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Biharmonic B-General Helices According to Bishop Frame in Heisenberg Group Heis^3

T. Körpınar, E. Turhan and V. Asil

Department of Mathematics 23119, Fırat University, Elazığ, Turkey

Abstract: In this paper, we study biharmonic B-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for B-general helices to be biharmonic according to Bishop frame. We characterize the biharmonic B-general helices in terms of Bishop frame in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

Mathematics Subject Classifications: 31B30 · 58E20.
Key words: Biharmonic curve · Bishop frame · Heisenberg group

INTRODUCTION

Helices arise in nanosprings, carbon nanotubes, α-helices, DNA double and collagen triple helix, the double helix shape is commonly associated with DNA, since the double helix is structure of DNA. This fact was published for the first time by Watson and Crick in 1953 [1]. They constructed a molecular model of DNA in which there were two complementary, antiparallel (side-by-side in opposite directions) strands of the bases guanine, adenine, thymine and cytosine, covalently linked through phosphodiesterase bonds. Each strand forms a helix and two helices are held together through hydrogen bonds, ionic forces, hydrophobic interactions and van der Waals fores forming a double helix, lipid bilayers, bacterial flagella in Salmonella and E. coli, aerial hyphae in actynomycetes, bacterial shape in spirochetes, horns, tendrils, vines, screws, springs, helical staircases and sea shells (helico-spiral structures) [2, 3].

A curve of constant slope or general helix in Euclidean 3-space E^3, is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by Lancret in 1802 and first proved by de Saint Venant in 1845 [4] is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant.

In the last decade there have been a growing interest in the theory of biharmonic functions which can be divided into two main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDE: biharmonic functions are solutions of a fourth order strongly elliptic semilinear PDE.

In this paper, we study biharmonic B-general helices according to Bishop frame in the Heisenberg group Heis^3. We give necessary and sufficient conditions for B-general helices to be biharmonic according to Bishop frame. We characterize the biharmonic B-general helices in terms of Bishop frame in the Heisenberg group Heis^3. Additionally, we illustrate our main theorem.

The Heisenberg Group Heis^3: Heisenberg group Heis^3 can be seen as the space R^3 endowed with the following multipilcation:

$$(\bar{x},\bar{y},\bar{z})(x,y,z)=(\bar{x}+x,\bar{y}+y,\bar{z}+z-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y}) \qquad (2.1)$$

Heis^3 is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g=dx^2+dy^2+(dz-xdy)^2$$

The Lie algebra of Heis^3 has an orthonormal basis

$$e_1=\frac{\partial}{\partial x},\ e_2=\frac{\partial}{\partial y}+x\frac{\partial}{\partial z},\ e_3=\frac{\partial}{\partial z}, \qquad (2.2)$$

Corresponding Author: T. Körpınar, Department of Mathematics 23119, Fırat University, Elazığ, Turkey.

for which we have the Lie products

$$[e_1, e_2] = e_3, [e_2, e_3] = [e_3, e_1] = 0$$

with

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

We obtain

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = 0, \\ \nabla_{e_1} e_2 &= -\nabla_{e_2} e_1 = \frac{1}{2} e_3, \\ \nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = -\frac{1}{2} e_2, \\ \nabla_{e_2} e_3 &= \nabla_{e_3} e_2 = \frac{1}{2} e_1. \end{aligned}$$

We adopt the following notation and sign convention for Riemannian curvature operator on Heis^3 defined by

$$R(X,Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_Y Z + \nabla_{[X,Y]} Z,$$

while the Riemannian curvature tensor is given by

$$R(X,Y,Z,W) = g(R(X,Y)Z,W),$$

where X,Y,Z,W are smooth vector fields on Heis^3

The components $\{R_{ijkl}\}$ of R relative to $\{e_1, e_1, e_3\}$ are defined by

$$g\left(R(e_i, e_j)e_k, e_l\right) = R_{ijkl}.$$

The non vanishing components of the above tensor fields are

$$\begin{aligned} &R_{121} = -\frac{3}{4} e_2, \quad R_{131} = \frac{1}{4} e_3, \quad R_{122} = \frac{3}{4} e_1, \\ &R_{232} = \frac{1}{4} e_3, \quad R_{133} = -\frac{1}{4} e_1, \quad R_{233} = -\frac{1}{4} e_2, \\ &R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}. \end{aligned} \tag{2.3}$$

3 Biharmonic B-general Helices with Bishop Frame in the Heisenberg Group Heis^3: Let $\gamma: I \to Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let {T,N,B} be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

T is the unit vector field γ' tangent to γ, N is the unit vector field in the direction of $\nabla_T T$ (normal to γ) and B is chosen so that {T,N,B} is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= \kappa \mathbf{N}, \\ \nabla_{\mathbf{T}} \mathbf{N} &= -\kappa \mathbf{T} + \tau \mathbf{B}, \\ \nabla_{\mathbf{T}} \mathbf{B} &= -\tau \mathbf{N}, \end{aligned} \tag{3.1}$$

where k is the curvature of γ and τ is its torsion and

$$\begin{aligned} &g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{N},\mathbf{N}) = 1, g(\mathbf{B},\mathbf{B}) = 1, \\ &g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $k \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{T}} \mathbf{T} &= k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2, \\ \nabla_{\mathbf{T}} \mathbf{M}_1 &= -k_1 \mathbf{T}, \\ \nabla_{\mathbf{T}} \mathbf{M}_2 &= -k_2 \mathbf{T}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{M}_1,\mathbf{M}_1) = 1, g(\mathbf{M}_2,\mathbf{M}_2) = 1, \\ &g(\mathbf{T},\mathbf{M}_1) = g(\mathbf{T},\mathbf{M}_2) = g(\mathbf{M}_1,\mathbf{M}_2) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set {T,M$_1$,M$_1$} as Bishop trihedra, k_1 and k_2 as Bishop curvatures. where $\theta(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'(s)$ and $\kappa(s) = \sqrt{k_2^2 + k_1^2}$. Thus, Bishop curvatures are defined by [5]

$$\begin{aligned} k_1 &= \kappa(s) \cos \theta(s), \\ k_2 &= \kappa(s) \sin \theta(s). \end{aligned} \tag{3.5}$$

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T^1 \mathbf{e}_1 + T^2 \mathbf{e}_2 + T^3 \mathbf{e}_3, \\ \mathbf{M}_1 &= M_1^1 \mathbf{e}_1 + M_1^2 \mathbf{e}_2 + M_1^3 \mathbf{e}_3, \\ \mathbf{M}_2 &= M_2^1 \mathbf{e}_1 + M_2^2 \mathbf{e}_2 + M_2^3 \mathbf{e}_3. \end{aligned} \tag{3.6}$$

Theorem 3.1: *$\gamma: I \to Heis^3$ is a biharmonic curve with Bishop frame if and only if* $k_1^2 + k_2^2 = \text{constant} = C \neq 0$,

$$\begin{aligned} k_1'' - Ck_1 &= k_1 \left[\frac{1}{4} - \left(M_2^3 \right)^2 \right] - k_2 M_1^3 M_2^3, \\ k_2'' - Ck_2 &= k_1 M_1^3 M_2^3 + k_2 \left[\frac{1}{4} - \left(M_1^3 \right)^2 \right]. \end{aligned} \tag{3.7}$$

Definition 3.2: *A regular curve $\gamma: I \to Heis^3$ is called a general helix provided the unit vector* T of the curve γ has constant angle θ with some fixed unit vector u that is

$$g\left(\mathbf{T}(s),u\right) = \cos\theta \text{ for all } s \in I. \tag{3.8}$$

To separate a general helix according to Bishop frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as B-general helix.

Theorem 3.3: *(Main Theorem) Let $\gamma_B : I \to Heis^3$ be a unit speed biharmonic B-general helix with non-zero natural curvatures. Then the parametric equation of γ_B are*

$$x_B(s) = \frac{\sin\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]+\zeta_2,$$

$$y_B(s) = \frac{\sin\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]+\zeta_3, \text{3.9}$$

$$z_B(s) = (\cos\theta)s + \frac{\sin^2\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\left(\frac{s}{2} - \frac{\sin 2\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]}{4\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\right.$$

$$\left.\frac{\zeta_1\sin\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]+\zeta_4, \tag{8}$$

where ζ_0, $\zeta_1,\zeta_2,\zeta_3,\zeta_4$ are constants of integration.

Proof: Without loss of generality, we take the axis of γ_B is parallel to the vector e_3. Then,

$$g\left(\mathbf{T},\mathbf{e}_3\right) = T_3 = \cos\theta, \tag{3.10}$$

where θ is constant angle.
Substituting the components T_1, T_2 and T_3 in the second equation of (3.6), we have the following equation

$$\mathbf{T} = \sin\theta\cos\Gamma(s)\mathbf{e}_1 + \sin\theta\sin\Gamma(s)\mathbf{e}_2 + \cos\theta\mathbf{e}_3. \tag{3.11}$$

Using Bishop formulas, we get

$$\left|\nabla_{\mathbf{T}}\mathbf{T}\right|^2 = k_1^2 + k_2^2$$

The covariant derivative of the vector field T is:

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' + T_2T_3)\mathbf{e}_1 + (T_2' - T_1T_3)\mathbf{e}_2 + T_3'\mathbf{e}_3. \tag{3.12}$$

We use (3.11) and (3.12), yields

$$\Gamma(s) = \left(\frac{k_1^2+k_2^2}{\sin^2\theta} - \cos\theta\right)^{\frac{1}{2}} s + \zeta_0, \tag{3.13}$$

where ζ_0 is constant of integration.
Substituting (3.13) in (3.11), we have

$$\mathbf{T} = \sin\theta\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]\mathbf{e}_1 + \sin\theta\sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]\mathbf{e}_2 \tag{3.14}$$

From orthonormal basis (2.2) and (3.14), we obtain

$$\mathbf{T} = \left(\sin\theta\cos\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right],\right.$$

$$\sin\theta\sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right],$$

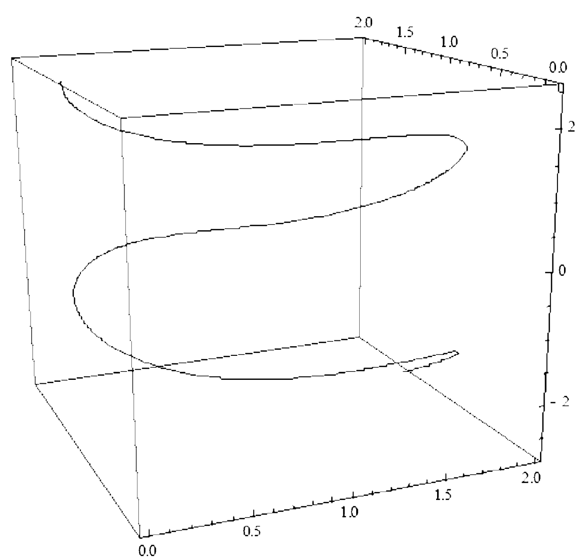
$$\cos\theta + \frac{\sin^2\theta}{\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}}\sin^2\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]$$

$$\left.+\zeta_1\sin\theta\sin\left[\left(\frac{k_1^2+k_2^2}{\sin^2\theta}-\cos\theta\right)^{\frac{1}{2}}s+\zeta_0\right]\right), \tag{3.15}$$

where ζ_1 is constant of integration.

If we integrate above equation, we have (3.9), the theorem is proved.

We can draw unit speed biharmonic B-general helices according to Bishop frame with helping the programme of Mathematica as follow:



$k_1^2 + k_2^2 = \sin\theta = 1$ and $\zeta_0 = \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0.$

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