta_-^2 + v\beta_+^2, \varepsilon = -1, \\ 8) a &= 1, \delta = v, \chi = \beta_+^2, \rho = v\beta_+^2, \varepsilon = -1, \\ 9) a &= v + 1, \delta = v, \chi = \beta_-^2 + v\beta_+^2, \rho = v\beta_+^2, \varepsilon = -1, \\ 10) a &= v + 1, \delta = v, \chi = \beta_-^2 + v\beta_+^2, \rho = \beta_-^2, \varepsilon = 1. \end{aligned}$$

Recall that the generalized factorization of the matrix $L(\lambda)$ in $L^2(R)$ is called the following representation

$$L(\lambda) = L_+(\lambda) \begin{pmatrix} \left(\frac{\lambda-i}{\lambda+i} \right)^{\chi_1} & 0 \\ 0 & \left(\frac{\lambda-i}{\lambda+i} \right)^{\chi_2} \end{pmatrix} L_-(\lambda), \quad (21)$$

where

- $\chi_1, \chi_2 \in \mathbb{Z}$, $L_{\pm}^{\pm 1} \in L^2(R, \rho)$ (i.e. each component of the matrix belong to $L^2(R, \rho)$, where $\rho(\lambda) = \frac{1}{\sqrt{\lambda^2 + 1}}$. The matrix functions $L_{\pm}^{\pm 1}(\lambda)$ in the upper half-plane $\text{Im}(\lambda) > 0$ and $L_{\pm}^{\pm 1}(\lambda)$ have the analytic continuations in the lower half-plane $\text{Im}(\lambda) < 0$.
- The components of the matrix function $\rho(\lambda)L_{\pm}^{\pm 1}(\lambda)$ belong to the $L^2(R)$ and have analytic continuations in the upper half-plane $\text{Im}(\lambda) > 0$ (lower half-plane $\text{Im}(\lambda) < 0$).

The factorization is called canonical, if $\chi_1 = \chi_2 = 0$. Let

$$R_+(\lambda) = \frac{\alpha \left(\lambda + \sqrt{k^2 + \frac{\chi}{\alpha}} \right)}{\delta \left(\lambda + \sqrt{k^2 + \frac{\chi}{\delta}} \right)}, \quad (22)$$

$$R_-(\lambda) = \frac{\lambda - \sqrt{k^2 + \frac{\chi}{\alpha}}}{\lambda - \sqrt{k^2 + \frac{\chi}{\delta}}}. \quad (23)$$

It is easy to see that the components $\rho(\lambda)L_{\pm}^{\pm 1}(\lambda)$ belong to the $L^2(R)$ and $\rho(\lambda)L_{\pm}^{\pm 1}(\lambda)$ have the analytic continuations in the upper half-plane $\text{Im} \lambda > 0$ (lower half-plane $\text{Im}(\lambda) < 0$).

Denote by

$$r_+(\lambda) = \frac{1}{2\pi i} \int_{-\infty + ic}^{\infty + ic} \frac{r(\xi) d\xi}{\xi - \lambda}, \quad r_-(\lambda) = \frac{1}{2\pi i} \int_{-\infty + id}^{\infty + id} \frac{r(\xi) d\xi}{\xi - \lambda}, \quad (24)$$

where

$$r(\lambda) = \frac{\varepsilon \sqrt{\lambda^2 - k^2}}{\alpha \left(\lambda + \sqrt{k^2 + \frac{\chi}{\alpha}} \right) \left(\lambda - \sqrt{k^2 + \frac{\rho}{\delta}} \right)}, \quad (25)$$

$k_* < c < d < k_{**}$. Here

$$k_* = \max \left\{ -\text{Im} \sqrt{k^2 + \frac{\chi}{\alpha}}, -\text{Im} \sqrt{k^2 + \frac{\rho}{\delta}} \right\}, \quad k_{**} = \min \left\{ \text{Im} \sqrt{k^2 + \frac{\chi}{\alpha}}, \text{Im} \sqrt{k^2 + \frac{\rho}{\delta}} \right\}$$

It is evident that $r_{\pm}(\lambda)$ belong to the $L^2(R)$ and $r_{\pm}(\lambda)$ have analytic continuation in the upper (lower) half-plane [6].

Since

$$R_+(\lambda)R_-(\lambda) = R(\lambda) = \frac{\alpha \gamma^2(\lambda) - \chi}{\delta \gamma^2(\lambda) - \rho}, \quad (26)$$

$$r_+(\lambda) + r_-(\lambda) = r(\lambda) \quad (27)$$

and

$$\begin{pmatrix} 1, & r_{\pm}(\lambda) \\ 0, & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1, & -r_{\pm}(\lambda) \\ 0, & 1 \end{pmatrix}, \quad \begin{pmatrix} R_{\pm}(\lambda), & 0 \\ 0, & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{R_{\pm}(\lambda)}, & 0 \\ 0, & 1 \end{pmatrix}, \quad (28)$$

therefore we have the equality

$$L(\lambda) = \begin{pmatrix} R_+(\lambda), & 0 \\ 0, & 1 \end{pmatrix} \begin{pmatrix} 1, & r_+(\lambda) \\ 0, & 1 \end{pmatrix} \begin{pmatrix} 1, & r_-(\lambda) \\ 0, & 1 \end{pmatrix} \begin{pmatrix} R_-(\lambda), & 0 \\ 0, & 1 \end{pmatrix}. \quad (29)$$

The representation (29) is canonical factorization of the matrix function $L(\lambda)$.

Using the equalities (28) and (29) we can write the Riemann-Hilbert problem (16) in the form

$$\begin{pmatrix} \frac{1}{R_+(\lambda)}, & -r_+(\lambda) \\ 0, & 1 \end{pmatrix} \bar{u}_+(\lambda) = \begin{pmatrix} R_-(\lambda), & r_-(\lambda) \\ 0, & 1 \end{pmatrix} \bar{u}_-(\lambda) + \begin{pmatrix} \frac{1}{R_+(\lambda)}, & -r_+(\lambda) \\ 0, & 1 \end{pmatrix} \bar{m}(\lambda). \quad (30)$$

In opened form we get the following system

$$\begin{cases} \frac{u_+(\lambda)}{R_+(\lambda)} - r_+(\lambda)w_+(\lambda) = R_-(\lambda)u_-(\lambda) + r_-(\lambda)w_-(\lambda) + \frac{m_1(\lambda)}{R_+(\lambda)} - r_+(\lambda)m_2(\lambda), \\ w_+(\lambda) = w_-(\lambda) + m_2(\lambda). \end{cases} \quad (31)$$

Hence we have

$$w_+(\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} m_2(x) e^{i\lambda x} dx, \quad w_-(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 m_2(x) e^{i\lambda x} dx,$$

So

$$\begin{aligned} u_+(\lambda) &= \frac{r_+(\lambda)R_+(\lambda)}{\sqrt{2\pi}} \int_0^{\infty} m_2(x) e^{i\lambda x} dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} (m_1(x) + r_+(x)R_+(x)m_2(x)) e^{i\lambda x} dx, \\ u_-(\lambda) &= \frac{r_-(\lambda)}{R_-(\lambda)\sqrt{2\pi}} \int_{-\infty}^0 m_2(x) e^{i\lambda x} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \left(\frac{m_1(x)}{R_-(x)} + \frac{r_+(x)m_2(x)}{R_-(x)} \right) e^{i\lambda x} dx. \end{aligned}$$

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ a(\lambda) \frac{\gamma(\lambda) + \beta_+ \tanh(\beta_+ y)}{\gamma(\lambda)} e^{-\gamma(\lambda)y} \chi_+(y) + b(\lambda) \frac{\gamma(\lambda) - \beta_- \tanh(\beta_- y)}{\gamma(\lambda)} e^{\gamma(\lambda)y} \chi_-(y) \right\} e^{i\lambda x} d\lambda, \quad (32)$$

Which is get the solution of the boundary value problem (2).

Theorem: The boundary value problem (2) has unique solution, which is given by the formula (32), where the functions $a(\lambda)$ and $b(\lambda)$ can be restored by the formula (13).

CONCLUSION

Solvability of boundary value problem reduces to solvability of Riman-Hilbrt boundary problems. The solvability analysis is based on the factorization problem of some matrix-function.

REFERENCES

1. Speck, F.O., 1986. Mixed boundary value problems of the type of Sommerfeld half-plane problem, Proc. of the Royal Soc. Edinburg, 104A., pp: 261-277.
2. Noble, B., 1958. Methods based on the Wiener-Hopf technique for the solution of partial differential equations, pergamon, London,
3. Rawlins, A.D., 1981. The explicit Wiener-Hopf factorization of a special matrix, Z. Angew. Math. Mech., 61: 527-528.
4. Daniel, V.G., 1984. On the solution of two coupled Wiener-Hopf equations, SIAM J. Appl. Math., 44: 667-680.

5. Eskin, G.I., Boundary value problems for elliptic pseudodifferential equations, NAUKA, Moscow, 1.
6. Muskhelishvili, N.I., 1968. Singular integral equations, NAUKA, Moscow.